INEQUALITIES FOR DIFFERENTIABLE REPRODUCING KERNELS AND AN APPLICATION TO POSITIVE INTEGRAL OPERATORS

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Let $I \subseteq \mathbb{R}$ be an interval and let $k : I^2 \to \mathbb{C}$ be a reproducing kernel on I. We show that if $k(x,y)$ is in the appropriate differentiability class, it satisfies a 2-parameter family of inequalities of which the diagonal dominance inequality for reproducing kernels is the 0th order case. We provide an application to integral operators: if k is a positive definite kernel on I (possibly unbounded) with differentiability class $\mathcal{S}_n(I^2)$ and satisfies an extra integrability condition, we show that eigenfunctions are $C^n(I)$ and provide a bound for its Sobolev H^n norm. This bound is shown to be optimal.

1. Introduction

Given a set E, a positive definite matrix in the sense of Moore (see, e.g., Moore [5, 6] and Aronszajn [1]) is a function $k : E \times E \to \mathbb{C}$ such that

$$\sum_{i,j=1}^{n} k(x_i,x_j)\overline{\xi_i}\xi_j \geq 0 \tag{1.1}$$

for all $n \in \mathbb{N}$, $(x_1,\dots,x_n) \in E^n$ and $(\xi_1,\dots,\xi_n) \in \mathbb{C}^n$; that is, all finite square matrices M of elements $m_{ij} = k(x_i,x_j)$, $i, j = 1,\dots,n$, are positive semidefinite.

From (1.1) it follows that a positive definite matrix in the sense of Moore has the following basic properties: (1) it is conjugate symmetric, that is, $k(x,y) = \overline{k(y,x)}$ for all $x, y \in E$, (2) it satisfies $k(x,x) \geq 0$ for all $x \in E$, and (3) $|k(x,y)|^2 \leq k(x,x)k(y,y)$ for all $x, y \in E$. We sometimes refer to this last basic inequality as the "diagonal dominance" inequality.

The theorem of Moore-Aronszajn [1, 5, 6] provides an equivalent characterization of positive definite matrices as *reproducing kernels:* $k : E \times E \to \mathbb{C}$ is a positive definite matrix in the sense of Moore if and only if there exists a (uniquely determined) Hilbert space H_k

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composed of functions on E such that

$$\begin{gathered} \forall y \in E, \quad k(x,y) \in H_k \text{ as a function of } x, \\ \forall x \in E \text{ and any } f \in H_k, \quad f(x) = \langle f(y), k(y,x) \rangle_{H_k}. \end{gathered} \tag{1.2}$$

Properties (1.2) are jointly called the *reproducing property of k in H_k*. The function k itself is called a *reproducing kernel on E* and the associated (and unique) Hilbert space H_k a *reproducing kernel Hilbert space*; see, for example, Saitoh [8].

Throughout this paper we deal exclusively with the case where $E = I \subseteq \mathbb{R}$ is a real interval, nontrivial but otherwise arbitrary; in particular I may be unbounded. Only in Section 3 we will need the further assumption that I is closed; this extra condition will at that point be explicitly required. If $x \in I$ is a boundary point of I, a limit at x will mean the one-sided limit as $y \to x$ with $y \in I$.

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. A function $k : I^2 \to \mathbb{C}$ is said to *be of class $\mathcal{S}_n(I^2)$* if, for every $m_1 = 0, 1, \ldots, n$ and $m_2 = 0, 1, \ldots, n$, the partial derivatives $\partial^{m_1+m_2}/\partial y^{m_2} \partial x^{m_1} k(x, y)$ are continuous in I^2.

Remark 1.2. Clearly from the definition $C^{2n}(I^2) \subset \mathcal{S}_n(I^2) \subset C^n(I^2)$. It is also clear that a function of class $\mathcal{S}_n(I^2)$ will not in general be in $C^{n+1}(I^2)$. Note however that in class $\mathcal{S}_n(I^2)$ equality of all intervening mixed partial derivatives holds.

In [4, Theorem 2.7], the following result is shown to hold for differentiable reproducing kernels as a nontrivial consequence of positive semidefiniteness of the matrices $k(x_i, x_j)$ in (1.1).

Theorem 1.3. *Let $I \subset \mathbb{R}$ be an interval and let $k(x, y)$ be a reproducing kernel on I of class $\mathcal{S}_n(I^2)$. Then for all $x, y \in I$ and all $0 \le m \le n$,*

$$\left| \frac{\partial^m k}{\partial x^m}(x,y) \right|^2 \le \frac{\partial^{2m} k}{\partial y^m \partial x^m}(x,x) k(y,y). \tag{1.3}$$

Remark 1.4. An immediate consequence of conjugate symmetry of k is that inequality (1.3) is equivalent to

$$\left| \frac{\partial^m k}{\partial y^m}(x,y) \right|^2 \le \frac{\partial^{2m} k}{\partial y^m \partial x^m}(y,y)\, k(x,x). \tag{1.4}$$

Remark 1.5. Observe that the 1-parameter family of inequalities (1.3) coupled with the condition $k(y, y) \ge 0$ for all $y \in I$ implies that

$$\frac{\partial^{2m} k}{\partial y^m \partial x^m}(x,x) \ge 0 \tag{1.5}$$

for all $x \in I$ and all $0 \le m \le n$.

2. Differentiable reproducing kernel inequalities

Let $I \subseteq \mathbb{R}$ be an interval and $k : I \times I \to \mathbb{C}$. Denote by I_R the set of all $x \in I$ such that $x + h$ is in I for $|h| < R$. For sufficiently small R, I_R is a nonempty open interval. For $|h| < R$ we define $\delta_h : I_R^2 \to \mathbb{C}$ by

$$\delta_h(x,y) = k(x+h,y+h) - k(x+h,y) - k(x,y+h) + k(x,y). \tag{2.1}$$

We then have the following lemma.

Lemma 2.1. *If $k(x,y)$ is a reproducing kernel on I^2 and $|h| < R$, then $\delta_h(x,y)$ is a reproducing kernel in I_R^2.*

Proof. Let $l \in \mathbb{N}$, $(x_1,\dots,x_l) \in I_h^l$ and $(\xi_1,\dots,\xi_l) \in \mathbb{C}^l$. We are required to show that $\sum_{i,j=1}^{l} \delta_h(x_i,x_j)\ \xi_i\overline{\xi_j} \geq 0$. Define $x_{l+i} = x_i + h$ and $\xi_{l+i} = -\xi_i$ for $i = 1,\dots,l$. Since k is a reproducing kernel on I^2, we have $\sum_{i,j=1}^{2l} k(x_i,x_j)\ \xi_i\overline{\xi_j} \geq 0$. Rewriting the left-hand side, we obtain

$$\begin{aligned}
\sum_{i,j=1}^{2l} k(x_i,x_j)\xi_i\overline{\xi_j} &= \sum_{i,j=1}^{l} k(x_i,x_j)\xi_i\overline{\xi_j} \\
&\quad + \sum_{i=1}^{l}\sum_{j=l+1}^{2l} k(x_i,x_j)\xi_i\overline{\xi_j} + \sum_{i=l+1}^{2l}\sum_{j=1}^{l} k(x_i,x_j)\xi_i\overline{\xi_j} + \sum_{i,j=l+1}^{2l} k(x_i,x_j)\xi_i\overline{\xi_j} \\
&= \sum_{i,j=1}^{l} k(x_i,x_j)\xi_i\overline{\xi_j} + \sum_{i,j=1}^{l} k(x_i,x_j+h)\xi_i(-\overline{\xi_j}) + \sum_{i,j=1}^{l} k(x_i+h,x_j)(-\xi_i)\overline{\xi_j} \\
&\quad + \sum_{i,j=1}^{l} k(x_i+h,x_j+h)(-\xi_i)(-\overline{\xi_j}) \\
&= \sum_{i,j=1}^{l} \left[k(x_i+h,x_j+h) - k(x_i+h,x_j) - k(x_i,x_j+h) + k(x_i,x_j)\right]\xi_i\overline{\xi_j} \\
&= \sum_{i,j=1}^{l} \delta_h(x_i,x_j)\xi_i\overline{\xi_j} \geq 0.
\end{aligned} \tag{2.2}$$

Thus $\delta_h(x,y)$ is a reproducing kernel on I_R^2 as stated. □

We will frequently denote, for ease of notation, $k_m(x,y) = (\partial^{2m}k/\partial y^m\partial x^m)(x,y)$.

Proposition 2.2. *Let $I \subset \mathbb{R}$ be an interval and let $k(x,y)$ be a reproducing kernel of class $\mathcal{S}_n(I^2)$. Then, for all $0 \leq m \leq n$, $k_m(x,y) = (\partial^{2m}/\partial y^m\partial x^m)k(x,y)$ is a reproducing kernel of class $\mathcal{S}_{n-m}(I^2)$.*

Proof. Since in the case $n = 0$ the statement is empty, we begin by concentrating on the case $m = n = 1$. Suppose k is of class $\mathcal{S}_1(I^2)$. Then, by [4, Lemma 2.5], if $|h| < R$, we have

$$k_1(x,y) = \lim_{h\to 0} \frac{\delta_h(x,y)}{h^2}, \tag{2.3}$$

for every $(x,y) \in I_R^2$. By Lemma 2.1, $\delta_h(x,y)$ is a reproducing kernel on I_R^2. Hence the last inequality in (2.2) implies that

$$\sum_{i,j=1}^{l} k_1(x_i,x_j)\xi_i\overline{\xi_j} \geq 0 \tag{2.4}$$

for any natural l, $(x_1,\dots,x_l) \in I_R^l$ and $(\xi_1,\dots,\xi_l) \in \mathbb{C}^l$. Therefore, $k_1(x,y)$ is a reproducing kernel on I_R^2. By continuity of k_1 inequality (2.4) holds for boundary points in I_2 (if they exist) with the interpretation of partial derivatives as appropriate one-sided limits. Thus (2.4) holds for all $(x_1,\dots,x_l) \in I^l$ and every choice of $l \in \mathbb{N}$ and $(\xi_1,\dots,\xi_l) \in \mathbb{C}^l$. Therefore k_1 is a reproducing kernel on I^2.

To conclude the proof, we now fix $n \in \mathbb{N}$, suppose that k is a reproducing kernel of class $\mathcal{S}_n(I^2)$ and that k_m is a reproducing kernel for some $m < n$. It is immediate to see that k_m is of class $\mathcal{S}_{n-m}(I^2)$. Repeating the argument used in the proof of the case $m = n = 1$, we conclude that k_{m+1} is a reproducing kernel. Therefore k_m is a reproducing kernel for all $0 \leq m \leq n$. This finishes the proof. □

Theorem 2.3. *Let $I \subseteq \mathbb{R}$ be an interval and $k(x,y)$ be a reproducing kernel of class $\mathcal{S}_n(I^2)$. Then, for every m_1, $m_2 = 0,1,\dots,n$ and all $x,y \in I$,*

$$\left|\frac{\partial^{m_1+m_2}}{\partial y^{m_2}\partial x^{m_1}}k(x,y)\right|^2 \leq \frac{\partial^{2m_1}}{\partial y^{m_1}\partial x^{m_1}}k(x,x)\frac{\partial^{2m_2}}{\partial y^{m_2}\partial x^{m_2}}k(y,y). \tag{2.5}$$

Proof. Since k is a reproducing kernel of class $\mathcal{S}_n(I^2)$, by Proposition 2.2 k_m is a reproducing kernel of class $\mathcal{S}_{n-m}(I^2)$ for every $0 \leq m \leq n$. Let $0 \leq m_1 \leq m_2 \leq n$. Then $k_{m_1}(x,y) = (\partial^{2m_1}/\partial y^{m_1}\partial x^{m_1})k(x,y)$ is a reproducing kernel of class $\mathcal{S}_{n-m_1}(I^2)$. We may write

$$\begin{aligned}\frac{\partial^{m_1+m_2}}{\partial y^{m_2}\partial x^{m_1}}k(x,y) &= \frac{\partial^{m_2-m_1}}{\partial y^{m_2-m_1}}\frac{\partial^{2m_1}}{\partial y^{m_1}\partial x^{m_1}}k(x,y)\\ &= \frac{\partial^{m_2-m_1}}{\partial y^{m_2-m_1}}k_{m_1}(x,y).\end{aligned} \tag{2.6}$$

Since $m_2 - m_1 \leq n - m_1$, application of Theorem 1.3 to k_{m_1} yields

$$\left|\frac{\partial^{m_2-m_1}}{\partial y^{m_2-m_1}}k_{m_1}(x,y)\right|^2 \leq k_{m_1}(x,x)\frac{\partial^{2(m_2-m_1)}}{\partial y^{(m_2-m_1)}\partial x^{(m_2-m_1)}}k_{m_1}(y,y). \tag{2.7}$$

Hence

$$\left|\frac{\partial^{m_2+m_1}}{\partial y^{m_2}\partial x^{m_1}}k(x,y)\right|^2 \leq \frac{\partial^{2m_1}}{\partial y^{m_1}\partial x^{m_1}}k(x,x)\frac{\partial^{2m_2}}{\partial y^{m_2}\partial x^{m_2}}k(y,y) \tag{2.8}$$

as stated. The proof of the case $0 \leq m_2 \leq m_1 \leq n$ can be obtained in a similar way using the corresponding inequalities derived by conjugate symmetry (see Remark 1.4). □

Remark 2.4. Setting $n = 0$ in Theorem 2.3 yields the statement that if the reproducing kernel $k(x,y)$ is continuous then the diagonal dominance inequality $|k(x,y)|^2 \leq k(x,x)k(y,y)$ holds. Even though continuity is not necessary, this means that the diagonal dominance inequality for reproducing kernels may be thought of as the particular case $n = 0$ in Theorem 2.3.

In this precise sense, Theorem 2.3 yields a 2-parameter family of inequalities which is the generalization of the diagonal dominance inequality for (sufficiently) differentiable reproducing kernels.

3. Sobolev bounds for eigenfunctions of positive integral operators

Throughout this section $I \subseteq \mathbb{R}$ will denote a closed, but not necessarily bounded, interval. A linear integral operator $K : L^2(I) \to L^2(I)$

$$K(\phi) = \int_I k(x,y)\phi(y)dy \tag{3.1}$$

with kernel $k(x,y) \in L^2(I^2)$ is said to be positive if

$$\iint_I k(x,y)\overline{\phi(x)}\phi(y)dx\,dy \geq 0 \tag{3.2}$$

for all $\phi \in L^2(I)$. The corresponding kernel $k(x,y)$ is an $L^2(I)$-*positive definite kernel*. A positive definite kernel is conjugate symmetric for almost all $x, y \in I$, so the associated operator K is self-adjoint. All eigenvalues of K are real and nonnegative as a consequence of (3.2).

Definition 3.1. A positive definite kernel $k(x,y)$ in an interval $I \subseteq \mathbb{R}$ is said to be in class $\mathcal{A}_0(I)$ if

(1) it is continuous in I^2,
(2) $k(x,x) \in L^1(I)$,
(3) $k(x,x)$ is uniformly continuous in I.

Remark 3.2. If I is compact, the first condition trivially implies the other two, so $\mathcal{A}_0(I)$ coincides with the continuous functions $C(I^2)$. Definition 3.1 is therefore especially meaningful in the case where I is unbounded. It has recently been shown [2] that, if k is a positive definite kernel in class $\mathcal{A}_0(I)$, then the corresponding operator is compact, trace class and satisfies (the analog of) Mercer's theorem [7], irrespective of whether I is bounded or unbounded. For this reason a positive definite kernel in class $\mathcal{A}_0(I)$ is sometimes called a Mercer-like kernel [4].

It may easily be shown [2] that, if I is unbounded, the simultaneous conditions of $k(x,x) \in L^1(I)$ and uniform continuity of $k(x,x)$ in I in Definition 3.1 may be equivalently replaced by $k(x,x) \in L^1(I)$ and $k(x,x) \to 0$ as $|x| \to +\infty$. This equivalent characterization of $\mathcal{A}_0(I)$ may sometimes be useful in applications (e.g., [3] or the proof of Theorem 3.5 below).

The following summarizes the properties of positive definite kernels relevant for this paper. If $k(x,y) \in L^2(I)$ is a positive definite kernel, then K is a Hilbert-Schmidt operator; in particular it is compact, so its eigenvalues have finite multiplicity and accumulate only at 0. The spectral expansion

$$k(x,y) = \sum_{i\geq 1} \lambda_i \phi_i(x)\overline{\phi_i(y)} \tag{3.3}$$

holds, where the $\{\phi_i\}_{i\geq 1}$ are an $L^2(I)$-orthonormal set of eigenfunctions spanning the range of K, the $\{\lambda_i\}_{i\geq 1}$ are the nonzero eigenvalues of K and convergence of the series (3.3) is in $L^2(I)$. If in addition k is in class $\mathcal{A}_0(I)$, then for all $x \in I$ $k(x,x) \geq 0$ and for all $x, y \in I$ $|k(x,y)|^2 \leq k(x,x)k(y,y)$, eigenfunctions ϕ_i associated to nonzero eigenvalues are uniformly continuous on I, convergence of the series (3.3) is absolute and uniform on I, and the operator K is trace class and satisfies the trace formula $\int_I k(x,x)dx = \sum_{i\geq 1}\lambda_i$. In the case where I is compact, the last statements are the classical theorem of Mercer; for proofs see, for example, [7] for compact I and [2] for noncompact I. Finally, it is not difficult to show that continuous positive definite kernels are reproducing kernels on I [4], so that the results of Section 2 apply.

Definition 3.3. Let $n \geq 1$ be an integer and $I \subseteq \mathbb{R}$. A positive definite kernel $k : I^2 \to \mathbb{C}$ is said to belong to class $\mathcal{A}_n(I)$ if $k \in \mathcal{S}_n(I)$ and

$$k(x,y), \frac{\partial^2 k}{\partial y \partial x}(x,y), \dots, \frac{\partial^{2n} k}{\partial y^n \partial x^n}(x,y) \tag{3.4}$$

are in class $\mathcal{A}_0(I)$.

Remark 3.4. Trivially $\mathcal{A}_n(I) \subset \mathcal{A}_{n-1}(I) \subset \cdots \subset \mathcal{A}_1(I) \subset \mathcal{A}_0(I)$. More significantly, observe that a positive definite kernel in class $\mathcal{A}_n(I)$ possesses a delicate but precise mix of local (differentiability class $\mathcal{S}_n(I)$) and global (integrability and uniform continuity of each k_m, $m = 0,\dots,n$, along the diagonal $y = x$) properties.

For k in class $\mathcal{A}_n(I)$, we set for each $m = 0,\dots,n$

$$\mathcal{K}_m \equiv \int_I k_m(x,x)dx. \tag{3.5}$$

From Theorem 2.3 it follows that $0 \leq |k_m(x,y)|^2 \leq k_m(x,x)k_m(y,y)$ for all $x, y \in I$. Thus for each $m = 0,\dots,n$, $\mathcal{K}_m > 0$ unless $k_m(x,y)$ is identically zero. In the result below $H^n(I)$ denotes, as usual, the Sobolev Hilbert space $W^{n,2}(I)$ normed by $\|\phi\|^2_{H^n(I)} = \sum_{m=0}^n \|\phi^{(m)}\|^2_{L^2(I)}$. For $0 \leq l \leq n$, we define

$$C_{n,l} = \mathcal{K}_l^{1/2}\left(\sum_{m=l}^{n}\mathcal{K}_m\right)^{1/2}. \tag{3.6}$$

Theorem 3.5. *Suppose $k(x,y)$ is a positive definite kernel in class $\mathcal{A}_n(I)$. Let $0 \leq l \leq n$ and let $\phi_i^{[l]}$ be a normalized eigenfunction of $k_l(x,y)$ associated with a nonzero eigenvalue $\lambda_i^{[l]}$. Then $\phi_i^{[l]}$ is in $C^{n-l}(I) \cap H^{n-l}(I)$ and*

$$\left\|\phi_i^{[l]}\right\|_{H^{n-l}(I)} \leq \frac{C_{n,l}}{\lambda_i^{[l]}}. \tag{3.7}$$

Proof. Let k be in $\mathcal{A}_n(I)$. Then k_l is in $\mathcal{A}_{n-l}(I)$. For fixed $l = 0,\dots,n$, suppose $\phi_i^{[l]}$ is a normalized eigenfunction of k_l associated to $\lambda_i^{[l]} \neq 0$, that is

$$\phi_i^{[l]}(x) = \frac{1}{\lambda_i^{[l]}} \int_I k_l(x,y)\phi_i^{[l]}(y)dy \tag{3.8}$$

with $\|\phi_i^{[l]}\|_{L^2(I)} = 1$. In the case where I is compact, differentiation of (3.8) $n-l$ times under the integral sign holds automatically, and so eigenfunctions are $C^{n-l}(I)$. For unbounded I this is no longer automatic. We will show, however, that in this case it is also true, but as specific consequence of k being a positive definite kernel in class $\mathcal{A}_n(I)$. Thus for the rest of the proof of the first statement I will, without loss of generality, be taken to be $\mathbb{R}$.

By hypothesis, for $0 \leq l \leq m \leq n$ the integrand function $(\partial^{m-l}k_l(x,y))/(\partial x^{m-l})\phi_i^{[l]}(y)$ corresponding to the $(m-l)$th differentiation under the integral sign exists and is continuous. We have

$$\begin{aligned}
\left|\frac{\partial^{m-l}}{\partial x^{m-l}}k_l(x,y)\phi_i^{[l]}(y)\right| &= \left|\frac{\partial^{m-l}}{\partial x^{m-l}}k_l(x,y)\right| \left|\phi_i^{[l]}(y)\right| \\
&\leq \left(\frac{\partial^{2(m-l)}}{\partial y^{m-l}\partial x^{(m-l)}}k_l(x,x)\right)^{1/2} k_l(y,y)^{1/2}\left|\phi_i^{[l]}(y)\right| \\
&\leq k_m(x,x)^{1/2}k_l(y,y)^{1/2}\left|\phi_i^{[l]}(y)\right|,
\end{aligned} \tag{3.9}$$

where we have used Theorem 2.3 with $m_1 = m-l$, $m_2 = 0$, and k replaced with k_l. The fact that $k_l(y,y)^{1/2}|\phi_i^{[l]}(y)|$ is in $L^1(I)$ follows from the Cauchy-Schwartz inequality since

$$\begin{aligned}
\int_I k_l(y,y)^{1/2}\left|\phi_i^{[l]}\right|dy &\leq \left(\int_I k_l(y,y)dy\right)^{1/2}\left\|\phi_i^{[l]}\right\|_{L^2(I)} \\
&= \left(\int_I k_l(y,y)dy\right)^{1/2} = \mathcal{K}_l^{1/2} < +\infty.
\end{aligned} \tag{3.10}$$

Thus differentiation under the integral sign holds, the integral (3.8) is $n-l$ times differentiable, and so are the eigenfunctions $\phi_i^{[l]}$. An analogous argument shows that the integral corresponding to the $(n-l)$th derivative under the integral sign is continuous in I. Thus eigenfunctions corresponding to nonzero eigenvalues are $C^{n-l}(I)$.

The norm estimates work identically for bounded or unbounded I, so from now on we need not make any assumption about it. By the Cauchy-Schwartz inequality and Theorem 2.3 we have

$$\begin{aligned}
\left\|\phi_i^{[l](m-l)}\right\|_{L^2(I)}^2 &= \int_I \left|\phi_i^{[l](m-l)}(x)\right|^2 dx \\
&= \int_I \left|\frac{1}{\lambda_i^{[l]}}\int_I \left(\frac{\partial^{m-l}}{\partial x^{m-l}}k_l(x,y)\right)\phi_i^{[l]}(y)dy\right|^2 dx \\
&\leq \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \int_{-\infty}^{+\infty}\left[\int_I \left|\frac{\partial^{m-l}}{\partial x^{m-l}}k_l(x,y)\right|^2 dy \int_I \left|\phi_i^{[l]}(y)\right|^2 dy\right]dx
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \int_I \left[\int_I \frac{\partial^{2(m-l)} k_l(x,x)}{\partial y^{m-l}\partial x^{m-l}} k_l(y,y)\,dy\right] dx \cdot \left\|\phi_i^{[l]}\right\|_{L^2(I)}^2 \\
&= \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \int_I k_m(x,x)\,dx \int_I k_l(y,y)\,dy = \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \mathcal{K}_m \mathcal{K}_l
\end{aligned} \tag{3.11}$$

for all $0 \leq l \leq m \leq n$ with $l+m \leq n$. Thus

$$\left\|\phi_i^{[l]}\right\|_{H^{n-l}(I)}^2 = \sum_{m=l}^{n} \left\|\phi_i^{[l](m-l)}\right\|_{L^2(I)}^2 \leq \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \sum_{m=l}^{n} \mathcal{K}_m \mathcal{K}_l \tag{3.12}$$

or, recalling definition (3.6), $\|\phi_i^{[l]}\|_{H^{n-l}(I)} \leq C_{n,l}/\lambda_i^{[l]}$ as asserted. □

Since the operators with kernels k_l are compact and positive, for each l the eigenvalue sequence $\{\lambda_i^{[l]}\}_{i\in\mathbb{N}}$ may be assumed to be decreasing to 0. We denote by $E_N^{[l]} = \oplus_{i=1}^{N} E_{\lambda_i^{[l]}}$ the direct sum of the eigenspaces associated with the first N eigenvalues of k_l.

Corollary 3.6. *Suppose $k(x,y)$ is a positive definite kernel in class $\mathcal{A}_n(I)$ and let $0 \leq l \leq n$. Suppose $\lambda_N^{[l]}$ is a nonzero eigenvalue of k_l. Then for any $\phi \in E_N^{[l]}$,*

$$\|\phi\|_{H^{n-l}(I)} \leq C_{n,l} \left[\sum_{i=1}^{N} \left(\frac{1}{\lambda_i^{[l]}}\right)^2\right]^{1/2} \|\phi\|_{L^2(I)}. \tag{3.13}$$

Proof. Since $\{\phi_i^{[l]}\}_{i=1}^{N}$ constitute an $L^2(I)$-orthonormal basis for $E_N^{[l]}$, we have $\phi = \sum_{i=1}^{N} c_i \phi_i^{[l]}$ with $\|\phi\|_{L^2(I)}^2 = \sum_{i=1}^{N} |c_i|^2$. For $l \leq m \leq n$,

$$\begin{aligned}
\|\phi^{(m)}\|_{L^2(I)}^2 &= \left\|\sum_{i=1}^{N} c_i \phi_i^{[l](m)}\right\|_{L^2(I)}^2 \leq \left(\sum_{i=1}^{N} |c_i| \left\|\phi_i^{[l](m)}\right\|_{L^2(I)}\right)^2 \leq \left(\sum_{i=1}^{N} |c_i|^2\right)\left(\sum_{i=1}^{N} \left\|\phi_i^{[l](m)}\right\|_{L^2(I)}^2\right) \\
&\leq \|\phi\|_{L^2(I)}^2 \sum_{i=1}^{N} \left(\frac{1}{\lambda_i^{[l]}}\right)^2 \mathcal{K}_m \mathcal{K}_l.
\end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned}
\|\phi\|_{H^{n-l}(I)} &= \left(\sum_{m=l}^{n} \|\phi^{(m)}\|_{L^2(I)}^2\right)^{1/2} \leq \mathcal{K}_l^{1/2} \left(\sum_{m=l}^{n} \mathcal{K}_m\right)^{1/2} \left[\sum_{i=1}^{N} \left(\frac{1}{\lambda_i^{[l]}}\right)^2\right]^{1/2} \|\phi\|_{L^2(I)} \\
&= C_{n,l} \left[\sum_{i=1}^{N} \left(\frac{1}{\lambda_i^{[l]}}\right)^2\right]^{1/2} \|\phi\|_{L^2(I)}
\end{aligned} \tag{3.15}$$

as stated. □

Remark 3.7. The norm bound obtained in (3.7) cannot, in general, be improved. To show this let $I \subset \mathbb{R}$ and choose $\phi \in C^{n-l}(I) \cap H^{n-l}(I)$ with $\|\phi\|_{L^2(I)} = 1$ and $\phi(x) \to 0$ as $|x| \to \infty$

if I is unbounded. By Remark 3.2 these choices imply that $k_l(x,y) = \phi(x)\overline{\phi(y)}$ is a rank-1 positive definite kernel in class $\mathcal{A}_{n-l}(I)$ irrespective of whether I is bounded or not. In particular the only nonzero eigenvalue is $\lambda^{[l]} = 1$ and the corresponding normalized eigenvector is ϕ. Recalling the definition (3.5) of $\mathcal{K}_m$, we have in this case

$$\mathcal{K}_m = \int_I k_m(x,x)dx = \int_I \left|\phi^{(m-l)}(x)\right|^2 dx = \left\|\phi^{(m-l)}\right\|^2_{L^2(I)} \tag{3.16}$$

for $0 \le l \le m \le n$. By our choice of k_l we have $\mathcal{K}_l = \|\phi\|^2_{L^2(I)} = 1$ and, since $\lambda^{[l]} = 1$, we may write

$$\|\phi\|^2_{H^{n-l}} = \sum_{m=l}^{n} \left\|\phi^{(m-l)}\right\|^2_{L^2(I)} = \sum_{m=l}^{n} \mathcal{K}_m = \frac{\mathcal{K}_l}{\lambda^{[l]}} \sum_{m=l}^{n} \mathcal{K}_m, \tag{3.17}$$

and so in this case equality holds in (3.11). This shows that the bound in Theorem 3.5 is sharp and cannot be improved.

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