

*Research Article*

## On the $(p, q)$ -Boundedness of Nonisotropic Spherical Riesz Potentials

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We introduced the concept of nonisotropic spherical Riesz potential operators generated by the  $\lambda$ -distance of variable order on  $\lambda$ -sphere and its  $(p, q)$ -boundedness were investigated.

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### 1. Introduction

Let

$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}. \quad (1.1)$$

In  $\mathbb{R}^n$  spaces,  $L_p$  and  $L_\infty$  are defined as follows:

$$L_p = L_p(\Omega_{n,\lambda}) = \left\{ f(x) : \|f\|_p = \left( \int_{\Omega_{n,\lambda}} |f(x)|^p dx \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty$$
$$L_\infty = L_\infty(\Omega_{n,\lambda}) = \left\{ f(x) : \|f\|_\infty = \operatorname{ess\,sup}_{x \in \Omega_{n,\lambda}} |f(x)| < \infty \right\}, \quad (1.2)$$

where  $\Omega_{n,\lambda}$  is the  $n$ -dimensional unite  $\lambda$ -sphere of  $\mathbb{R}^n$  which is dependent on the  $\lambda$ -distance. The  $\lambda$ -distance between points  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  is defined by the following formula given in [1–10]:

$$|x - y|_\lambda := \left( |x_1 - y_1|^{1/\lambda_1} + |x_2 - y_2|^{1/\lambda_2} + \dots + |x_n - y_n|^{1/\lambda_n} \right)^{\lambda/n}, \quad (1.3)$$

where  $x, y \in \Omega_{n,\lambda}$ ,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\lambda_k > 0$ ,  $k = 1, 2, \dots, n$ ,  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$ . Note that this distance has the following properties of homogeneity for any positive  $t$ ,

$$\left( |t^{\lambda_1} x_1|^{1/\lambda_1} + \dots + |t^{\lambda_n} x_n|^{1/\lambda_n} \right)^{|\lambda|/n} = t^{|\lambda|/n} |x|_\lambda. \tag{1.4}$$

This equality give us that nonisotropic  $\lambda$ -distance is the order of a homogeneous function  $|\lambda|/n$ . So the nonisotropic  $\lambda$ -distance has the following properties:

- (1)  $|x|_\lambda = 0 \Leftrightarrow x = \theta$ ,
- (2)  $|t^\lambda x|_\lambda = |t|^{|\lambda|/n} |x|_\lambda$ ,
- (3)  $|x + y|_\lambda \leq 2^{(1+1/\lambda_{\min})|\lambda|/n} (|x|_\lambda + |y|_\lambda)$ .

Here we consider  $\lambda$ -spherical coordinates by the following formulas:

$$x_1 = (\rho \cos \theta_1)^{2\lambda_1}, \dots, x_n = (\rho \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1})^{2\lambda_n}. \tag{1.5}$$

We obtained that  $|x|_\lambda = \rho^{2|\lambda|/n}$ . It can be seen that the Jacobian  $J_\lambda(\rho, \varphi)$  of this transformation is  $J_\lambda(\rho, \theta) = \rho^{2|\lambda|-1} W_\lambda(\theta)$ , where  $W_\lambda(\theta)$  is the bounded function, which only depends on angles  $\theta_1, \theta_2, \dots, \theta_{n-1}$ . It is clear that if  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 1/2$ , then the  $\lambda$ -distance is the Euclidean distance.

We define angle

$$\cos |x - y|_\lambda = x \cdot y, \tag{1.6}$$

where  $x$  and  $y$  are vectors on the  $n$ -dimensional unite  $\lambda$ -sphere.

For  $f \in L(\Omega_{n,\lambda})$ ,  $0 < \alpha(x) < n$ , we will consider the following nonisotropic spherical Riesz potential operator generated by the  $\lambda$ -distance of variable order:

$$I_\lambda^{\alpha(x)} f(x) = \int_{\Omega_{n,\lambda}} |x - y|_\lambda^{\alpha(x)-n} f(y) dy, \quad x \in \Omega_{n,\lambda}. \tag{1.7}$$

The aim of this paper to show that the well-known properties of classical Riesz potentials may be formulated for our generalization (1.7). We will study the  $(p, q)$ -boundedness of operators (1.7). Note that our results are the generalization of corresponding results for classical Riesz potentials, given in [11]. The important properties of the nonisotropic Riesz potentials and theirs generalizations were studied by many authors. We refer to papers [1–9, 12]. The nonisotropic spherical Riesz potential generated by  $\lambda$ -distance is the classical Riesz potential for  $\lambda_i = 1/2$ ,  $i = 1, 2, \dots, n$  and  $\alpha(x) = \alpha$ . Here particular importance of the nonisotropic kernel is that it does not have the classical triangle inequality.

It is well known that the classical Riesz potentials  $I_\alpha \varphi = \varphi * |x|^{\alpha-n}$  are bounded operators from  $L_p(R^n)$  to  $L_q(R^n)$  for  $1/q = 1/p - \alpha/n$ ,  $0 < \alpha < n$ ,  $1 \leq p < q < \infty$  [10].

LEMMA 1.1. Let  $J_\lambda(x) = \int_{\Omega_{n,\lambda}^1} f(x) K(x, y) dy$ ,  $x \in \Omega_{n,\lambda}^2$ ,

$$k_1 = \sup_{y \in \Omega_{n,\lambda}^1} \left( \int_{\Omega_{n,\lambda}^2} |K(x, y)|^q dx \right)^{1/r} < \infty, \quad k_2 = \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} |K(x, y)|^q dy \right)^{1/q-1/r} < \infty \tag{1.8}$$

and the following conditions are carried out:  $1 \leq p \leq r \leq \infty$ ,  $1 - 1/p + 1/r = 1/q$ ,  $f \in L_p(\Omega_{n,\lambda}^1)$ . Then

$$\|J_\lambda\|_{L_p(\Omega_{n,\lambda}^1)} \leq \|f\|_{L_r(\Omega_{n,\lambda}^2)} k_1 k_2. \quad (1.9)$$

*Proof.* Let  $\lambda, \mu, \nu$  be positive numbers such that  $1/\lambda + 1/\mu + 1/\nu = 1$ . We write

$$J_\lambda(x) = \int_{\Omega_{n,\lambda}^1} f^{p(1/p-1/\mu)}(x) f^{p/\mu}(x) K^{q(1/q-1/\nu)}(x, y) K^{q/\nu}(x, y) dy. \quad (1.10)$$

By Hölder's inequality with exponents  $\lambda, \mu$ , and  $\nu$ , we obtain

$$J_\lambda(x) \leq \left( \int_{\Omega_{n,\lambda}^1} f(y)^{p\lambda(1/p-1/\mu)} K(x, y)^{\lambda q(1/q-1/\nu)} dy \right)^{1/\lambda} \left( \int_{\Omega_{n,\lambda}^1} f(y)^p dy \right)^{1/\mu} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{1/\nu}. \quad (1.11)$$

Since we want to have  $f^p$  and  $K^q$  in the integrand above, we note that we can choose  $\lambda, \mu, \nu$  in such a way

$$\frac{1}{\lambda} = \left( \frac{1}{p} - \frac{1}{\mu} \right), \quad \frac{1}{\lambda} = \left( \frac{1}{q} - \frac{1}{\nu} \right), \quad \frac{1}{\lambda} = \frac{1}{r}. \quad (1.12)$$

With these choices of  $\lambda, \mu$  and  $\nu$ , we can rewrite expression last inequality,

$$J_\lambda(x) \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{p/\mu} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{1/\nu} \left( \int_{\Omega_{n,\lambda}^1} f(y)^p K(x, y)^q dy \right)^{1/\lambda}. \quad (1.13)$$

Taking  $r$ th powers and integrating in  $x$ ,

$$\begin{aligned} & \int_{\Omega_{n,\lambda}^2} |J_\lambda(x)|^r dx \\ & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{r p/\mu} \int_{\Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{r/\nu} \left( \int_{\Omega_{n,\lambda}^1} f(y)^p K(x, y)^q dy \right)^{r/\lambda} dx \\ & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{r p/\mu} \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{r/\nu} \int_{\Omega_{n,\lambda}^2} \sup_{y \in \Omega_{n,\lambda}^1} K(x, y)^q \left( \int_{\Omega_{n,\lambda}^1} f(y)^p dy \right) dx \\ & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{r p/\mu} \|f\|_{L_p(\Omega_{n,\lambda}^1)}^p \sup_{y \in \Omega_{n,\lambda}^1} \left( \int_{\Omega_{n,\lambda}^2} K(x, y)^q dx \right) \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{r/\nu}. \end{aligned} \quad (1.14)$$

Hence

$$\|J_\lambda(x)\|_{L_r(\Omega_{n,\lambda}^2)}^r \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{p(r/\mu+1)} \sup_{y \in \Omega_{n,\lambda}^1} \left( \int_{\Omega_{n,\lambda}^2} K(x, y)^q dx \right) \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x, y)^q dy \right)^{r/\nu}. \quad (1.15)$$

Taking  $r$ th roots, we have the following inequality:

$$\begin{aligned}
 & \|J_\lambda(x)\|_{L_r(\Omega_{n,\lambda}^2)} \\
 & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)}^{p(1/\mu+1/r)} \sup_{y \in \Omega_{n,\lambda}^1} \left( \int_{\Omega_{n,\lambda}^2} K(x,y)^q dx \right)^{1/r} \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x,y)^q dy \right)^{1/v} \\
 & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)} \sup_{y \in \Omega_{n,\lambda}^1} \left( \int_{\Omega_{n,\lambda}^2} K(x,y)^q dx \right)^{1/r} \sup_{x \in \Omega_{n,\lambda}^2} \left( \int_{\Omega_{n,\lambda}^1} K(x,y)^q dy \right)^{1/q-1/r} \\
 & \leq \|f\|_{L_p(\Omega_{n,\lambda}^1)} k_1 k_2. \quad \square
 \end{aligned} \tag{1.16}$$

**THEOREM 1.2** (Riesz-Therin interpolation theorem, [13]). *Suppose  $T$  is simultaneously of weak types  $(p_0, q_0)$  and  $(p_1, q_1)$ ,  $1 \leq p_i, q_i \leq \infty$ . If  $0 < t < 1$  and  $1/p_t = (1-t)/p_0 + t/p_1$ ,  $1/q_t = (1-t)/q_0 + t/q_1$ , then  $T$  is of type  $(p_t, q_t)$ , and*

$$\|T\|_{(p_t, q_t)} \leq \|T\|_{(p_0, q_0)}^{1-t} \|T\|_{(p_1, q_1)}^t. \tag{1.17}$$

The following theorem gives the condition of absolute convergence of the potential  $I_\lambda^{\alpha(x)} f$ .

**THEOREM 1.3.** *Let  $0 < m \leq \alpha(x) < n$ ,  $f \in L_1(\Omega_{n,\lambda})$ . Then the integral (1.7) is absolutely convergent for almost every  $x$ .*

*Proof.* Let  $L_{y,\theta} = \{x \in \Omega_{n,\lambda} : y \cdot x = \cos \theta\}$ ,  $|L_{y,\theta}| = |\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta$ . Hence we have

$$\begin{aligned}
 & \int_{\Omega_{n,\lambda}} |I_\lambda^{\alpha(x)} f(x)| dx \\
 & \leq \iint_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy dx \\
 & = \int_{\Omega_{n,\lambda}} |f(y)| \int_{\Omega_{n,\lambda}} \frac{1}{|x-y|_\lambda^{n-\alpha(x)}} dx dy \\
 & = \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^\pi \left( \int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))}} dL_{y,\theta}(x) \right) d\theta \right] dy \\
 & = \int_{\Omega_{n,\lambda}} |f(y)| \left[ \left( \int_0^1 + \int_1^\pi \right) \left( \int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))}} dL_{y,\theta}(x) \right) d\theta \right] dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^1 \left( \int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-m)}} dL_{y,\theta}(x) \right) d\theta + \int_1^\pi \left( \int_{L_{y,\theta}} dL_{y,\theta}(x) \right) d\theta \right] dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^1 \frac{|\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta}{\theta^{(2|\lambda|/n)(n-m)}} d\theta + \int_1^\pi |\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta d\theta \right] dy \\
 & \leq |\Omega_{n-1,\lambda}| \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^1 \frac{1}{\theta^{1-(2|\lambda|/n)m}} d\theta + \int_1^\pi d\theta \right] dy \leq M \|f\|_1 < \infty.
 \end{aligned} \tag{1.18}$$

The proof is completed. □

**THEOREM 1.4.** *Let  $0 < m \leq \alpha(x) < n$ ,  $1 \leq p < \infty$ . Then  $I_\lambda^{\alpha(x)} f$  is of type  $(p, p)$ , that is,*

$$\|I_\lambda^{\alpha(x)} f\|_p \leq M \|f\|_p, \tag{1.19}$$

where the constant  $M$  is dependent on  $\lambda$ ,  $m$ , and  $n$ .

*Proof.* Let

$$S_\theta f(x) = \frac{1}{|L_{x,\theta}|} \int_{L_{x,\theta}} f(y) dL_{x,\theta}(y). \tag{1.20}$$

Thus we have

$$\|S_\theta f\|_p \leq M \|f\|_p. \tag{1.21}$$

By the Minkowsky inequality for integrals, we have the following inequality:

$$\begin{aligned} & \left( \int_{\Omega_{n,\lambda}} |I_\lambda^{\alpha(x)} f(x)|^p dx \right)^{1/p} \\ &= \left( \int_{\Omega_{n,\lambda}} \left| \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \right|^p dx \right)^{1/p} \\ &\leq \left( \int_{\Omega_{n,\lambda}} \left| \int_0^\pi \frac{|\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta}{\theta^{(2|\lambda|/n)(n-\alpha(x))}} \frac{1}{|L_{x,\theta}|} \int_{L_{x,\theta}} f(y) dL_{x,\theta} d\theta \right|^p dx \right)^{1/p} \\ &\leq M \left( \int_{\Omega_{n,\lambda}} \left| \int_0^\pi \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} |S_\theta(f)| d\theta \right|^p dx \right)^{1/p} \\ &\leq M \left( \int_0^1 + \int_1^\pi \right) \left( \int_{\Omega_{n,\lambda}} \frac{1}{\theta^{(1-(2|\lambda|/n)\alpha(x))p}} |S_\theta(f)|^p dx \right)^{1/p} d\theta \\ &\leq M \int_0^1 \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} \left( \int_{\Omega_{n,\lambda}} |S_\theta(f)|^p dx \right)^{1/p} d\theta + M \int_1^\pi \left( \int_{\Omega_{n,\lambda}} |S_\theta(f)|^p dx \right)^{1/p} d\theta \\ &\leq M \int_0^1 \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} \|S_\theta(f)\|_p d\theta + M \int_1^\pi \|S_\theta(f)\|_p d\theta \\ &\leq M \|f\|_p \left( \int_0^1 \frac{d\theta}{\theta^{1-(2|\lambda|/n)\alpha(x)}} + \int_1^\pi d\theta \right) \leq M \|f\|_p. \end{aligned} \tag{1.22}$$

The proof is completed. □

The following theorem is an expanded form of Theorem 1.4.

**THEOREM 1.5.** *Let  $0 < m \leq \alpha(x) < n$ ,  $1 < p \leq r$ ,  $n/p - n/r < m$ . Then  $I_\lambda^{\alpha(x)} f$  is of type  $(p, r)$ , that is,*

$$\|I_\lambda^{\alpha(x)} f\|_r \leq M \|f\|_p, \tag{1.23}$$

where the constant  $M$  is dependent on  $\lambda$ ,  $m$ , and  $n$ .

*Proof.* Let  $q = pr/(pr + p - r)$ ,  $1/q + 1/q' = 1$ . We show that  $I_\lambda^{\alpha(x)} f$  is of type  $(1, q)$  and  $(q', \infty)$ . By the Minkowsky inequality for integrals, we have the following inequality:

$$\begin{aligned}
 & \|I_\lambda^{\alpha(x)} f\|_q \\
 & \leq \left( \int_{\Omega_{n,\lambda}} \left| \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \right|^q dx \right)^{1/q} \\
 & \leq \int_{\Omega_{n,\lambda}} \left( \int_{\Omega_{n,\lambda}} \frac{|f(y)|^q}{|x-y|_\lambda^{(n-\alpha(x))q}} dx \right)^{1/q} dy \\
 & = \int_{\Omega_{n,\lambda}} |f(y)| \left( \int_{\Omega_{n,\lambda}} \frac{1}{|x-y|_\lambda^{(n-\alpha(x))q}} dx \right)^{1/q} dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^\pi \left( \int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))q}} dL_{y,\theta}(x) \right) d\theta \right] dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^1 \left( \int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-m)q}} dL_{y,\theta}(x) \right) d\theta + \int_1^\pi |\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta d\theta \right] dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ \int_0^1 \frac{|\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1} \theta}{\theta^{(2|\lambda|/n)(n-m)q}} d\theta + M \right] dy \\
 & \leq \int_{\Omega_{n,\lambda}} |f(y)| \left[ |\Omega_{n-1,\lambda}| \int_0^1 \frac{1}{\theta^{(2|\lambda|/n)(n-m)q-2|\lambda|+1}} d\theta + M \right] dy \leq M \|f\|_1.
 \end{aligned}
 \tag{1.24}$$

Thus the last integral is convergence where

$$\begin{aligned}
 & \frac{pr}{pr + p - r} < \frac{n}{n - m} \quad \text{for } \frac{n}{p} - \frac{n}{r} < m, \\
 & q < \frac{n}{n - m} \implies \frac{2|\lambda|}{n} (n - m)q + 1 - 2|\lambda| < 1.
 \end{aligned}
 \tag{1.25}$$

This shows that  $I_\lambda^{\alpha(x)} f$  is of type  $(1, q)$ . On the other hand, from Hölder’s inequality, we have

$$\begin{aligned}
 |I_\lambda^{\alpha(x)} f| & < \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \\
 & \leq \left( \int_{\Omega_{n,\lambda}} |f(y)|^{q'} dy \right)^{1/q'} \left( \int_{\Omega_{n,\lambda}} \frac{1}{|x-y|_\lambda^{(n-\alpha(x))q}} dy \right)^{1/q} \leq M \|f\|_{q'}.
 \end{aligned}
 \tag{1.26}$$

Therefore we have

$$\|I_\lambda^{\alpha(x)} f\|_\infty \leq M \|f\|_{q'}.
 \tag{1.27}$$

This shows that  $I_\lambda^{\alpha(x)} f$  is of type  $(q', \infty)$ .

Let  $t = q(1 - 1/p)$ , then from Theorem 1.2,  $I_\lambda^{\alpha(x)} f$  is of type  $(p, r)$  where  $1/p = (1 - t)/1 + t/q'$ ,  $1/r = (1 - t)/q'$ . The proof is completed.  $\square$

THEOREM 1.6. Let  $0 < m \leq \alpha(x) < n$ ,  $1 < p < r$ ,  $n/p - n/r = m$ . Then  $I_\lambda^{\alpha(x)} f$  is of type  $(p, r)$ .

Proof. Firstly, for a constant  $m$  we will consider the  $\alpha(x) = m$ . Thus, by using Lemma 1.1 for  $K(x, y) = |x - y|_\lambda^{m-n}$ , we obtain the following inequality:

$$\|I_\lambda^m f\|_r < M \|f\|_p. \tag{1.28}$$

This shows that  $I_\lambda^m$  is of  $(p, r)$  type.

Let

$$\Omega_{n,\lambda,x} = \{y \in \Omega_{n,\lambda} : |x - y|_\lambda \geq 1\}, \quad \overline{\Omega}_{n,\lambda,x} = \Omega_{n,\lambda} \setminus \Omega_{n,\lambda,x}. \tag{1.29}$$

Then

$$\begin{aligned} |I_\lambda^{\alpha(x)} f| &\leq \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x - y|_\lambda^{n-\alpha(x)}} dy \leq \int_{\Omega_{n,\lambda,x}} |f(y)| dy + \int_{\overline{\Omega}_{n,\lambda,x}} \frac{|f(y)|}{|x - y|_\lambda^{n-m}} dy \\ &\leq \int_{\Omega_{n,\lambda,x}} |f(y)| dy + I_\lambda^m f(x) \leq M' \|f\|_p + I_\lambda^m f(x). \end{aligned} \tag{1.30}$$

Therefore we have

$$\begin{aligned} \|I_\lambda^{\alpha(x)} f\|_r &\leq \|M' \|f\|_p + I_\lambda^m f(x)\|_r \leq M' \|f\|_p + \|I_\lambda^m f(x)\|_r \\ &\leq M' \|f\|_p + M \|f\|_p = C \|f\|_p. \end{aligned} \tag{1.31}$$

Thus  $I_\lambda^{\alpha(x)} f$  is of type  $(p, r)$ .

The proof is completed. □

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