DUAL L_p AFFINE ISOPERIMETRIC INEQUALITIES

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We establish some inequalities for the dual p-centroid bodies which are the dual forms of the results by Lutwak, Yang, and Zhang. Further, we establish a Brunn-Minkowski-type inequality for the polar of dual p-centroid bodies.

1. Introduction

Corresponding to each convex (or more general) subset of n-dimensional Euclidean space, $\mathbb{R}^n$, there is a unique ellipsoid with the following property. The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the Legendre ellipsoid of the convex set. The Legendre ellipsoid is a well-known concept from classical mechanics. For a star-shaped (about the origin) set $K \subset \mathbb{R}^n$, it is easy to see that its Legendre ellipsoid, usually denoted by $\Gamma_2 K$, is an object of the dual Brunn-Minkowski theory. In [6], the dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory is introduced. For a convex body (i.e., a compact, convex subset with nonempty interior) K in $\mathbb{R}^n$, its dual analog of $\Gamma_2 K$ is dented by $\Gamma_{-2} K$. More in general, in [8], the L_p analog of centroid bodies, $\Gamma_p K$ for a convex body K also being investigated, and, in [7], the dual of $\Gamma_p K$, $\Gamma_{-p} K$ are defined. The main aim of this article is to establish some affine inequalities for $\Gamma_{-p} K$, which are dual analog of the main results in [5, 8]. The techniques developed by Lutwak, Yang, and Zhang play a critical role throughout our paper.

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Let B denote the unit ball (the convex hull of S^{n-1}) in $\mathbb{R}^n$, and write ω_n for the n-dimensional volume of B. Note that

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \tag{1.1}$$

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defines ω_n for all nonnegative real n (not just the positive integer). For real $p \geq 1$, define $c_{n,p}$ by

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}. \tag{1.2}$$

If K is a convex body in $\mathbb{R}^n$ that contained the origin in its interior and $p > 0$, then the p-dual centroid body of K, $\Gamma_{-p}K$, is defined as the body whose radial function, for $u \in S^{n-1}$, is given by

$$\rho_{\Gamma_{-p}K}(u)^{-p} = \frac{1}{nc_{n-2,p}V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v), \tag{1.3}$$

where $S_p(K,v)$ denote the p-surface area measure.

For $p \geq 1$ the body $\Gamma_{-p}K$ is a convex body. The normalization is chosen so that for the standard unit ball B in $\mathbb{R}^n$, we have $\Gamma_{-p}B = B$ and this definition of $\Gamma_{-p}K$ is different from the definition given by Lutwak et al. in [7].

The main results of ours are the following Theorems 1.1, 1.4, and 1.5.

Theorem 1.1. *If K is a convex body in $\mathbb{R}^n$, then for $p \geq 1$,*

$$V(\Gamma_{-p}K) \leq V(K), \tag{1.4}$$

with equality if and only if K is an ellipsoid centered at the origin.

The dual analog of Theorem 1.1 for $\Gamma_p K$ has been established by Lutwak et al. in [5] (see Campi and Gronchi [1] for an alternate approach), that is, the following holds.

Theorem 1.2. *If K is a star body (about the origin) in $\mathbb{R}^n$, then for $p \geq 1$,*

$$V(\Gamma_p K) \geq V(K), \tag{1.5}$$

with equality if and only if K is an ellipsoid centered at the origin.

One of the most important affine isoperimetric inequalities is the Blaschke-Santaló inequality, that is,

$$V(K)V(K^*) \leq \omega_n^2, \tag{1.6}$$

with equality if and only if K is an ellipsoid.

Here the polar of a convex body K in $\mathbb{R}^n$ is defined by

$$K^* = \{x \in \mathbb{R}^n \mid x \cdot y \leq 1 \, \forall y \in K\}, \tag{1.7}$$

where $x \cdot y$ denotes the standard inner product of x and y.

In [8], Lutwak and Zhang generalized this result and get the following theorem.

Theorem 1.3. *If K is a star body (about the origin) in $\mathbb{R}^n$, then for $1 \leq p \leq \infty$,*

$$V(K)V(\Gamma_p^* K) \leq \omega_n^2, \tag{1.8}$$

with equality if and only if K is an ellipsoid centered at the origin.

Obviously, let $p \to \infty$, one can just get the Blaschke-Santaló inequality. Note that we use $\Gamma_p^* K$ rather than $(\Gamma_p K)^*$ to denote the polar of $\Gamma_p K$.

In this paper, we establish the weak dual analog of Theorem 1.3 for $\Gamma_{-p}K$ and get the following inequality.

Theorem 1.4. *If K is a convex body in $\mathbb{R}^n$ such that $\Gamma_{-p}^* K$ is an ellipsoid, then for $p \geq 1$,*

$$V(K)V(\Gamma_{-p}^* K) \geq \omega_n^2, \tag{1.9}$$

with equality if and only if K is a centered ellipsoid.

Here we use $\Gamma_{-p}^* K$ to denote the polar of $\Gamma_{-p}K$ and a centered ellipsoid is the ellipsoid whose symmetric center is the origin.

Note. The general inequality with the form of Theorem 1.4 does not exist since we can get a contradiction to the Blaschke-Santaló inequality if $p \to \infty$.

Finally, we establish the following Brunn-Minkowski-type inequality for the polar of $\Gamma_{-p}K$. Here $\dot{+}_p$ denote the p-Blaschke sum.

Theorem 1.5. *If K and L are centered convex bodies in $\mathbb{R}^n$, then for $p > 1$ and $n \neq p$,*

$$V(K \dot{+}_p L)\, V(\Gamma_{-p}^*(K \dot{+}_p L))^{p/n} \geq V(K)V(\Gamma_{-p}^* K)^{p/n} + V(L)V(\Gamma_{-p}^* L)^{p/n}. \tag{1.10}$$

and the equality holds if and only if $V(K)\Gamma_{-p}^* K$ and $V(L)\Gamma_{-p}^* L$ are dilates, that is,

$$V(K)\Gamma_{-p}^* K = rV(L)\Gamma_{-p}^* L \quad \text{for some } r > 0. \tag{1.11}$$

Let $\Pi_p K$ denote the p-projection of K. Theorem 1.5 is equivalent to the following.

Theorem 1.6. *If K and L are centered convex bodies in $\mathbb{R}^n$, then for $p > 1$ and $n \neq p$,*

$$V(\Pi_p(K \dot{+}_p L))^{p/n} \geq V(\Pi_p K)^{p/n} + V(\Pi_p L)^{p/n}, \tag{1.12}$$

and the equality holds if and only if $\Pi_p K$ and $\Pi_p L$ are dilates.

2. Mixed and dual mixed volumes and the operator Γ_{-p}

For quick reference, we recall some basic properties regarding the L_p-mixed volume and its dual theory, and some properties of the operator Γ_{-p} also being established by different method from [7]. For general reference of convex body and mixed volume, the reader may wish to consult Gardner [3], Schneider [9] and Thompson [10].

If K is a convex body in $\mathbb{R}^n$, then its support function $h_K(\cdot) : S^{n-1} \to \mathbb{R}$ is defined by

$$h_K(u) = \max\{u \cdot x : x \in K\}. \tag{2.1}$$

The radial function, $\rho_K(\cdot) : \mathbb{R} - \{0\} \to [0, \infty)$, of a compact, star-shaped (about the origin) $K \subset \mathbb{R}$, is defined, for $x \neq 0$, by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}. \tag{2.2}$$

If ρ_K is positive and continuous, then we call K a star body (about the origin).

It follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_{K^*} = \frac{1}{\rho_K}, \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.3}$$

For $p \geq 1$, convex bodies K, L and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L$ is defined as the convex body whose support function is given by

$$h^p_{K+_p\varepsilon\cdot L}(\cdot) = h^p_K(\cdot) + \varepsilon h^p_L(\cdot). \tag{2.4}$$

Firey combinations of convex bodies were defined and studied by Firey [2] (who called them p-means of convex bodies).

For $p \geq 1$, the L_p-mixed volume, $V_p(K,L)$, of the convex bodies K, L can be defined by

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.5}$$

That this limit exists was demonstrated in [4].

It was shown in [4] that corresponding to each convex body K containing the origin in its interior in $\mathbb{R}^n$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h^p_Q(u)\,dS_p(K,u), \tag{2.6}$$

for each convex body Q. The measure $S_1(K,\cdot)$ is just the classical surface area measure of K and usually denoted by $S(K,\cdot)$ or S_K.

In [4], a solution to the even L_p-Minkowski problem in $\mathbb{R}^n$ was given for all $p \geq 1$, except for $p = n-1$. From this, the p-Blaschke addition was defined in [4]. For centered convex bodies K and L in $\mathbb{R}^n$, and $n \neq p \geq 1$, define $K \dotplus_p L$, p-Blaschke sum of K and L, by

$$S_p(K \dotplus_p L) = S_p(K,\cdot) + S_p(L,\cdot). \tag{2.7}$$

For the L_p-mixed volume V_p, it has been shown in [5] that

$$V_p(\phi K, L) = V_p(K, \phi^{-1}L), \tag{2.8}$$

where $\phi \in \mathrm{SL}(n)$ and K, L are convex bodies.

If K is a convex body in $\mathbb{R}^n$ that contained the origin in its interior and $p > 0$, then the dual p-centroid body of K, $\Gamma_{-p}K$, is defined as the body whose radial function, for $u \in S^{n-1}$, is given by

$$\rho_{\Gamma_{-p}K}(u)^{-p} = \frac{1}{nc_{n-2,p}V(K)}\int_{S^{n-1}} |u\cdot v|^p\,dS_p(K,v). \tag{2.9}$$

For $p \geq 1$ the body $\Gamma_{-p}K$ is a convex body. Note that our definition of $\Gamma_{-p}K$ is different from the definition given by Lutwak et al. in [7]. That is for $K = B$, we have

$$\Gamma_{-p}B = B. \tag{2.10}$$

For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, $u \in S^{n-1}$, and $1 \leq p \leq \infty$, the L_p-centroid body of K, which is dual to $\Gamma_{-p}K$, is defined in [8] by

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p}V(K)} \int_K |u \cdot x|^p dx. \tag{2.11}$$

It has been known that in [5], for $\phi \in \mathrm{SL}(n)$,

$$\Gamma_p \phi K = \phi \Gamma_p K. \tag{2.12}$$

For star bodies K, L, and $p \geq 1$, $\varepsilon > 0$, the L_p-harmonic radial combination $K +_{-p} \varepsilon \cdot L$ is defined as the star body whose radial function is given by

$$\rho^{-p}_{K+_{-p}\varepsilon\cdot L}(\cdot) = \rho_K^{-p}(\cdot) + \varepsilon\rho_L^{-p}(\cdot). \tag{2.13}$$

The dual mixed volume $V_{-p}(K,L)$ of the star bodies K, L can be defined by

$$-\frac{n}{p}V_{-p}(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K +_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.14}$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $V_{-p}(K,L)$ of the star bodies K, L:

$$V_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_L^{-p}(v)dS(v), \tag{2.15}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

For the L_{-p}-mixed volume V_{-p}, it has been shown in [5] that

$$V_{-p}(\phi K,L) = V_{-p}(K,\phi^{-1}L), \tag{2.16}$$

where $\phi \in \mathrm{SL}(n)$ and K, L are star bodies.

A connection between the operators Γ_p and Γ_{-p} is given in the following identity.

Lemma 2.1. *Suppose $K, L \subset \mathbb{R}^n$. If K is a convex body that contains the origin in its interior and L is a star body about the origin, then*

$$\frac{V_p(L,\Gamma_p K)}{V(L)} = \frac{V_{-p}(K,\Gamma_{-p}L)}{V(K)}. \tag{2.17}$$

Proof. From the integral representation (2.6), definition (2.11), Fubini's theorem, definition (2.9), the integral representation (2.15), and the property of Γ-function, it follows that

$$
\begin{aligned}
V_p(L,\Gamma_p K)
&= \frac{1}{n}\int_{S^{n-1}} h^p_{\Gamma_p K}(u)\,dS_p(L,u)\\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{c_{n,p}V(K)}\int_K |u\cdot x|^p\,dx\,dS_p(L,u)\\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}} |u\cdot v|^p\rho_K^{n+p}(v)\,dS(v)\,dS_p(L,u)\\
&= \frac{c_{n-2,p}V(L)}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}} \frac{1}{nc_{n-2,p}V(L)}\int_{S^{n-1}} |u\cdot v|^p\,dS_p(L,u)\rho_K^{n+p}(v)\,dS(v)\\
&= \frac{c_{n-2,p}V(L)}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}} \rho^{-p}_{\Gamma_{-p}L}(v)\rho_K^{n+p}(v)\,dS(v)\\
&= \frac{nc_{n-2,p}V(L)}{(n+p)c_{n,p}V(K)}V_{-p}(K,\Gamma_{-p}L)\\
&= \frac{V(L)}{V(K)}V_{-p}(K,\Gamma_{-p}L).
\end{aligned}
\tag{2.18}
$$

□

A connection between the operators Γ_2 and Γ_{-2}, which is similar to the above lemma, has been established in [6].

From the above lemma, we can get the following proposition which has been obtained in [7] by different method.

Proposition 2.2. *If $p>0$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then for $\phi\in \mathrm{GL}(n)$,*

$$\Gamma_{-p}\phi K=\phi\Gamma_{-p}K. \tag{2.19}$$

Proof. From Lemma 2.1, (2.8), (2.12), Lemma 2.1 again, and (2.16), we have for each star body Q and $\phi\in \mathrm{SL}(n)$

$$
\begin{aligned}
\frac{V_{-p}(Q,\Gamma_{-p}\phi K)}{V(Q)} &= \frac{V_p(\phi K,\Gamma_p Q)}{V(\phi K)} = \frac{V_p(K,\phi^{-1}\Gamma_p Q)}{V(K)}\\
&= \frac{V_p(K,\Gamma_p\phi^{-1}Q)}{V(K)} = \frac{V_{-p}(\phi^{-1}Q,\Gamma_{-p}K)}{V(\phi^{-1}Q)}\\
&= \frac{V_{-p}(Q,\phi\Gamma_{-p}K)}{V(Q)}.
\end{aligned}
\tag{2.20}
$$

But $V_{-p}(Q,\Gamma_{-p}\phi K)/V(Q) = V_{-p}(Q,\phi\Gamma_{-p}K)/V(Q)$ for all star bodies Q implies that

$$\Gamma_{-p}\phi K = \phi\Gamma_{-p}K. \tag{2.21}$$

Combing with the fact (from the definition of $\Gamma_{-p}K$)

$$\Gamma_{-p}\lambda K = \lambda\Gamma_{-p}K \quad \text{for } \lambda > 0, \tag{2.22}$$

we can get the conclusion. □

For each convex body K, in [5] the support function of L_p-projection body $\Pi_p K$ is defined by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p \, dS_p(K,v). \tag{2.23}$$

From the above definitions (2.3) and (2.9), we can get the following.

Proposition 2.3. *Suppose $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior, then*

$$\Pi_p K = \left(\frac{V(K)}{\omega_n}\right)^{1/p} \Gamma^*_{-p}K. \tag{2.24}$$

The following proposition given in [5] will be used as a lemma.

Lemma 2.4. *If K is a convex body in $\mathbb{R}^n$, then for $p \geq 1$,*

$$V(K)^{(n-p)/p} V(\Pi^*_p K) \leq \omega_n^{n/p}, \tag{2.25}$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.1. From (2.3), Proposition 2.3, and Lemma 2.4, we have

$$V(K)^{(n-p)/p} V\left(\left(\frac{\omega_n}{V(K)}\right)^{1/p} \Gamma_{-p}K\right) \leq \omega_n^{n/p}. \tag{2.26}$$

By the volume formula of convex body,

$$V(K)^{(n-p)/p} \left(\frac{\omega_n}{V(K)}\right)^{n/p} V(\Gamma_{-p}K) \leq \omega_n^{n/p}, \tag{2.27}$$

that is,

$$V(\Gamma_{-p}K) \leq V(K), \tag{2.28}$$

with equality if and only if K is an ellipsoid centered at the origin. □

3. Mixed volume inequalities and the operator Γ^*_{-p}

We will require some basic inequalities regarding the L_p-mixed volumes V_p and the dual mixed volume V_{-p}. The L_p analog of the classical Minkowski inequality states that for convex bodies K, L,

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{3.1}$$

with equality if and only if K and L are dilates. The L_p-Minkowski inequality was established in [4] by using the Minkowski inequality. The basic inequality for dual mixed volume V_{-p} is that for star bodies K, L,

$$V_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{3.2}$$

with equality if and only if K and L are dilates. This inequality is an immediate consequence of the Hölder inequality and the integral representation (2.15).

Lemma 3.1. *If K and Q are convex bodies in $\mathbb{R}^n$ and $p \geq 1$, then*

$$\frac{V_p(K,\Gamma^*_{-p}Q)}{V(K)} = \frac{V_p(Q,\Gamma^*_{-p}K)}{V(Q)}. \tag{3.3}$$

Proof. From the integral representation (2.3), (2.6), and (2.9), we have for $p \geq 1$ that

$$\begin{aligned}
\frac{V_p(K,\Gamma^*_{-p}Q)}{V(K)} &= \frac{1}{nV(K)} \int_{S^{n-1}} h^p_{\Gamma^*_{-p}Q}(u)\,dS_p(K,u) \\
&= \frac{1}{nV(K)} \int_{S^{n-1}} \rho^{-p}_{\Gamma_{-p}Q}(u)\,dS_p(K,u) \\
&= \frac{1}{n^2 c_{n-2,p} V(K)V(Q)} \iint_{S^{n-1}} |u\cdot v|^p\,dS_p(Q,v)\,dS_p(K,u) \\
&= \frac{1}{nV(Q)} \int_{S^{n-1}} \rho^{-p}_{\Gamma_{-p}K}(v)\,dS_p(Q,v) \\
&= \frac{V_p(Q,\Gamma^*_{-p}K)}{V(Q)}.
\end{aligned} \tag{3.4}$$

The dual analog of the above equality has been established in [5]. □

Lemma 3.2. *If $p \geq 1$ and K is a convex body in $\mathbb{R}^n$, then*

$$V(\Gamma^*_{-p}\Gamma^*_{-p}K) \leq V(K), \tag{3.5}$$

with equality if and only if K and $\Gamma^*_{-p}\Gamma^*_{-p}K$ are dilates.

Proof. In Lemma 3.1, let $Q = \Gamma^*_{-p}K$, then we get

$$\frac{V_p(K,\Gamma^*_{-p}\Gamma^*_{-p}K)}{V(K)} = \frac{V_p(\Gamma^*_{-p}K,\Gamma^*_{-p}K)}{V(\Gamma^*_{-p}K)}. \tag{3.6}$$

Note that $V_p(\Gamma^*_{-p}K,\Gamma^*_{-p}K) = V_p(\Gamma^*_{-p}K)$, so

$$V(K) = V_p(K,\Gamma^*_{-p}\Gamma^*_{-p}K). \tag{3.7}$$

By (3.1), we have

$$V_p(K,\Gamma^*_{-p}\Gamma^*_{-p}K) \geq V(K)^{(n-p)/n}V^{p/n}(\Gamma^*_{-p}\Gamma^*_{-p}K), \tag{3.8}$$

with equality if and only if K and $\Gamma^*_{-p}\Gamma^*_{-p}K$ are dilates.

That is

$$V(\Gamma^*_{-p}\Gamma^*_{-p}K) \leq V(K), \tag{3.9}$$

with equality if and only if K and $\Gamma^*_{-p}\Gamma^*_{-p}K$ are dilates. □

Proof of Theorem 1.4. Because that $\Gamma^*_{-p}K$ is an ellipsoid, there exist $\phi \in \mathrm{GL}(n)$ such that $\Gamma^*_{-p}K = \phi B$. By Proposition 2.2 and the definition of $\Gamma_{-p}K$, it follows that

$$\Gamma_{-p}(\Gamma^*_{-p}K) = \Gamma_{-p}(\phi B) = \phi\Gamma_{-p}(B) = \phi B = \Gamma^*_{-p}K. \tag{3.10}$$

Thus

$$\Gamma^*_{-p}(\Gamma^*_{-p}K) = (\Gamma^*_{-p}K)^*. \tag{3.11}$$

With the fact that the product of the volumes of centered polar reciprocal ellipsoid is ω_n^2, we get

$$V(\Gamma^*_{-p}\Gamma^*_{-p}K) = V((\Gamma^*_{-p}K)^*) = \frac{\omega_n^2}{V(\Gamma^*_{-p}K)}. \tag{3.12}$$

By Lemma 3.2, we prove the inequality

$$V(K)V(\Gamma^*_{-p}K) \geq \omega_n^2. \tag{3.13}$$

From the equality condition of Lemma 3.2, it follows that K and $\Gamma^*_{-p}\Gamma^*_{-p}K$ are dilates. But $\Gamma^*_{-p}\Gamma^*_{-p}K = (\Gamma^*_{-p}K)^*$ is a centered ellipsoid. Hence, in Theorem 1.4, the equality implies that K is a centered ellipsoid. □

Proof of Theorem 1.1. Second method. In Lemma 2.1, let $K = \Gamma_{-p}L$, and note that $V_{-p}(K, K) = V(K)$, then we can get

$$V(L) = V_p(L,\Gamma_p\Gamma_{-p}L). \tag{3.14}$$

By (2.23), we get

$$V(L) = V_p(L,\Gamma_p\Gamma_{-p}L) \geq V(L)^{(n-p)/n}V(\Gamma_p\Gamma_{-p}L)^{p/n}. \tag{3.15}$$

In Theorem 1.2, let $K = \Gamma_{-p}L$, then we get

$$V(L) \geq V(L)^{(n-p)/n}V(\Gamma_p\Gamma_{-p}L)^{p/n} \geq V(L)^{(n-p)/n}V(\Gamma_{-p}L)^{p/n}, \tag{3.16}$$

that is

$$V(L) \geq V(\Gamma_{-p}L). \tag{3.17}$$

□

Proof of Theorem 1.6. First, we established the following inequality for centered convex bodies K, L in $\mathbb{R}^n$:

$$V(\Pi_p(K\dot{+}_pL))^{p/n} \geq V(\Pi_pK)^{p/n} + V(\Pi_pL)^{p/n}. \tag{3.18}$$

From (2.6), (2.23), (3.1), and the definition of p-Blaschke addition, we have for $n \neq p > 1$, and any convex body Q

$$\begin{aligned} V_p(Q,\Pi_p(K\dot{+}_pL)) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_p(K\dot{+}_pL)}(u)dS_p(Q,u) \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_pK}(u)dS_p(Q,u) + \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_pL}(u)dS_p(Q,u) \\ &= V_p(Q,\Pi_pK) + V_p(Q,\Pi_pL) \\ &\geq V(Q)^{(n-p)/n}V(\Pi_pK)^{p/n} + V(Q)^{(n-p)/n}V(\Pi_pL)^{p/n} \\ &= V(Q)^{(n-p)/n}\left(V(\Pi_pK)^{p/n} + V(\Pi_pL)^{p/n}\right). \end{aligned} \tag{3.19}$$

Let $Q = \Pi_p(K\dot{+}_pL)$ in the above inequality, then we get

$$V(\Pi_p(K\dot{+}_pL))^{p/n} \geq V(\Pi_pK)^{p/n} + V(\Pi_pL)^{p/n}, \tag{3.20}$$

with equality if and only if Π_pK and Π_pL are dilates.

By Proposition 2.3 and (3.20), we can get Theorem 1.5 immediately. □

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