CONTINUITY OF MULTILINEAR OPERATORS ON TRIEBEL-LIZORKIN SPACES

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The continuity of some multilinear operators related to certain convolution operators on the Triebel-Lizorkin space is obtained. The operators include Littlewood-Paley operator and Marcinkiewicz operator.

1. Introduction

Let T be the Calderón-Zygmund singular integral operator, a well-known result of Coifman et al. (see [6]) states that the commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in$ BMO) is bounded on $L^p(R^n)$ $(1 < p < \infty)$; Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator; in [8, 9], these results on the Triebel-Lizorkin spaces and the case $b \in \text{Lip}\,\beta$ (where $\text{Lip}\,\beta$ is the homogeneous Lipschitz space) are obtained. The main purpose of this paper is to study the continuity of some multilinear operators related to certain convolution operators on the Triebel-Lizorkin spaces. In fact, we will obtain the continuity on the Triebel-Lizorkin spaces for the multilinear operators only under certain conditions on the size of the operators. As the applications, the continuity of the multilinear operators related to the Littlewood-Paley operator and Marcinkiewicz operator on the Triebel-Lizorkin spaces are obtained.

2. Notations and results

Throughout this paper, Q will denote a cube of R^n with side parallel to the axes, and for a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $1 \le r < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\delta r/n}}\int_Q |f(y)|^r\,dy\right)^{1/r}, \tag{2.1}$$

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we denote $M_{\delta,r}(f) = M_r(f)$ if $\delta = 0$, which is the Hardy-Littlewood maximal function when $r = 1$ (see [10]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space, and let the Lipschitz space $\dot{\wedge}_\beta$ be the space of functions f such that

$$\|f\|_{\dot{\wedge}_\beta} = \sup_{x,\, h\in R^n,\, h\neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta < \infty}, \tag{2.2}$$

where Δ_h^k denotes the kth difference operator (see [9]).

We are going to study the multilinear operator as follows.

Let m be a positive integer and let A be a function on R^n. We denote

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leq m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha. \tag{2.3}$$

Definition 2.1. Let $F(x,t)$ define on $R^n \times [0,+\infty)$, denote

$$\begin{aligned} F_t(f)(x) &= \int_{R^n} F(x-y,t) f(y)\,dy, \\ F_t^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} F(x-y,t) f(y)\,dy. \end{aligned} \tag{2.4}$$

Let H be the Hilbert space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||; \tag{2.5}$$

and also define $T(f)(x) = \|F_t(f)(x)\|$.

In particular, consider the following two sublinear operators.

Definition 2.2. Fix $\varepsilon > 0$, $n > \delta \geq 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int \psi(x)\,dx = 0$;

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$;

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$g_\delta^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{2.6}$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y) f(y)\,dy \tag{2.7}$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Denote that $F_t(f) = \psi_t * f$, and also define that

$$g_\delta(f)(x) = \left(\int_0^\infty \left| F_t(f)(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{2.8}$$

which is the Littlewood-Paley g function when $\delta = 0$ (see [11]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$, then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\delta(f)(x) = \|F_t(f)(x)\|, \qquad g_\delta^A(f)(x) = \|F_t^A(f)(x)\|. \tag{2.9}$$

Definition 2.3. Let $0 \le \delta < n$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\delta^A(f)(x) = \left(\int_0^\infty \left| F_t^A(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{2.10}$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y)dy; \tag{2.11}$$

denote

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy, \tag{2.12}$$

and also define that

$$\mu_\delta(f)(x) = \left(\int_0^\infty \left| F_t(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{2.13}$$

which is the Marcinkiewicz operator when $\delta = 0$ (see [12]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that

$$\mu_\delta(f)(x) = \|F_t(f)(x)\|, \qquad \mu_\delta^A(f)(x) = \|F_t^A(f)(x)\|. \tag{2.14}$$

It is clear that Definitions 2.2 and 2.3 are the particular examples of Definition 2.1. Note that when $m = 0$, T^A is just the commutator of F_t and A, while when $m > 0$, it is nontrivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5, 7]). The main purpose of this paper is to study the continuity for the multilinear operators on the Triebel-Lizorkin spaces. We will prove the following theorems in Section 3.

Theorem 2.4. *Let g_δ^A be the multilinear Littlewood-Paley operator as in Definition 2.2. If $0 < \beta < \min(1, \varepsilon)$ and $D^\alpha A \in \dot{\wedge}_\beta$ for $|\alpha| = m$, then*

(a) g_δ^A maps $L^p(R^n)$ continuously into $\dot{F}_q^{\beta,\infty}(R^n)$, for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$;
(b) g_δ^A maps $L^p(R^n)$ continuously into $L^q(R^n)$ for $1 < p < n/(\delta+\beta)$ and $1/p - 1/q = (\delta+\beta)/n$.

Theorem 2.5. *Let μ_δ^A be the multilinear Marcinkiewicz operator as in Definition 2.3. If $0 < \beta < \min(1/2, \gamma)$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$, then*
(a) μ_δ^A maps $L^p(R^n)$ continuously into $\dot{F}_q^{\beta,\infty}(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$,
(b) μ_δ^A maps $L^p(R^n)$ continuously into $L^q(R^n)$ for $1 < p < n/(\delta+\beta)$ and $1/p - 1/q = (\delta+\beta)/n$.

3. Main theorem and proof

We first prove a general theorem.

Theorem 3.1 (main theorem). *Let $0 \le \delta < n$, $0 < \beta < 1$, and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Suppose F_t, T, and T^A are the same as in Definition 2.1, if T is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, and T satisfies the following size condition:*

$$||F_t^A(f)(x) - F_t^A(f)(x_0)|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1} f(x) \tag{3.1}$$

for any cube Q with $\operatorname{supp} f \subset (2Q)^c$ and $x \in Q$, then
(a) T^A is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta,\infty}(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$,
(b) T^A is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/(\delta+\beta)$ and $1/q = 1/p - (\delta+\beta)/n$.

To prove the theorem, we need the following lemmas.

Lemma 3.2 (see [9]). *For $0 < \beta < 1$, $1 < p < \infty$,*

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_{Q} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \right\|_{L^p} \\ &\approx \left\| \sup_{\cdot\in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|\, dx \right\|_{L^p}. \end{aligned} \tag{3.2}$$

Lemma 3.3 (see [9]). *For $0 < \beta < 1$, $1 \le p \le \infty$,*

$$\begin{aligned} \|f\|_{\dot{\Lambda}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p\, dx \right)^{1/p}. \end{aligned} \tag{3.3}$$

Lemma 3.4 (see [1, 2]). *Suppose that $1 \le r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

$$||M_{\delta,r}(f)||_{L^q} \le C \|f\|_{L^p}. \tag{3.4}$$

Lemma 3.5 (see [5]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q}, \tag{3.5}$$

where $\tilde{Q}(x,y)$ is the cube centered at x and has side length $5\sqrt{n}|x-y|$.

Proof of Theorem 3.1 (main theorem). Fix a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}(1/\alpha!)(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} F(x-y,t) f(y)dy \\ &= \int_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} F(x-y,t) f_2(y)dy \\ &\quad + \int_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} F(x-y,t) f_1(y)dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{F(x-y,t)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y)dy, \end{aligned} \tag{3.6}$$

then

$$\begin{aligned} |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)| &= \big| \|F_t^A(f)(x)\| - \|F_t^{\tilde{A}}(f_2)(x_0)\| \big| \\ &\le \left\| F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1 \right)(x) \right\| \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right)(x) \right\| \\ &\quad + \|F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)\| = A(x) + B(x) + C(x), \end{aligned} \tag{3.7}$$

thus,

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |T^A(f)(x) - T^{\tilde{A}}(f)(x_0)| dx \\ &\quad \le \frac{1}{|Q|^{1+\beta/n}} \int_Q A(x)dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q B(x)dx \\ &\qquad + \frac{1}{|Q|^{1+\beta/n}} \int_Q C(x)dx := I + II + III. \end{aligned} \tag{3.8}$$

Now, let us estimate I, II, and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemmas 3.3 and 3.5, we get

$$\begin{aligned}|R_m(\tilde{A};x,y)| &\le C|x-y|^m \sum_{|\alpha|=m} \sup_{x\in\tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}| \\ &\le C|x-y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta},\end{aligned} \tag{3.9}$$

thus, taking r, s such that $1 \le r < p$ and $1/s = 1/r - \delta/n$, by the (L^r, L^s) boundedness of T and Holder' inequality, we obtain

$$\begin{aligned}I &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \frac{1}{|Q|}\int_Q |T(f_1)(x)|\,dx \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|T(f_1)\|_{L^s} |Q|^{-1/s} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|f_1\|_{L^r} |Q|^{-1/s} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}).\end{aligned} \tag{3.10}$$

Secondly, using the following inequality (see [9]):

$$\|(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{\tilde{Q}}\|_{L^r} \le C|Q|^{1/s+\beta/n} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(x), \tag{3.11}$$

and similar to the proof of I, we gain

$$II \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \tag{3.12}$$

For III, using the size condition of T, we have

$$III \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \tag{3.13}$$

We now put these estimates together; and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemmas 3.2 and 3.4, we obtain

$$\|T^A(f)\|_{\dot{F}_q^{\beta,\infty}} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}. \tag{3.14}$$

This completes the proof of (a).

(b) By same argument as in proof of (a), we have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)|\,dx \\ &\quad\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \big(M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)\big),\end{aligned} \tag{3.15}$$

thus,

$$(T^A(f))^{\#} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \big(M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)\big). \tag{3.16}$$

Now, using Lemma 3.4, we gain

$$
\begin{aligned}
||T^A(f)||_{L^q} &\le C||(T^A(f))^{\#}||_{L^q} \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}(||M_{\delta+\beta,r}(f)||_{L^q}+||M_{\delta+\beta,1}(f)||_{L^q})\le C\|f\|_{L^p}.
\end{aligned}
\tag{3.17}
$$

This completes the proof of (b) and the theorem. □

To prove Theorems 2.4 and 2.5, since g_δ and μ_δ are all bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1<p<n/\delta$ and $1/q=1/p-\delta/n$ (see [11, 12]), it suffices to verify that g_δ^A and μ_δ^A satisfy the size condition in *Theorem 3.1 (main theorem)*.

Suppose $\operatorname{supp} f\subset(2Q)^c$ and $x\in Q=Q(x_0,l)$. Note that $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$.

For g_δ^A, we write

$$
\begin{aligned}
&F_t^{\tilde{A}}(f)(x)-F_t^{\tilde{A}}(f)(x_0)\\
&\quad=\int_{R^n\setminus\tilde{Q}}\left[\frac{\psi_t(x-y)}{|x-y|^m}-\frac{\psi_t(x_0-y)}{|x_0-y|^m}\right]R_m(\tilde{A};x,y)f(y)dy\\
&\qquad+\int_{R^n\setminus\tilde{Q}}\frac{\psi_t(x_0-y)f(y)}{|x_0-y|^m}[R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)]dy\\
&\qquad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n\setminus\tilde{Q}}\left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m}\right]D^\alpha\tilde{A}(y)f(y)dy\\
&\quad=I_1+I_2+I_3.
\end{aligned}
\tag{3.18}
$$

By the condition on ψ, we obtain

$$
\begin{aligned}
||I_1||&\le C\int_{R^n\setminus\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{m+1}}|R_m(\tilde{A};x,y)||f(y)|\left(\int_0^\infty\frac{tdt}{(t+|x_0-y|)^{2(n+1-\delta)}}\right)^{1/2}dy\\
&\quad+C\int_{R^n\setminus\tilde{Q}}\frac{|x-x_0|^\varepsilon}{|x_0-y|^m}|R_m(\tilde{A};x,y)||f(y)|\left(\int_0^\infty\frac{tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus2^{k+1}\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^\infty(2^{-k}+2^{-k\varepsilon})\left(\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(y)|dy\right)\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x).
\end{aligned}
\tag{3.19}
$$

For I_2, by the formula (see [5]):

$$R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \widetilde{A};x,x_0)(x-y)^\eta \tag{3.20}$$

and Lemma 3.5, we get

$$|R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y)| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} |x-x_0| \, |x_0-y|^{m-1}, \tag{3.21}$$

thus, similar to the proof of I_1,

$$\begin{aligned} ||I_2|| &\le C \int_{R^n \setminus \widetilde{Q}} \frac{|R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y)|}{|x_0-y|^{m+n-\delta}} |f(y)| \, dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} |f(y)| \, dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \end{aligned} \tag{3.22}$$

For I_3, similar to the proof of I_1, we obtain

$$\begin{aligned} ||I_3|| &\le C \sum_{|\alpha|=m} \int_{R^n \setminus \widetilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f(y)| \, |D^\alpha \widetilde{A}(y)| \, dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} \left(2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)}\right) M_{\delta,1}(f)(x) \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x) \end{aligned} \tag{3.23}$$

so that

$$||F_t^{\widetilde{A}}(f)(x) - F_t^{\widetilde{A}}(f)(x_0)|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \tag{3.24}$$

For μ_δ^A, we write

$$
\begin{aligned}
&||F_t^{\widetilde{A}}(f)(x) - F_t^{\widetilde{A}}(f)(x_0)|| \\
&\quad\le \left(\int_0^\infty \left[\int_{|x-y|\le t,\ |x_0-y|>t} \frac{|\Omega(x-y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}} |f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left[\int_{|x-y|>t,\ |x_0-y|\le t} \frac{|\Omega(x_0-y)|\,|R_m(\widetilde{A};x_0,y)|}{|x_0-y|^{m+n-1-\delta}} |f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left[\int_{|x-y|\le t,|x_0-y|\le t} \left|\frac{\Omega(x-y)R_m(\widetilde{A};x,y)}{|x-y|^{m+n-1-\delta}}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \frac{\Omega(x_0-y)R_m(\widetilde{A};x_0,y)}{|x_0-y|^{m+n-1-\delta}}\right| |f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + C\sum_{|\alpha|=m} \left(\int_0^\infty \left|\int_{|x-y|\le t}\left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1-\delta}} - \int_{|x_0-y|\le t}\frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1-\delta}}\right)\right.\right. \\
&\qquad\qquad \left.\left. \times D^\alpha\widetilde{A}(y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} := J_1 + J_2 + J_3 + J_4.
\end{aligned}
\tag{3.25}
$$

Then

$$
\begin{aligned}
J_1 &\le C\int_{R^n\setminus\widetilde{Q}} \frac{|f(y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}} \left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2} dy \\
&\le C\int_{R^n\setminus\widetilde{Q}} \frac{|f(y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=1}^\infty 2^{-k/2} \frac{1}{|2^k\widetilde{Q}|^{1-\delta/n}} \int_{2^k\widetilde{Q}} |f(y)|dy \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x),
\end{aligned}
\tag{3.26}
$$

similarly, we have $J_2 \le C\sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x)$.

For J_3, by the following inequality (see [12]):

$$
\left|\frac{\Omega(x-y)}{|x-y|^{m+n-1-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{m+n-1-\delta}}\right| \le C\left(\frac{|x-x_0|}{|x_0-y|^{m+n-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{m+n-1-\delta+\gamma}}\right),
\tag{3.27}
$$

we gain

$$\begin{aligned} J_3 &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot\Lambda_\beta}|Q|^{\beta/n}\int_{R^n\setminus\tilde Q}\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1-\delta+\gamma}}\right)\\ &\qquad\times\left(\int_{|x_0-y|\le t,\ |x-y|\le t}\frac{dt}{t^3}\right)^{1/2}|f(y)|\,dy\\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot\Lambda_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}(2^{-k}+2^{-\gamma k})M_{\delta,1}(f)(x)\\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot\Lambda_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x). \end{aligned} \tag{3.28}$$

For J_4, similar to the proof of J_1, J_2, and J_3, we obtain

$$\begin{aligned} J_4 &\le C\sum_{|\alpha|=m}\int_{R^n\setminus\tilde Q}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2-\delta}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta}}\right)\\ &\qquad\times|D^\alpha\tilde A(y)||f(y)|\,dy\\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot\Lambda_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}\left(2^{k(\beta-1)}+2^{k(\beta-1/2)}+2^{k(\beta-\gamma)}\right)\frac{1}{|2^k\tilde Q|}\int_{2^k\tilde Q}|f(y)|\,dy\\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot\Lambda_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x). \end{aligned} \tag{3.29}$$

These yield the desired results.

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