ON RANDOM COINCIDENCE AND FIXED POINTS FOR A PAIR OF MULTIVALUED AND SINGLE-VALUED MAPPINGS

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Let (X,d) be a Polish space, $\mathrm{CB}(X)$ the family of all nonempty closed and bounded subsets of X, and (Ω,Σ) a measurable space. A pair of a hybrid measurable mappings $f:\Omega\times X\to X$ and $T:\Omega\times X\to \mathrm{CB}(X)$, satisfying the inequality (1.2), are introduced and investigated. It is proved that if X is complete, $T(\omega,\cdot)$, $f(\omega,\cdot)$ are continuous for all $\omega\in\Omega$, $T(\cdot,x)$, $f(\cdot,x)$ are measurable for all $x\in X$, and $f(\omega\times X)=X$ for each $\omega\in\Omega$, then there is a measurable mapping $\xi:\Omega\to X$ such that $f(\omega,\xi(w))\in T(\omega,\xi(w))$ for all $\omega\in\Omega$. This result generalizes and extends the fixed point theorem of Papageorgiou (1984) and many classical fixed point theorems.

1. Introduction and preliminaries

Random fixed point theorems are stochastic generalizations of classical fixed point theorems. Random fixed point theorems for contraction mappings on separable complete metric spaces have been proved by several authors (Zhang and Huang [25], Hanš [6, 7], Itoh [8], Lin [12], Papageorgiou [13, 14], Shahzad and Hussian [19, 20], Špaček [22], and Tan and Yuan [23]). The stochastic version of the well known Schauder's fixed point theorem was proved by Sehgal and Singh [18].

Let (X,d) be a metric space and $T:X\to X$ a mapping. The class of mappings T satisfying the following contractive condition:

$$d(Tx,Ty)\le \alpha\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}\\ +\beta\max\{d(x,Tx),d(y,Ty)\}+\gamma[d(x,Ty)+d(y,Tx)] \tag{1.1}$$

for all $x,y\in X$, where α, β, γ are nonnegative real numbers such that $\beta>0$, $\gamma>0$, and $\alpha+\beta+2\gamma=1$, was introduced and investigated by Ćirić [1]. Ćirić proved that in a complete

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metric space such mappings have a unique fixed point. This class of mappings was further studied by many authors (Ćirić [2, 3], Singh and Mishra [21], and Rhoades et al. [16]). Singh and Mishra [21] have generalized Ćirić's [2] fixed point theorem to a common fixed point theorem of a pair of mappings and presented some application of such theorems to dynamic programming.

Let (Ω,Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let (X,d) be a metric space. We denote by 2^X the family of all subsets of X, by $\mathrm{CB}(X)$ the family of all nonempty closed and bounded subsets of X, and by H the Hausdorff metric on $\mathrm{CB}(X)$, induced by the metric d. For any $x \in X$ and $A \subseteq X$, by $d(x,A)$ we denote the distance between x and A, that is, $d(x,A) = \inf\{d(x,a) : a \in A\}$.

A mapping $T : \Omega \to 2^X$ is called Σ-*measurable* if for any open subset U of X, $T^{-1}(U) = \{\omega : T(w) \cap U \neq \varnothing\} \in \Sigma$. In what follows, when we speak of measurability we will mean Σ-measurability. A mapping $f : \Omega \times X \to X$ is called a *random operator* if for any $x \in X$, $f(\cdot,x)$ is measurable. A mapping $T : \Omega \times X \to \mathrm{CB}(X)$ is called a *multivalued random operator* if for every $x \in X$, $T(\cdot,x)$ is measurable. A mapping $s : \Omega \to X$ is called a *measurable selector* of a measurable multifunction $T : \Omega \to 2^X$ if s is measurable and $s(\omega) \in T(\omega)$ for all $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a *random fixed point* of a random multifunction $T : \Omega \times X \to \mathrm{CB}(X)$ if $\xi(w) \in T(w,\xi(w))$ for every $w \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a *random coincidence* of $T : \Omega \times X \to \mathrm{CB}(X)$ and $f : \Omega \times X \to X$ if $f(\omega,\xi(w)) \in T(w,\xi(w))$ for every $w \in \Omega$.

The aim of this paper is to prove a stochastic analog of the Ćirić [1] fixed point theorem for single-valued mappings, extended to a coincidence theorem for a pair of a random operator $f : \Omega \times X \to X$ and a multivalued random operator $T : \Omega \times X \to \mathrm{CB}(X)$, satisfying the following nonexpansive-type condition: for each $\omega \in \Omega$,

$$
\begin{aligned}
&H\big(T(\omega,x),T(\omega,y)\big)\\
&\quad\le \alpha(\omega)\max\Big\{d\big(f(\omega,x),f(\omega,y)\big),d\big(f(\omega,x),T(\omega,x)\big),d\big(f(\omega,y),T(\omega,y)\big),\\
&\qquad\qquad \Big(\frac{1}{2}\Big)\big[d\big(f(\omega,x),T(\omega,y)\big)+d\big(f(\omega,y),T(\omega,x)\big)\big]\Big\}\\
&\qquad+\beta(\omega)\max\big\{d\big(f(\omega,x),T(\omega,x)\big),d\big(f(\omega,y),T(\omega,y)\big)\big\}\\
&\qquad+\gamma(\omega)\big[d\big(f(\omega,x),T(\omega,y)\big)+d\big(f(\omega,y),T(\omega,x)\big)\big]
\end{aligned}
\tag{1.2}
$$

for every $x,y \in X$, where $\alpha,\beta,\gamma : \Omega \to [0,1)$ are measurable mappings such that for all $\omega \in \Omega$,

$$\beta(\omega) > 0, \qquad \gamma(\omega) > 0, \tag{1.3}$$

$$\alpha(\omega)+\beta(\omega)+2\gamma(\omega) = 1. \tag{1.4}$$

2. Main results

Now we are proving our main result.

Theorem 2.1. *Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $T:\Omega\times X\to \mathrm{CB}(X)$ and $f:\Omega\times X\to X$ be mappings such that*

(i) *$T(\omega,\cdot)$, $f(\omega,\cdot)$ are continuous for all $\omega\in\Omega$,*

(ii) *$T(\cdot,x)$, $f(\cdot,x)$ are measurable for all $x\in X$,*

(iii) *they satisfy (1.2), where $\alpha(\omega),\beta(\omega),\gamma(\omega):\Omega\to X$ satisfy (1.3) and (1.4).*

If $f(\omega\times X)=X$ for each $\omega\in\Omega$, then there is a measurable mapping $\xi:\Omega\to X$ such that $f(\omega,\xi(w))\in T(w,\xi(w))$ for all $\omega\in\Omega$ (i.e., T and f have a random coincidence point).

Proof. Let $\Psi=\{\xi:\Omega\to X\}$ be a family of measurable mappings. Define a function $g:\Omega\times X\to R^+$ as follows:

$$g(\omega,x)=d(x,T(\omega,x)). \tag{2.1}$$

Since $x\to T(\omega,x)$ is continuous for all $\omega\in\Omega$, we conclude that $g(\omega,\cdot)$ is continuous for all $\omega\in\Omega$. Also, since $\omega\to T(\omega,x)$ is measurable for all $x\in X$, we conclude that $g(\cdot,x)$ is measurable (see Wagner [24, page 868]) for all $\omega\in\Omega$. Thus $g(\omega,x)$ is the Caratheodory function. Therefore, if $\xi:\Omega\to X$ is a measurable mapping, then $\omega\to g(\omega,\xi(w))$ is also measurable (see [17]).

Now we will construct a sequence of measurable mappings $\{\xi_n\}$ in Ψ and a sequence $\{f(\omega,\xi_n(\omega))\}$ in X as follows. Let $\xi_0\in\Psi$ be arbitrary. Then the multifunction $G:\Omega\to \mathrm{CB}(X)$ defined by $G(\omega)=T(w,\xi_0(w))$ is measurable.

From the Kuratowski and Ryll-Nardzewski [11] selector theorem, there is a measurable selector $\mu_1:\Omega\to X$ such that $\mu_1(\omega)\in T(w,\xi_0(w))$ for all $\omega\in\Omega$. Since $\mu_1(\omega)\in T(w,\xi_0(w))\subseteq X=f(\omega\times X)$, let $\xi_1\in\Psi$ be such that $f(\omega,\xi_1(\omega))=\mu_1(\omega)$. Thus $f(\omega,\xi_1(\omega))\in T(\omega,\xi_0(\omega))$ for all $\omega\in\Omega$.

Let $k:\Omega\to(1,\infty)$ be defined by

$$k(\omega)=1+\frac{\beta(\omega)\gamma(\omega)}{2} \tag{2.2}$$

for all $\omega\in\Omega$. Then $k(\omega)$ is measurable. Since $k(\omega)>1$ and $f(\omega,\xi_1(\omega))$ is a selector of $T(w,\xi_0(w))$, from Papageorgiou [13, Lemma 2.1] there is a measurable selector $\mu_2(\omega)=f(\omega,\xi_2(\omega))$; $\xi_2\in\Psi$, such that for all $\omega\in\Omega$,

$$\begin{gathered} f(\omega,\xi_2(\omega))\in T(\omega,\xi_1(\omega)),\\ d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega)))\le k(\omega)H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))). \end{gathered} \tag{2.3}$$

Similarly, as $f(\omega,\xi_2(\omega))$ is a selector of $T(w,\xi_1(w))$, there is a measurable selector $\mu_3(\omega)=f(\omega,\xi_3(\omega))$ of $T(\omega,\xi_2(\omega))\subseteq f(\omega\times X)$ such that

$$d(f(\omega,\xi_2(\omega)),f(\omega,\xi_3(\omega)))\le k(\omega)H(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))). \tag{2.4}$$

Continuing this process we can construct a sequence of measurable mappings $\mu_n:\Omega\to X$, defined by $\mu_n(\omega)=f(\omega,\xi_n(\omega))$; $\xi_n\in\Psi$, such that

$$f(\omega,\xi_{n+1}(\omega))\in T(\omega,\xi_n(\omega)), \tag{2.5}$$

$$d(f(\omega,\xi_n(\omega)),f(\omega,\xi_{n+1}(\omega)))\le k(\omega)H(T(\omega,\xi_{n-1}(\omega)),T(\omega,\xi_n(\omega))). \tag{2.6}$$

Observe that condition (1.2) is clumsy. So, for simplicity, in the rest of the paper we will use this condition in the following form:

$$\begin{aligned} H(T(\omega,x),T(\omega,y)) \le \alpha(\omega)\max&\left\{d(f(\omega,x),f(\omega,y)),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\ &+\beta(\omega)\max\{d(f(\omega,x),T(\omega,x)),d(f(\omega,y),T(\omega,y))\} \\ &+\gamma(\omega)[d(f(\omega,x),T(\omega,y))+d(f(\omega,y),T(\omega,x))]. \end{aligned} \tag{2.7}$$

From (2.7),

$$\begin{aligned} &H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))) \\ &\quad\le \alpha(\omega)\max\left\{d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\ &\qquad+\beta(\omega)\max\{d(f(\omega,\xi_0(\omega)),T(\omega,\xi_0(\omega))),d(f(\omega,\xi_1(\omega)),T(\omega,\xi_1(\omega)))\} \\ &\qquad+\gamma(\omega)[d(f(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))+d(f(\omega,\xi_1(\omega)),T(\omega,\xi_0(\omega)))]. \end{aligned} \tag{2.8}$$

Since $f(\omega,\xi_1(\omega))\in T(\omega,\xi_0(\omega))$, then

$$\begin{aligned} d(f(\omega,\xi_1(\omega)),T(\omega,\xi_0(\omega)))&=0, \\ d(f(\omega,\xi_0(\omega)),T(\omega,\xi_0(\omega)))&\le d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))), \\ d(f(\omega,\xi_1(\omega)),T(\omega,\xi_1(\omega)))&\le H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))). \end{aligned} \tag{2.9}$$

Thus from (2.8),

$$\begin{aligned} &H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))) \\ &\quad\le \alpha(\omega)\max\left\{d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\ &\qquad+\beta(\omega)\max\{d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))),H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\} \\ &\qquad+\gamma(\omega)[d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))+H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))]. \end{aligned} \tag{2.10}$$

If we assume that $H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))>d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))$, then we have, as $\gamma(\omega)>0$,

$$\begin{aligned} &\gamma(\omega)[d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))+H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))] \\ &\quad<2\gamma(\omega)H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))). \end{aligned} \tag{2.11}$$

Thus, from (1.4) and (2.10), we have

$$\begin{aligned}
&H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\\
&\quad< \alpha(\omega)H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))+\beta(\omega)H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\\
&\qquad+2\gamma(\omega)H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\\
&\quad=(\alpha(\omega)+\beta(\omega)+2\gamma(\omega))H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\\
&\quad=H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))),
\end{aligned} \tag{2.12}$$

a contradiction. Therefore,

$$H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\le d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \tag{2.13}$$

Since $d(f(\omega,\xi_1(\omega)),T(\omega,\xi_1(\omega)))\le H(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))$, we have

$$d(f(\omega,\xi_1(\omega)),T(\omega,\xi_1(\omega)))\le d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \tag{2.14}$$

By induction, we can show that

$$H(T(\omega,\xi_n(\omega)),T(\omega,\xi_{n+1}(\omega)))\le d(f(\omega,\xi_n(\omega)),f(\omega,\xi_{n+1}(\omega))), \tag{2.15}$$

$$d(f(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\le d(f(\omega,\xi_{n-1}(\omega)),f(\omega,\xi_n(\omega))) \tag{2.16}$$

for each $n\ge 1$ and all $\omega\in\Omega$. From (2.6) and (2.15),

$$d(f(\omega,\xi_n(\omega)),f(\omega,\xi_{n+1}(\omega)))\le k(\omega)d(f(\omega,\xi_{n-1}(\omega)),f(\omega,\xi_n(\omega))). \tag{2.17}$$

By (2.17), we get

$$\begin{aligned}
d(f(\omega,\xi_0(\omega)),f(\omega,\xi_2(\omega)))&\le d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))\\
&\qquad+d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega)))\\
&\le(1+k(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))).
\end{aligned} \tag{2.18}$$

From (2.7),

$$\begin{aligned}
&H(T(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega)))\\
&\quad\le\alpha(\omega)\max\left\{d(f(\omega,\xi_0(\omega)),f(\omega,\xi_2(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\}\\
&\qquad+\beta(\omega)\max\{d(f(\omega,\xi_0(\omega)),T(\omega,\xi_0(\omega))),d(f(\omega,\xi_2(\omega)),T(\omega,\xi_2(\omega)))\}\\
&\qquad+\gamma(\omega)[d(f(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega)))+d(f(\omega,\xi_2(\omega)),T(\omega,\xi_0(\omega)))].
\end{aligned} \tag{2.19}$$

Using (2.15), (2.16), (2.17), and (2.18) and the triangle inequality, we get

$$\begin{aligned} d(f(\omega,\xi_2(\omega)),T(\omega,\xi_0(\omega))) &\leq H(T(\omega,\xi_1(\omega)),T(\omega,\xi_0(\omega))) \\ &\leq d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))), \end{aligned} \tag{2.20}$$

$$\begin{aligned} d(f(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega))) &\leq d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))+d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega))) \\ &\quad + d(f(\omega,\xi_2(\omega)),T(\omega,\xi_2(\omega))) \\ &\leq (1+k(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\quad + d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega))) \\ &\leq (1+2k(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \end{aligned} \tag{2.21}$$

Now from (1.4), (2.17), (2.18), and (2.19), we have

$$\begin{aligned} &H(T(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega))) \\ &\quad \leq \alpha(\omega)(1+k(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\qquad + \beta(\omega)k(\omega)d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\qquad + 2\gamma(\omega)(1+k(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\quad = [(1+k(\omega))(\alpha(\omega)+\beta(\omega)+2\gamma(\omega))-\beta(\omega)]d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\quad = (1+k(\omega)-\beta(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \end{aligned} \tag{2.22}$$

Hence we get, as $1+k(\omega)<2k(\omega)$,

$$H(T(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega))) \leq (2k(\omega)-\beta(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \tag{2.23}$$

From (1.4) and (2.7) we have, as $f(\omega,\xi_2(\omega)) \in T(\omega,\xi_1(\omega))$,

$$\begin{aligned} &H(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))) \\ &\quad \leq \alpha(\omega)\max\left\{d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\ &\qquad + \beta(\omega)\max\{d(f(\omega,\xi_1(\omega)),T(\omega,\xi_1(\omega))),d(f(\omega,\xi_2(\omega)),T(\omega,\xi_2(\omega)))\} \\ &\qquad + \gamma(\omega)d(f(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))). \end{aligned} \tag{2.24}$$

Since $f(\omega,\xi_1(\omega)) \in T(\omega,\xi_0(\omega))$, by (2.23) we have

$$\begin{aligned} d(f(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))) &\leq H(T(\omega,\xi_0(\omega)),T(\omega,\xi_2(\omega))) \\ &\leq (2k(\omega)-\beta(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \end{aligned} \tag{2.25}$$

Thus from (2.17) and (2.24), we get

$$\begin{aligned} &H(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))) \\ &\quad\le \alpha(\omega)k(\omega)d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega)))+\beta(\omega)k(\omega)d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\qquad+\gamma(\omega)(2k(\omega)-\beta(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))) \\ &\quad=[k(\omega)(\alpha(\omega)+\beta(\omega)+2\gamma(\omega))-\beta(\omega)\gamma(\omega)]d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \end{aligned} \tag{2.26}$$

Hence, as $\alpha(\omega)+\beta(\omega)+2\gamma(\omega)=1$,

$$H(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega)))\le(k(\omega)-\beta(\omega)\gamma(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \tag{2.27}$$

From (2.6) and (2.27),

$$\begin{aligned} d(f(\omega,\xi_2(\omega)),f(\omega,\xi_3(\omega)))&\le k(\omega)H(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))) \\ &\le k(\omega)(k(\omega)-\beta(\omega)\gamma(\omega))d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \end{aligned} \tag{2.28}$$

Since $k(\omega)=1+\beta(\omega)\gamma(\omega)/2$, we have

$$\begin{aligned} k(\omega)(k(\omega)-\beta(\omega)\gamma(\omega))&=\left(1+\frac{\beta(\omega)\gamma(\omega)}{2}\right)\left(1+\frac{\beta(\omega)\gamma(\omega)}{2}-\beta(\omega)\gamma(\omega)\right) \\ &=\left(1+\frac{\beta(\omega)\gamma(\omega)}{2}\right)\left(1-\frac{\beta(\omega)\gamma(\omega)}{2}\right) \\ &=1-\frac{\beta^2(\omega)\gamma^2(\omega)}{4}. \end{aligned} \tag{2.29}$$

Thus from (2.28),

$$d(f(\omega,\xi_2(\omega)),f(\omega,\xi_3(\omega)))\le\left(1-\frac{\beta^2(\omega)\gamma^2(\omega)}{4}\right)d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))). \tag{2.30}$$

Analogously,

$$d(f(\omega,\xi_3(\omega)),f(\omega,\xi_4(\omega)))\le(1-\beta^2(\omega)\gamma^2(\omega)/4)d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega))). \tag{2.31}$$

By induction,

$$\begin{aligned} &d(f(\omega,\xi_n(\omega)),f(\omega,\xi_{n+1}(\omega))) \\ &\quad\le\left(1-\frac{\beta^2(\omega)\gamma^2(\omega)}{4}\right)^{[n/2]} \\ &\qquad\times\max\{d(f(\omega,\xi_0(\omega)),f(\omega,\xi_1(\omega))),d(f(\omega,\xi_1(\omega)),f(\omega,\xi_2(\omega)))\}, \end{aligned} \tag{2.32}$$

where $[n/2]$ stands for the greatest integer not exceeding $n/2$. Since $\beta(\omega)\gamma(\omega) > 0$ for all $\omega \in \Omega$, from (2.32), we conclude that $\{f(\omega,\xi_n(\omega))\}$ is a Cauchy sequence in $f(\omega \times X)$. Since $f(\omega \times X) = X$ is complete, there is a measurable mapping $f(\omega,\xi(\omega)) \in f(\omega \times X)$ such that

$$\lim_{n\to\infty} f(\omega,\xi_n(\omega)) = f(\omega,\xi(\omega)). \tag{2.33}$$

Now by the triangle inequality and (1.2), we have

$$\begin{aligned}
& d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) \\
&\quad \le d(f(\omega,\xi(\omega)),f(\omega,\xi_{n+1}(\omega))) + d(f(\omega,\xi_{n+1}(\omega)),T(\omega,\xi(\omega))) \\
&\quad \le d(f(\omega,\xi(\omega)),f(\omega,\xi_{n+1}(\omega))) + H(T(\omega,\xi_n(\omega)),T(\omega,\xi(\omega))) \\
&\quad \le d(f(\omega,\xi(\omega)),f(\omega,\xi_{n+1}(\omega))) \\
&\qquad + \alpha(\omega)\max\left\{d(f(\omega,\xi_n(\omega)),f(\omega,\xi(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\
&\qquad + \beta(\omega)\max\{d(f(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega))),d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega)))\} \\
&\qquad + \gamma(\omega)[d(f(\omega,\xi_n(\omega)),T(\omega,\xi(\omega))) + d(f(\omega,\xi(\omega)),T(\omega,\xi_n(\omega)))].
\end{aligned} \tag{2.34}$$

Thus

$$\begin{aligned}
& d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) \\
&\quad \le d(f(\omega,\xi(\omega)),f(\omega,\xi_{n+1}(\omega))) \\
&\qquad + \alpha(\omega)\max\left\{d(f(\omega,\xi_n(\omega)),f(\omega,\xi(\omega))),\cdot,\cdot,\left(\frac{1}{2}\right)[\cdot+\cdot]\right\} \\
&\qquad + \beta(\omega)\max\{d(f(\omega,\xi_n(\omega)),f(\omega,\xi_{n+1}(\omega))),d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega)))\} \\
&\qquad + \gamma(\omega)[d(f(\omega,\xi_n(\omega)),T(\omega,\xi(\omega))) + d(f(\omega,\xi(\omega)),f(\omega,\xi_{n+1}(\omega)))].
\end{aligned} \tag{2.35}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) &\le \alpha(\omega)d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) \\
&\quad + \beta(\omega)d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) \\
&\quad + \gamma(\omega)d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) \\
&= (1-\gamma(\omega))d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))).
\end{aligned} \tag{2.36}$$

Hence $d(f(\omega,\xi(\omega)),T(\omega,\xi(\omega))) = 0$, as $1-\gamma(\omega) < 1$ for all $\omega \in \Omega$. Hence, as $T(\omega,\xi(\omega))$ is closed,

$$f(\omega,\xi(\omega)) \in T(\omega,\xi(\omega)) \quad \forall \omega \in \Omega. \tag{2.37}$$

□

Remark 2.2. If in Theorem 2.1, $f(\omega,x) = x$ for all $(\omega,x) \in \Omega\times X$, then we get the following random fixed point theorem.

Corollary 2.3. *Let (X,d) be a separable complete metric space, let (Ω,Σ) be a measurable space, and let a mapping $T:\Omega\times X\to \mathrm{CB}(X)$ be such that $T(\omega,\cdot)$ is continuous for all $\omega\in\Omega$, $T(\cdot,x)$ is measurable for all $x\in X$, and*

$$\begin{aligned}
&H(T(\omega,x),T(\omega,y))\\
&\quad\le \alpha(\omega)\max\left\{d(x,y),d(x,T(\omega,x)),d(y,T(\omega,y)),\left(\frac{1}{2}\right)[d(x,T(\omega,y))+d(y,T(\omega,x))]\right\}\\
&\qquad+\beta(\omega)\max\{d(x,T(\omega,x)),d(y,T(\omega,y))\}+\gamma(\omega)[d(x,T(\omega,y))+d(y,T(\omega,x))]
\end{aligned}\tag{2.38}$$

for every $x,y\in X$, where $\alpha,\beta,\gamma:\Omega\to(0,1)$ are measurable mappings satisfying (1.2). Then there is a measurable mapping $\xi:\Omega\to X$ such that $\xi(w)\in T(w,\xi(w))$ for all $\omega\in\Omega$.

Corollary 2.4. *Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $f:\Omega\times X\to X$ and $T:\Omega\times X\to\mathrm{CB}(X)$ be two mappings satisfying the conditions (i) and (ii) in Theorem 2.1. If $f(\omega\times X)=X$ for each $\omega\in\Omega$ and f and T satisfy the following condition:*

$$\begin{aligned}
&H(T(\omega,x),T(\omega,y))\\
&\quad\le\lambda(\omega)\max\left\{d(f(\omega,x),f(\omega,y)),d(f(\omega,x),T(\omega,x)),d(f(\omega,y),T(\omega,y)),\right.\\
&\qquad\qquad\left.\frac{d(f(\omega,x),T(\omega,y))+d(f(\omega,y),T(\omega,x))}{2}\right\},
\end{aligned}\tag{2.39}$$

where $\lambda:\Omega\to(0,1)$ is a measurable function, then there is a measurable mapping $\xi:\Omega\to X$ such that $f(\omega,\xi(w))\in T(w,\xi(w))$ for all $\omega\in\Omega$.

Proof. It is clear that if f and T satisfy (2.39), then f and T satisfy (1.2) with

$$\alpha(\omega)=\lambda(\omega),\qquad \beta(\omega)=\frac{1-\lambda(\omega)}{2},\qquad \gamma(\omega)=\frac{1-\lambda(\omega)}{4}.\tag{2.40}$$

□

Remark 2.5. If in Corollary 2.4, $f(\omega,x)=x$ for all $(\omega,x)\in\Omega\times X$, then we obtain the corresponding theorems of Hadžić [5] and Papageorgiou [13].

Finally, we give a simple example which shows that Theorem 2.1 and Corollaries 2.3 and 2.4 are actually an improvement of the results of Kubiak [10] and Papageorgiou [13].

Example 2.6. Let (Ω,Σ) be any measurable space and let $K=\{0,1,2,4,6\}$ be the subset of the real line. Let the mappings $f:\Omega\times K\to K$ and $T:\Omega\times K\to K$ be defined such that for each $\omega \in \Omega$,

$$\begin{aligned} &f(\omega,0)=2, \quad f(\omega,1)=4, \quad f(\omega,2)=6, \quad f(\omega,4)=0, \quad f(\omega,6)=1, \\ &T(\omega,0)=1, \quad T(\omega,1)=2, \quad T(\omega,2)=4, \quad T(\omega,4)=0, \quad T(\omega,6)=0. \end{aligned} \tag{2.41}$$

Then f and T do not satisfy the contractive-type condition (2.39). Indeed, for $x = 1$ and $y = 2$, we have

$$d(T(\omega,1),T(\omega,2)) = 2 > \lambda(\omega)\max\left\{\|4-6\|,\|4-2\|,\|6-4\|,\frac{0+\|6-2\|}{2}\right\} = 2\lambda(\omega) \tag{2.42}$$

for any $\lambda(\omega) < 1$. On the other hand,

$$d(T(\omega,1),T(\omega,2)) = \frac{4}{5}\cdot 2+\frac{1}{10}\cdot 2+\frac{1}{20}(4+0). \tag{2.43}$$

Thus, for $x = 1$ and $y = 2$, f and T satisfy (1.2) with $\alpha(\omega) = 4/5$, $\beta(\omega) = 1/10$, and $\gamma(\omega) = 1/20$. It is easy to show that f and T satisfy (1.2) for all $x, y \in K$, with the same $\alpha(\omega)$, $\beta(\omega)$, and $\gamma(\omega)$. Also, the rest of assumptions of Theorem 2.1 is satisfied and for $\xi(\omega) = 4$ we have

$$f(\omega,\xi(\omega)) = 0 = T(\omega,\xi(\omega)). \tag{2.44}$$

Note that T does not satisfy (2.38) either, as for instance, for $x = 0$ and $y = 2$, we have

$$\begin{aligned} &\alpha(\omega)\max\left\{\|0-2\|,\|0-1\|,\|2-4\|,\frac{\|0-4\|+\|2-1\|}{2}\right\} \\ &\qquad +\beta(\omega)\max\{\|0-1\|,\|2-4\|\}+\gamma(\omega)[\|0-4\|+\|2-1\|] \\ &\quad = \frac{5}{2}\alpha(\omega)+2\beta(\omega)+5\gamma(\omega) < 3[\alpha(\omega)+\beta(\omega)+2\gamma(\omega)] = 3 = d(T(\omega,0),T(\omega,2)). \end{aligned} \tag{2.45}$$

Remark 2.7. Corollary 2.4 is a stochastic generalization and improvement of the corresponding fixed point theorems for contractive-type multivalued mappings of Ćirić [2], Ćirić and Ume [4], Kubiaczyk [9], Kubiak [10], Papageorgiou [14], and several other authors. Also Theorem 2.1 generalizes and extends the corresponding fixed point theorems for nonexpansive-type single-valued mappings of Ćirić [1] and Rhoades [15].

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