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Research Article

On a Hilbert-Type Operator with a Symmetric Homogeneous Kernel of −1-Order and Applications

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Some character of the symmetric homogenous kernel of -1-order in Hilbert-type operator $T: l^r \to l^r$ $(r > 1)$ is obtained. Two equivalent inequalities with the symmetric homogenous kernel of $-\lambda$-order are given. As applications, some new Hilbert-type inequalities with the best constant factors and the equivalent forms as the particular cases are established.

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1. Introduction

If the real function $k(x,y)$ is measurable in $(0,\infty)\times(0,\infty)$, satisfying $k(y,x) = k(x,y)$, for $x, y \in (0,\infty)$, then one calls $k(x,y)$ the symmetric function. Suppose that $p > 1$, $1/p + 1/q = 1$, l^r $(r = p, q)$ are two real normal spaces, and $k(x,y)$ is a nonnegative symmetric function in $(0,\infty)\times(0,\infty)$. Define the operator T as follows: for $a = \{a_m\}_{m=1}^{\infty} \in l^p$,

$$(Ta)(n) := \sum_{m=1}^{\infty} k(m,n)a_m, \quad n \in \mathbb{N}; \tag{1.1}$$

or for $b = \{b_n\}_{n=1}^{\infty} \in l^q$,

$$(Tb)(m) := \sum_{n=1}^{\infty} k(m,n)b_n, \quad m \in \mathbb{N}. \tag{1.2}$$

The function $k(x,y)$ is said to be the symmetric kernel of T.

If $k(x,y)$ is a symmetric function, for $\varepsilon(\geq 0)$ small enough and $x>0$, set $\widetilde{k}_r(\varepsilon,x)$ as

$$\widetilde{k}_r(\varepsilon,x):=\int_0^\infty k(x,t)\left(\frac{x}{t}\right)^{(1+\varepsilon)/r}dt \quad (r=p,q). \tag{1.3}$$

In 2007, Yang [1] gave three theorems as follows.

Theorem 1.1. (i) *If for fixed $x>0$, and $r=p,q$, the functions $k(x,t)(x/t)^{1/r}$ are decreasing in $t\in(0,\infty)$, and*

$$\widetilde{k}_r(0,x):=\int_0^\infty k(x,t)\left(\frac{x}{t}\right)^{1/r}dt=k_p \quad (r=p,q), \tag{1.4}$$

where k_p is a positive constant independent of x, then $T\in B(l^r\to l^r)$, T is called the Hilbert-type operator and $\|T\|r\leq k_p$ $(r=p,q)$;

(ii) *if for fixed $x>0$, $\varepsilon\geq 0$ and $r=p,q$, the functions $k(x,t)(x/t)^{(1+\varepsilon)/r}$ are decreasing in $t\in(0,\infty)$; $\widetilde{k}_r(\varepsilon,x)=k_p(\varepsilon)$ $(r=p,q;\ \varepsilon\geq 0)$ is independent of x, satisfying $k_p(\varepsilon)=k_p+o(1)$ $(\varepsilon\to 0^+)$, and*

$$\sum_{m=1}^\infty\frac{1}{m^{1+\varepsilon}}\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{(1+\varepsilon)/r}dt=O(1) \quad (\varepsilon\to 0^+;\ r=p,q), \tag{1.5}$$

then $\|T\|_r=k_p$ $(r=p,q)$.

Theorem 1.2. *Suppose that $p>1$, $1/p+1/q=1$, and $\widetilde{k}_r(0,x)$ $(r=p,q;\ x>0)$ in (1.3) satisfy condition (i) in Theorem 1.1. If $a_m,b_n\geq 0$ and $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, then one has the following two equivalent inequalities:*

$$\begin{gathered}\sum_{n=1}^\infty\sum_{m=1}^\infty k(m,n)a_mb_n\leq k_p\|a\|_p\|b\|_q;\\ \left\{\sum_{n=1}^\infty\left(\sum_{m=1}^\infty k(m,n)a_m\right)^p\right\}^{1/p}\leq k_p\|a\|_p,\end{gathered} \tag{1.6}$$

where the positive constant factor $k_p(=\int_0^\infty k(x,t)(x/t)^{1/q}dt)$ is independent of $x>0$.

Theorem 1.3. *Suppose that $p>1$, $1/p+1/q=1$, and $\widetilde{k}_r(\varepsilon,x)$ $(r=p,q;\ x>0,\ \varepsilon\geq 0)$ in (1.3) satisfy condition (ii) in Theorem 1.1. If $a_m,b_n\geq 0$ and $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, and $\|a\|_p,\|b\|_q>0$, T is defined by (1.1), and the formal inner product of Ta and b is defined by*

$$(Ta,b):=\sum_{n=1}^\infty\sum_{m=1}^\infty k(m,n)a_mb_n=(a,Tb), \tag{1.7}$$

then one has the following two equivalent inequalities:

$$(Ta,b) < \|T\|_p \|a\|_p \|b\|_q; \\ \|Ta\|_p < \|T\|_p \|a\|_p, \tag{1.8}$$

where the constant factor $\|T\|_p = \int_0^\infty k(x,t)(x/t)^{1/q} dt (> 0)$ *is the best possible.*

Recently, Yang [2] also considered some frondose character of the symmetric kernel for $p = q = 2$; Yang et al. [3–6] considered the character of the norm in Hilbert-type integral operator and some applications.

Definition 1.4. If $k(x,y)$ is a nonnegative function in $(0,\infty)\times(0,\infty)$, and there exists $\lambda > 0$, satisfying $k(xu,xv) = x^{-\lambda}k(u,v)$, for any $x,u,v \in (0,\infty)$, then $k(x,y)$ is said to be the homogeneous function of $-\lambda$-order.

In this paper, for keeping on research of the thesis in [1, 2], some frondose character of the symmetric homogeneous kernel of -1-order satisfying condition (ii) of Theorem 1.1 is considered. One also considers two equivalent inequalities with the symmetric homogeneous kernel of $-\lambda$-order. As applications, some new Hilbert-type inequalities with the best constant factors and the equivalent forms as the particular cases of the kernel are established.

For this, one needs the formula of the Beta function $B(u,v)$ as (see [7])

$$B(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{-u+1} du = B(v,u) \quad (u,v > 0). \tag{1.9}$$

2. A lemma and a theorem

Suppose that the symmetric kernel $k(x,y)$ is homogeneous function of -1-order. Setting $u = t/x$ in (1.3), one finds $\widetilde{k}_r(\varepsilon,x)$ is independent of $x > 0$ and $k_r(\varepsilon) := \int_0^\infty k(1,u)u^{-(1+\varepsilon)/r} du = \widetilde{k}_r(\varepsilon,x)$ $(r = p,q)$. If $k_p := \widetilde{k}_r(0,x)$ is a positive constant, then setting $v = 1/u$, one obtains $k_q = \int_0^\infty k(1,u)u^{-1/q} du = \int_0^\infty k(v,1)v^{-1/p} dv = k_p > 0$, and $\widetilde{k}_r(0,x) = k_p$ $(r = p,q)$. Hence based on the above conditions, if for fixed $x > 0$ and $r = p,q$, the functions $k(x,t)(x/t)^{1/r}$ are decreasing in $t \in (0,\infty)$, then the kernel $k(x,y)$ satisfies condition (i) of Theorem 1.1 and suits using Theorem 1.2.

Lemma 2.1. *Let* $p > 1$, $1/p + 1/q = 1$, *let the symmetric kernel* $k(x,y)$ *be homogeneous function of* -1*-order, and for fixed* $x > 0$, $r = p,q$, *the functions* $k(x,t)(x/t)^{1/r}$ *be decreasing in* $t \in (0,\infty)$. *If* $k(1,u)$ *is positive and continuous in* $(0,1]$, *and there exist constant* $\eta < \min\{1/p,1/q\}$ *and* $C \geq 0$, *such that* $\lim_{u\to 0^+} u^\eta k(1,u) = C$, *then for* $\varepsilon \in [0,\min\{p,q\}(1-\eta)-1)$, $k_r(\varepsilon) := \int_0^\infty k(1,u)u^{-(1+\varepsilon)/r} du$ *are positive constants satisfying* $k_p(\varepsilon) = k_p + o(1)$ $(\varepsilon\to 0^+;\ r = p,q)$, *and expression (1.5) is valid. Hence* $k(x,y)$ *satisfies condition (ii) of Theorem 1.1 and suits using Theorem 1.3.*

Proof. For fixed $x > 0$, $\varepsilon \geq 0$, and $r = p,q$, the functions $k(x,t)(x/t)^{(1+\varepsilon)/r} = k(x,t)(x/t)^{1/r}(x/t)^{\varepsilon/r}$ are still decreasing in $t \in (0,\infty)$. Since $\lim_{u\to 0^+} u^\eta k(1,u) = C$ and $u^\eta k(1,u)$

is positive and continuous in $(0,1]$, there exists a constant $L>0$, such that $u^{\eta}k(1,u)\leq L$ $(u\in[0,1])$. Setting $u=1/v$ in the following second integral, since $k(1,1/v)=vk(v,1)$, one finds

$$\begin{aligned}0<k_p(\varepsilon)&=\int_0^1 k(1,u)u^{-(1+\varepsilon)/p}du+\int_1^{\infty}k(1,u)u^{-(1+\varepsilon)/p}du\\&=\int_0^1 k(1,u)u^{-(1+\varepsilon)/p}du+\int_0^1 k(v,1)v^{(1+\varepsilon)/p-1}dv\\&=\int_0^1[u^{\eta}k(1,u)][u^{-(1+\varepsilon)/p-\eta}+u^{(1+\varepsilon)/p-\eta-1}]du\\&\leq L\int_0^1(u^{-(1+\varepsilon)/p-\eta}+u^{(1+\varepsilon)/p-\eta-1})du=L\Big[\Big(\frac{1}{q}-\frac{\varepsilon}{p}-\eta\Big)^{-1}+\Big(\frac{1+\varepsilon}{p}-\eta\Big)^{-1}\Big].\end{aligned}\tag{2.1}$$

Hence the integral $k_p(\varepsilon)=\int_0^{\infty}k(1,u)u^{-(1+\varepsilon)/p}du$ is a positive constant. Since by (2.1), one obtains

$$\begin{aligned}0\leq|k_p(\varepsilon)-k_p|&=\Big|\int_0^1 k(1,u)(u^{-(1+\varepsilon)/p}-u^{-1/p}+u^{(1+\varepsilon)/p-1}-u^{-1/q})du\Big|\\&\leq\int_0^1[u^{\eta}k(1,u)]\,|u^{-(1+\varepsilon)/p-\eta}-u^{-1/p-\eta}+u^{(1+\varepsilon)/p-1-\eta}-u^{-1/q-\eta}|\,du\\&\leq L\int_0^1[\,|u^{-(1+\varepsilon)/p-\eta}-u^{-1/p-\eta}|+|u^{-1/q-\eta}-u^{(1+\varepsilon)/p-1-\eta}|\,]du\\&=L\Big[\Big|\int_0^1(u^{-(1+\varepsilon)/p-\eta}-u^{-1/p-\eta})du\Big|+\Big|\int_0^1(u^{-1/q-\eta}-u^{(1+\varepsilon)/p-1-\eta})du\Big|\Big]\\&=L\Big[\Big|\Big(\frac{1}{q}-\frac{\varepsilon}{p}-\eta\Big)^{-1}-\Big(\frac{1}{q}-\eta\Big)^{-1}\Big|+\Big|\Big(\frac{1}{p}-\eta\Big)^{-1}-\Big(\frac{1+\varepsilon}{p}-\eta\Big)^{-1}\Big|\Big].\end{aligned}\tag{2.2}$$

Then $|k_p(\varepsilon)-k_p|\to 0$ $(\varepsilon\to 0^+)$ and $k_p(\varepsilon)=k_p+o(1)$ $(\varepsilon\to 0^+)$. Similarly, $k_q(\varepsilon)$ is also a positive constant and $k_q(\varepsilon)=k_q+o(1)=k_p+o(1)$ $(\varepsilon\to 0^+)$. Hence $k_r(\varepsilon)$ is a positive constant with $k_r(\varepsilon)=k_p+o(1)$ $(\varepsilon\to 0^+;\ r=p,q)$. Since for $\varepsilon\in[0,\min\{p,q\}(1-\eta)-1)$ and $r=p,q$, one obtains

$$\begin{aligned}0&<\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1 k(m,t)\Big(\frac{m}{t}\Big)^{(1+\varepsilon)/r}dt=\sum_{m=1}^{\infty}\frac{1}{m^{2+\varepsilon}}\int_0^1 k\Big(1,\frac{t}{m}\Big)\Big(\frac{m}{t}\Big)^{(1+\varepsilon)/r}dt\\&=\sum_{m=1}^{\infty}\frac{1}{m^{2+\varepsilon}}\int_0^1\Big(\frac{t}{m}\Big)^{\eta}k\Big(1,\frac{t}{m}\Big)\Big(\frac{t}{m}\Big)^{-(1+\varepsilon)/r-\eta}dt\\&\leq L\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1\Big(\frac{t}{m}\Big)^{-(1+\varepsilon)/r-\eta}d\Big(\frac{t}{m}\Big)=\frac{L}{1-(1+\varepsilon)/r-\eta}\sum_{m=1}^{\infty}\frac{1}{m^{2-(1+\varepsilon)/r-\eta}}<\infty,\end{aligned}\tag{2.3}$$

and then (1.5) is valid. The lemma is proved. □

Note. In applying Lemma 2.1, if $k(1,u)$ is continuous in $[0,1]$, then one can set $\eta=0$ and does not consider the limit.

If $k_\lambda(x,y)$ is the homogeneous function of $-\lambda$-order ($\lambda>0$), then $k(x,y)=k_\lambda(x,y)(xy)^{(1/2)(\lambda-1)}$ is obviously homogeneous function of -1-order. Suppose that $k(x,y)$ satisfies the conditions of Lemma 2.1, setting $\omega_r(x)=x^{(r/2)(1-\lambda)}$ $(r=p,q)$, since

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(m,n)a_mb_n=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k(m,n)\left(\omega_p^{1/p}(m)a_m\right)\left(\omega_q^{1/q}(n)b_n\right);\\ \sum_{n=1}^{\infty}\left(\omega_q^{1-p}(n)\left(\sum_{m=1}^{\infty}k_\lambda(m,n)a_m\right)^p=\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}k(m,n)\left(\omega_p^{1/p}(m)a_m\right)\right]^p,\right.\end{gathered}\tag{2.4}$$

by (1.8), one has the following theorem.

Theorem 2.2. *Let $p>1$, $1/p+1/q=1$, let the symmetric kernel $k_\lambda(x,y)$ be homogeneous function of $-\lambda$-order ($\lambda>0$), and let the functions $k(x,y)=k_\lambda(x,y)(xy)^{(1/2)(\lambda-1)}$ satisfy the conditions of Lemma 2.1. If $\omega_r(x)=x^{(r/2)(1-\lambda)}$ $(r=p,q)$, $a_m,b_n\geq 0$, $a=\{a_m\}_{m=1}^{\infty}\in l_{\omega_p}^p$, $b=\{b_n\}_{n=1}^{\infty}\in l_{\omega_q}^q$, such that $\|a\|_{p,\omega_p}=\{\sum_{n=1}^{\infty}n^{(p/2)(1-\lambda)}a_n^p\}^{1/p}>0,\|b\|_{q,\omega_q}=\{\sum_{n=1}^{\infty}n^{(q/2)(1-\lambda)}b_n^q\}^{1/q}>0$, then one has the following two equivalent inequalities:*

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(m,n)a_mb_n<k_p\|a\|_{p,\omega_p}\|b\|_{q,\omega_q};\\ \left\{\sum_{n=1}^{\infty}\left(\omega_q^{1-p}(n)\left(\sum_{m=1}^{\infty}k_\lambda(m,n)a_m\right)^p\right\}^{1/p}<k_p\|a\|_{p,\omega_p},\right.\end{gathered}\tag{2.5}$$

where the constant factor $k_p=\int_0^\infty k(1,u)u^{-1/p}dt$ is the best possible.

3. Applications to some Hilbert-type inequalities

In the following, suppose that $p>1$, $1/p+1/q=1$, $\omega_r(n)=n^{(r/2)(1-\lambda)}$ $(r=p,q)$, $a_m,b_n\geq 0$, $a=\{a_m\}_{m=1}^{\infty}\in l_{\omega_p}^p$, $b=\{b_n\}_{n=1}^{\infty}\in l_{\omega_q}^q$, such that $\|a\|_{p,\omega_p}=\{\sum_{n=1}^{\infty}\omega_p(n)a_n^p\}^{1/p}>0$, $\|b\|_{q,\omega_q}=\{\sum_{n=1}^{\infty}\omega_q(n)b_n^q\}^{1/q}>0$, and one omits the words that the constant factors are the best possible.

(a) Let $k_\lambda(x,y)=(1/(x^\alpha+y^\alpha)^{\lambda/\alpha})(\alpha>0,0\leq 1-2\min\{1/p,1/q\}<\lambda\leq 1+2\min\{1/p,1/q\})$, and $k(x,y)=(xy)^{(\lambda-1)/2}/(x^\alpha+y^\alpha)^{\lambda/\alpha}$. Then for fixed $x>0$ and $r=p,q$, $((xt)^{(\lambda-1)/2}/(x^\alpha+t^\alpha)^{\lambda/\alpha})(x/t)^{1/r}=(x^{(1/2)(\lambda-1)+1/r}/(x^\alpha+t^\alpha)^{\lambda/\alpha})(1/t)^{1/r+(1/2)(1-\lambda)}$ are decreasing in $t\in(0,\infty)$. Since $k(1,u)=u^{(\lambda-1)/2}/(1+y^\alpha)^{\lambda/\alpha}$ is continuous in $(0,1]$, there exists $\eta=(1/2)(1-\lambda)<\min\{1/p,1/q\}$, such that $\lim_{u\to 0^+}u^\eta k(1,u)=1$; setting $t=u^\alpha$ in the following, one obtains

$$\begin{aligned}k_p&=\int_0^\infty\frac{1}{(1+u^\alpha)^{\lambda/\alpha}}u^{(\lambda-1)/2-1/p}du=\frac{1}{\alpha}\int_0^\infty\frac{1}{(1+t)^{\lambda/\alpha}}t^{(1/\alpha)[(\lambda+1)/2-1/p]-1}dt\\&=\frac{1}{\alpha}B\left(\frac{1}{\alpha}\left(\frac{\lambda+1}{2}-\frac{1}{p}\right),\frac{1}{\alpha}\left(\frac{\lambda+1}{2}-\frac{1}{q}\right)\right)=:k_p\left(\alpha,\lambda\right).\end{aligned}\tag{3.1}$$

Then by (2.5), one has the following corollary.

Corollary 3.1. *The following inequalities are equivalent:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^{\lambda/\alpha}}<k_p(\alpha,\lambda)\|a\|_{p,\omega_p}\|b\|_{q,\omega_q};$$
$$\left\{\sum_{n=1}^{\infty}n^{(p/2)(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^{\lambda/\alpha}}\right]^p\right\}^{1/p}<k_p(\alpha,\lambda)\|a\|_{p,\omega_p}. \tag{3.2}$$

In particular, (i) *for $\alpha=1$, one has $k_p(1,\lambda)=B((\lambda+1)/2-1/p,(\lambda+1)/2-1/q)$ and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda}<B\left(\frac{\lambda+1}{2}-\frac{1}{p},\frac{\lambda+1}{2}-\frac{1}{q}\right)\|a\|_{p,\omega_p}\|b\|_{q,\omega_q}; \tag{3.3}$$

$$\left\{\sum_{n=1}^{\infty}n^{(p/2)(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^\lambda}\right]^p\right\}^{1/p}<B\left(\frac{\lambda+1}{2}-\frac{1}{p},\frac{\lambda+1}{2}-\frac{1}{q}\right)\|a\|_{p,\omega_p}; \tag{3.4}$$

(ii) *for $\alpha=\lambda$, one has $k_p(\lambda,\lambda)=(1/\lambda)B((1/\lambda)((\lambda+1)/2-1/p),(1/\lambda)((\lambda+1)/2-1/q))$ and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}<\frac{1}{\lambda}B\left(\frac{1}{\lambda}\left(\frac{\lambda+1}{2}-\frac{1}{p}\right),\frac{1}{\lambda}\left(\frac{\lambda+1}{2}-\frac{1}{q}\right)\right)\|a\|_{p,\omega_p}\|b\|_{q,\omega_q}; \tag{3.5}$$

$$\left\{\sum_{n=1}^{\infty}n^{(p/2)(\lambda-1)}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\right)^p\right\}^{1/p}<\frac{1}{\lambda}B\left(\frac{1}{\lambda}\left(\frac{\lambda+1}{2}-\frac{1}{p}\right),\frac{1}{\lambda}\left(\frac{\lambda+1}{2}-\frac{1}{q}\right)\right)\|a\|_{p,\omega_p}. \tag{3.6}$$

(b) Let $k_\lambda(x,y)=(\ln(x/y)/(x^\lambda-y^\lambda))$ $(0\le 1-2\min\{1/p,1/q\}<\lambda\le 1+2\min\{1/p,1/q\})$, $k(x,y)=(\ln(x/y)/(x^\lambda-y^\lambda))(xy)^{(1/2)(\lambda-1)}$. Since $\ln(t/x)/((t/x)^\lambda-1)$ is decreasing in $t\in(0,\infty)$ (see [8]), then for fixed $x>0$ and $r=p,q$,

$$\frac{\ln(x/t)}{x^\lambda-t^\lambda}(xt)^{(1/2)(\lambda-1)}\left(\frac{x}{t}\right)^{1/r}=x^{-(1/2)(\lambda+1)+1/r}\frac{\ln(t/x)}{(t/x)^\lambda-1}\left(\frac{1}{t}\right)^{1/r+(1/2)(1-\lambda)} \tag{3.7}$$

are decreasing in $t\in(0,\infty)$. Since $k(1,u)=(\ln u)u^{(\lambda-1)/2}/(u^\lambda-1)$ is continuous in $(0,1](k(1,1)=\lim_{u\to 1}k(1,u))$, and $(1-\lambda)/2<\min\{1/p,1/q\}$, there exists $\varepsilon>0$, such that $\eta=(1/2)(1-\lambda)+\varepsilon<\min\{1/p,1/q\}$, and $\lim_{u\to 0^+}u^\eta k(1,u)=0$, then setting $t=u^\lambda$ in the following, and using the formula as (see [9])

$$\int_0^\infty\frac{\ln t}{t-1}t^{a-1}du=\left[\frac{\pi}{\sin a\pi}\right]^2=[B(a,1-a)]^2\quad(0<a<1), \tag{3.8}$$

one obtains

$$\begin{aligned} k_p &= \int_0^\infty \frac{\ln u}{u^\lambda - 1} u^{(\lambda-1)/2-1/p} du = \frac{1}{\lambda^2}\int_0^\infty \frac{\ln t}{t-1} t^{1/2+(1/\lambda)(1/q-1/2)-1} dt \\ &= \left[\frac{1}{\lambda}B\left(\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{q}-\frac{1}{2}\right),\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{2}\right)\right)\right]^2. \end{aligned} \tag{3.9}$$

Then by (2.5), one has the following corollary.

Corollary 3.2. *The following inequalities are equivalent:*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{\ln(m/n)a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{1}{\lambda}B\left(\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{q}-\frac{1}{2}\right),\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{2}\right)\right)\right]^2 \|a\|_{p,\omega_p}\|b\|_{q,\omega_q}; \tag{3.10}$$

$$\begin{aligned} &\left\{\sum_{n=1}^\infty n^{(p/2)(\lambda-1)}\left[\sum_{m=1}^\infty \frac{\ln(m/n)a_m}{m^\lambda - n^\lambda}\right]^p\right\}^{1/p} \\ &\quad < \left[\frac{1}{\lambda}B\left(\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{q}-\frac{1}{2}\right),\frac{1}{2}+\frac{1}{\lambda}\left(\frac{1}{p}-\frac{1}{2}\right)\right)\right]^2 \|a\|_{p,\omega_p}. \end{aligned} \tag{3.11}$$

(c) Let $k_\lambda(x,y) = 1/\max\{x^\lambda, y^\lambda\}$ $(0 \le 1 - 2\min\{1/p,1/q\} < \lambda \le 1 + 2\min\{1/p,1/q\})$, and $k(x,y) = (1/\max\{x^\lambda,y^\lambda\})(xy)^{(1/2)(\lambda-1)}$. Then for fixed $x > 0$ and $r = p,q$,

$$\frac{1}{\max\{x^\lambda,t^\lambda\}}(xt)^{(1/2)(\lambda-1)}\left(\frac{x}{t}\right)^{1/r} = x^{(1/2)(\lambda-1)+1/r}\frac{1}{\max\{x^\lambda,t^\lambda\}}\left(\frac{1}{t}\right)^{1/r+(1/2)(1-\lambda)} \tag{3.12}$$

are decreasing in $t \in (0,\infty)$. Since $k(1,u) = (u^{(\lambda-1)/2}/\max\{1,u^\lambda\})$ $(u \in (0,1])$ is continuous in $(0,1]$, there exists $\eta = (1/2)(1-\lambda) < \min\{1/p,1/q\}$, and $\lim_{u\to 0^+} u^\eta k(1,u) = 1$, one finds

$$\begin{aligned} k_p &= \int_0^\infty \frac{1}{\max\{1,u^\lambda\}} u^{(\lambda-1)/2-1/p} du = \int_0^1 u^{(\lambda-1)/2-1/p} du + \int_1^\infty u^{(\lambda-1)/2-\lambda-1/p} du \\ &= \left[\left(\frac{\lambda-1}{2}+\frac{1}{q}\right)^{-1} + \left(\frac{\lambda-1}{2}+\frac{1}{p}\right)^{-1}\right]. \end{aligned} \tag{3.13}$$

Then by (2.5), one has the following corollary.

Corollary 3.3. *The following inequalities are equivalent:*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} < \left[\left(\frac{\lambda-1}{2}+\frac{1}{q}\right)^{-1} + \left(\frac{\lambda-1}{2}+\frac{1}{p}\right)^{-1}\right]\|a\|_{p,\omega_p}\|b\|_{q,\omega_q}; \tag{3.14}$$

$$\left\{\sum_{n=1}^\infty n^{(p/2)(\lambda-1)}\left[\sum_{m=1}^\infty \frac{\ln(m/n)a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p\right\}^{1/p} < \left[\left(\frac{\lambda-1}{2}+\frac{1}{q}\right)^{-1} + \left(\frac{\lambda-1}{2}+\frac{1}{p}\right)^{-1}\right]\|a\|_{p,\omega_p}. \tag{3.15}$$

Remarks 3.4. (i) For $p = q = 2$ in (3.3), (3.5), (3.10), and (3.14), setting $\omega(n) = n^{1-\lambda}$ $(0 < \lambda \leq 2)$, one has some Hilbert-type inequalities with a parameter (see [8, 10–12]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\|a\|_{2,\omega}\|b\|_{2,\omega}; \tag{3.16}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < \frac{\pi}{\lambda}\|a\|_{2,\omega}\|b\|_{2,\omega}; \tag{3.17}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m^{\lambda}-n^{\lambda}} < \left(\frac{\pi}{\lambda}\right)^2\|a\|_{2,\omega}\|b\|_{2,\omega}; \tag{3.18}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda},n^{\lambda}\}} < \frac{4}{\lambda}\|a\|_{2,\omega}\|b\|_{2,\omega}. \tag{3.19}$$

(ii) For $\lambda = 1$ in (3.17), (3.18), and (3.19), one has the following base Hilbert-type inequalities (see [9]):

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\|a\|_2\|b\|_2;\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \pi^2\|a\|_2\|b\|_2;\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < 4\|a\|_2\|b\|_2.\end{gathered} \tag{3.20}$$

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