DIFFERENTIAL INEQUALITIES METHOD TO nTh-ORDER BOUNDARY VALUE PROBLEMS

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By the theory of differential inequality, bounding function method, and the theory of topological degree, this paper presents the existence criterions of solutions for the general nth-order differential equations under nonlinear boundary conditions, and extends many existing results.

1. Introduction

From Nagumo [10], there have been many accomplishments on the study of the existence of solutions for boundary value problems (BVPs) using the theory of differential inequality (cf. [1–9, 11–17]). However, for the nth-order nonlinear differential equations with the nonlinear boundary conditions, results are very few. The authors made some attempts to solve the nth-order Robin problem [14]. Now we are concerned with the nth-order nonlinear BVP:

$$\begin{gathered} y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right), \\ P_i\Big(y(a), y'(a), \ldots, y^{(n-1)}(a)\Big) = 0, \quad i = 1, \ldots, n-1, \\ P_n\Big(y(b), y'(b), \ldots, y^{(n-1)}(b)\Big) = 0, \end{gathered} \tag{1.1}$$

where $t \in I = [a,b]$, $f(t,\xi_0,\xi_1,\ldots,\xi_{n-1}) \in C(I \times \mathbb{R}^n,\mathbb{R})$, $P_i(\eta_0,\eta_1,\ldots,\eta_{n-1}) \in C(\mathbb{R}^n,\mathbb{R})$, $P_n(\zeta_0,\zeta_1,\ldots,\zeta_{n-1}) \in C(\mathbb{R}^n,\mathbb{R})$.

Our method is not only modifying the nonlinear function in the original equations, but also transforming the original nonlinear boundary conditions into some new boundary conditions which are easy to discuss. Thus, we get the new BVP which will be discussed firstly, then the judgement of the existence of solutions for the original BVP will be attained naturally. This technique dealing with the nonlinear problem is simpler and

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clearer compared with the method of shooting. However, it has scarcely been used in the available reference materials.

The paper is organized as follows. In Section 2, we give out some basic concepts and the preparative theorem. In Section 3, the main result is presented and proved. In Section 4, a more general BVP is studied. Finally, in Section 5, we use the results to solve an example which cannot be solved by [1–17].

2. Preparative theorem

2.1. Basic concepts. We first define a function

$$\delta(r,x,s) \equiv \begin{cases} r & \text{if } x < r, \\ x & \text{if } r \le x \le s, \\ s & \text{if } s < x, \end{cases} \tag{2.1}$$

where $r,x,s \in \mathbb{R}$, $r \le s$.

Definition 2.1. Assume that $\alpha(t),\beta(t) \in C^n(I,\mathbb{R})$. The pair of functions $(\alpha(t),\beta(t))$ is called a bounding function pair (or simply, a bounding pair) of BVP (1.1) in case there exists $N > 0$ such that for all $u(t) \in C^n(I,\mathbb{R})$:

(i) $\alpha^{(j)}(t) \le \beta^{(j)}(t)$, $t \in I$, $j = 0,1,\dots,n-2$;

(ii) $\alpha^{(n)}(t) \ge f(t,\overline{u}(t),\overline{u}'(t),\dots,\overline{u}^{(n-3)}(t),\alpha^{(n-2)}(t),\alpha^{(n-1)}(t))$, $\beta^{(n)}(t) \le f(t,\overline{u}(t),\overline{u}'(t),\dots,\overline{u}^{(n-3)}(t),\beta^{(n-2)}(t),\beta^{(n-1)}(t))$, where $\overline{u}^{(j)}(t) = \delta(\alpha^{(j)}(t),u^{(j)}(t),\beta^{(j)}(t))$, $j = 0,1,\dots,n-3$;

(iii) $P_i(\overline{u}(a),\dots,\overline{u}^{(i-2)}(a),\alpha^{(i-1)}(a),\alpha^{(i)}(a),\overline{u}^{(i+1)}(a),\dots,\overline{u}^{(n-1)}(a)) \le 0 \le P_i(\overline{u}(a),\dots,\overline{u}^{(i-2)}(a),\beta^{(i-1)}(a),\beta^{(i)}(a),\overline{u}^{(i+1)}(a),\dots,\overline{u}^{(n-1)}(a))$, $P_n(\overline{u}(b),\dots,\overline{u}^{(n-3)}(b),\alpha^{(n-2)}(b),\alpha^{(n-1)}(b)) \le 0 \le P_n(\overline{u}(b),\dots,\overline{u}^{(n-3)}(b),\beta^{(n-2)}(b),\beta^{(n-1)}(b))$, where $i = 1,2,\dots,n-1$, $\overline{u}^{(n-2)}(a) = \delta(\alpha^{(n-2)}(a),u^{(n-2)}(a),\beta^{(n-2)}(a))$, $\overline{u}^{(n-1)}(a) = \delta(-N,u^{(n-1)}(a),N)$.

Definition 2.2. A continuous function $f(t,\xi_0,\dots,\xi_{n-1})$ is said to satisfy a Nagumo condition with respect to variable ξ_{n-1} on the set

$$\mathcal{D} = \{(t,\xi_0,\dots,\xi_{n-1}) \mid t \in I; |\xi_j| \le r_j,\ j=0,1,\dots,n-2,\ r_j \text{ is a positive constant; } \xi_{n-1} \in R\} \tag{2.2}$$

in case there exists function $\Phi(t) \in C([0,+\infty],(0,+\infty))$, such that

$$\begin{gathered} |f(t,\xi_0,\dots,\xi_{n-1})| \le \Phi(|\xi_{n-1}|), \\ \int^{+\infty} \frac{s\,ds}{\Phi(s)} = +\infty. \end{gathered} \tag{2.3}$$

2.2. The modified problem. Assume that there are two functions $\alpha(t)$, $\beta(t)$ satisfying

$$\alpha^{(j)}(t) \le \beta^{(j)}(t), \quad j = 0,1,\dots,n-2. \tag{2.4}$$

We define function

$$\overline{f}(t,y,y',\dots,y^{(n-1)}) \equiv f(t,\overline{y},\overline{y}',\dots,\overline{y}^{(n-1)}) + h(y^{(n-2)}), \tag{2.5}$$

where $\overline{y}^{(j)}(t) = \delta(\alpha^{(j)}(t), y^{(j)}(t), \beta^{(j)}(t))$ $(j = 0,1,\dots,n-2)$ and $\overline{y}^{(n-1)}(t) = \delta(-N, y^{(n-1)}(t), N)$. N is a positive constant such that

$$N > \max_{t\in I}\left\{\frac{2M}{b-a}, |\alpha^{(n-1)}(t)|, |\beta^{(n-1)}(t)|\right\}, \tag{2.6}$$

$$\int_{2M/(b-a)}^{N} \frac{sds}{\Phi(s)} > 2M, \tag{2.7}$$

in which $M > \max_{t\in I}\{|\alpha^{(n-2)}(t)|, |\beta^{(n-2)}(t)|\}$. $h(y^{(n-2)})$ is continuous, bounded, and

$$h(y^{(n-2)}) \begin{cases} < 0 & \text{if } y^{(n-2)} < \alpha^{(n-2)}, \\ = 0 & \text{if } \alpha^{(n-2)} \le y^{(n-2)} \le \beta^{(n-2)}, \\ > 0 & \text{if } y^{(n-2)} > \beta^{(n-2)}. \end{cases} \tag{2.8}$$

Such function $h(\cdot)$ is easy to obtain, for example, let

$$h(y^{(n-2)}) \equiv \frac{y^{(n-2)} - \overline{y}^{(n-2)}}{1 + |y^{(n-2)} - \overline{y}^{(n-2)}|}. \tag{2.9}$$

In addition, we define

$$\begin{aligned} &\overline{P}_i\left(y(t), y'(t),\dots, y^{(n-1)}(t)\right) \\ &\quad\equiv \delta\left(\alpha^{(i-1)}(t), y^{(i-1)}(t) - P_i\left(y(t), y'(t),\dots, y^{(n-1)}(t)\right), \beta^{(i-1)}(t)\right), \quad i = 1,2,\dots,n-1, \\ &\overline{P}_n\left(y(t), y'(t),\dots, y^{(n-1)}(t)\right) \\ &\quad\equiv \delta\left(\alpha^{(n-2)}(t), y^{(n-2)}(t) - P_n\left(y(t), y'(t),\dots, y^{(n-1)}(t)\right), \beta^{(n-2)}(t)\right). \end{aligned} \tag{2.10}$$

Then we consider the following modified problem:

$$\begin{aligned} &y^{(n)} = \overline{f}\left(t, y, y',\dots, y^{(n-1)}\right), \\ &y^{(i-1)}(a) = \overline{P}_i\left(y(a), y'(a),\dots, y^{(n-1)}(a)\right), \quad i = 1,\dots,n-1, \\ &y^{(n-2)}(b) = \overline{P}_n\left(y(b), y'(b),\dots, y^{(n-1)}(b)\right). \end{aligned} \tag{2.11}$$

2.3. Preparative theorem.

LEMMA 2.3. *Assume that*

(A1) *BVP (1.1) has a bounding pair* $(\alpha(t),\beta(t))$ *on the interval* I *by Definition 2.1;*

(A2) *the function* $f(t,y,y',\dots,y^{(n-1)})$ *in BVP (1.1) satisfies the Nagumo condition with respect to* $y^{(n-1)}(t)$ *by Definition 2.2.*

Then BVP (2.11) has a solution $y(t)\in C^n(I,\mathbb{R})$ *such that*

$$\begin{gathered}\alpha^{(i)}(t)\le y^{(i)}(t)\le\beta^{(i)}(t),\quad i=0,1,\dots,n-2,\\ \left|y^{(n-1)}(t)\right|\le N,\quad t\in I,\end{gathered}\tag{2.12}$$

where N *is the positive constant given in the definition of* $\overline{f}$.

The proof of Lemma 2.3 is a simple consequence of the following three propositions.

PROPOSITION 2.4. *The modified BVP (2.11) has a solution* $y(t)\in C^n(I,\mathbb{R})$.

Proof. Consider

$$\begin{aligned}&y^{(n)}=\lambda\overline{f}\left(t,y,\dots,y^{(n-1)}\right)\equiv g(t),\\ &y^{(i-1)}(a)=\lambda\overline{P}_i\left(y(a),\dots,y^{(n-1)}(a)\right)\equiv g_i(a),\qquad i=1,2,\dots,n-1,\\ &y^{(n-2)}(b)=\lambda\overline{P}_n\left(y(b),\dots,y^{(n-1)}(b)\right)\equiv g_n(b),\end{aligned}\tag{2.13}$$

where $\lambda\in[0,1]$. From the representations of $\overline{f}$, $\overline{P}_i$, and $\overline{P}_n$, we know that $y^{(n)}(t)$, $y^{(i-1)}(a)$ ($i=1,2,\dots,n-1$), and $y^{(n-2)}(b)$ all are bounded. Also, by the mean value theorem, we may ensure that $y^{(n-1)}(t),\dots,y'(t),y(t)$ all are bounded functions in I. In fact, by the mean value theorem, there exists some $\xi\in(a,b)$ satisfying

$$y^{(n-2)}(b)-y^{(n-2)}(a)=y^{(n-1)}(\xi)(b-a),\tag{2.14}$$

then $y^{(n-1)}(\xi)$ is bounded. From

$$y^{(n-1)}(t)-y^{(n-1)}(\xi)=y^{(n)}(\eta)(t-\xi)\quad\forall t\in[a,b],\tag{2.15}$$

$y^{(n-1)}(t)$ is bounded. Thus, from

$$y^{(i)}(t)-y^{(i)}(a)=y^{(i+1)}(\zeta)(t-a),\quad \eta\in(a,b),\ i=0,1,\dots,n-2,\tag{2.16}$$

it is easy to see that $y^{(n-2)}(t),\dots,y'(t),y(t)$ all are bounded in I.

Let $\Omega=\{y(t)\in C^n(I,\mathbb{R})\mid \|y^{(i)}(t)\|<K,$ for all $t\in I, i=0,1,\dots,n-1, K$ is some sufficiently large positive constant$\}$. Then Ω is a bounded open set. BVP (2.13) can be equivalently written as the following integral equation:

$$y(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1}+\int_a^t\int_a^{t_{n-1}}\cdots\int_b^{t_1}g(s)ds\,dt_1\cdots dt_{n-1}\equiv T_\lambda y,\tag{2.17}$$

where T_λ is an integral operator with a parameter λ and $(c_1,\dots,c_n)$ is determined by the system of equations

$$\begin{gathered}
c_1+c_2a+c_3a^2+\cdots+c_na^{n-1}=g_1(a),\\
c_2+c_3\cdot 2a+\cdots+c_n(n-1)a^{n-2}=g_2(a),\\
\vdots\\
c_{n-1}(n-2)(n-3)\cdots 3+c_n(n-1)!a=g_{n-1}(a),\\
c_{n-1}(n-2)(n-3)\cdots 3+c_n(n-1)!b=g_n(b)-\int_a^b\int_b^{t_1}g(s)ds\,dt_1.
\end{gathered}\tag{2.18}$$

Let $H(\lambda,y)=(I-T_\lambda)(y)$, then $H:[0,1]\times\overline{\Omega}\to\mathbb{R}^n$ is continuous, where I is identity mapping. Let $h_\lambda(y)=H(\lambda,y)$, then $0\notin h_\lambda(\partial\Omega)$. In fact, for all $y\in\partial\Omega$, $\| y \|\geq K$. Noticing that K is sufficiently large, we have

$$\| h_\lambda(y)\| =\| y-T_\lambda y\|\geq\| y\|-\| T_\lambda y\|\geq K-\| T_\lambda y\|>0\quad\forall\lambda\in[0,1].\tag{2.19}$$

Thus, $0\notin h_\lambda(\partial\Omega)$. By the homotopy invariance theorem of topological degree, $\deg(h_\lambda,\Omega,0)$ is a constant, in particular, $\deg(h_1,\Omega,0)=\deg(h_0,\Omega,0)$. Noticing that $0\in\Omega$, by the normality of topological degree, we have

$$\deg(h_1,\Omega,0)=\deg(h_0,\Omega,0)=\deg(I-T_0,\Omega,0)=\deg(I,\Omega,0)=1.\tag{2.20}$$

Hence, by the solvability theorem of topological degree, it is clear that there exists some $y(t)$ satisfying (2.17), then this proposition is proved. □

Proposition 2.5. *Every solution $y(t)$ of the modified BVP (2.11) satisfies*

$$\alpha^{(i)}(t)\leq y^{(i)}(t)\leq\beta^{(i)}(t),\quad t\in I,\ i=0,1,\dots,n-2.\tag{2.21}$$

Proof. First, we show that

$$\alpha^{(n-2)}(t)\leq y^{(n-2)}(t)\leq\beta^{(n-2)}(t),\quad t\in I.\tag{2.22}$$

If $\alpha^{(n-2)}(t)\leq y^{(n-2)}(t)$ is not true, then there exists some $\xi\in[a,b]$, such that

$$\max_{t\in I}\left(\alpha^{(n-2)}(t)-y^{(n-2)}(t)\right)=\alpha^{(n-2)}(\xi)-y^{(n-2)}(\xi)>0.\tag{2.23}$$

Then $\xi\neq a,b$ by the boundary conditions of BVP (2.11). Thus

$$\alpha^{(n-1)}(\xi)-y^{(n-1)}(\xi)=0,\tag{2.24}$$

$$\alpha^{(n)}(\xi)-y^{(n)}(\xi)\leq 0.\tag{2.25}$$

However, on the other hand, from the definition of $\alpha(t)$ and that $y(t)$ is a solution of (2.11), we have

$$\begin{aligned}\alpha^{(n)}(\xi)-y^{(n)}(\xi) &\geq f\left(\xi,\overline{y}(\xi),\dots,\overline{y}^{(n-3)}(\xi),\alpha^{(n-2)}(\xi),\alpha^{(n-1)}(\xi)\right)\\ &\qquad - f\left(\xi,\overline{y}(\xi),\dots,\overline{y}^{(n-3)}(\xi),\overline{y}^{(n-2)}(\xi),\overline{y}^{(n-1)}(\xi)\right)-h\left(y^{(n-2)}(\xi)\right)\\ &= -h\left(y^{(n-2)}(\xi)\right)>0.\end{aligned} \tag{2.26}$$

This contradicts (2.25). Hence,

$$\alpha^{(n-2)}(t)\leq y^{(n-2)}(t),\quad t\in I. \tag{2.27}$$

A similar proof shows that

$$y^{(n-2)}(t)\leq \beta^{(n-2)}(t),\quad t\in I. \tag{2.28}$$

To sum up, (2.22) is true. From (2.22), the function $y^{(n-3)}(t)-\alpha^{(n-3)}(t)$ is increasing in I. Noticing

$$\alpha^{(n-3)}(a)\leq y^{(n-3)}(a), \tag{2.29}$$

we know that $\alpha^{(n-3)}(t)\leq y^{(n-3)}(t)$. A similar proof shows $y^{(n-3)}(t)\leq\beta^{(n-3)}(t)$. Using the same argument, it follows that $\alpha^{(i)}(t)\leq y^{(i)}(t)\leq\beta^{(i)}(t)$, $i=n-4,n-5,\dots,2,1$. Thus, the proof of Proposition 2.5 is completed. □

Proposition 2.6. *For every solution $y(t)$ of the modified BVP (2.11) holds*

$$\left|y^{(n-1)}(t)\right|\leq N,\quad t\in I. \tag{2.30}$$

Proof. Suppose that there exists some $\tau\in[a,b]$ such that

$$\left|y^{(n-1)}(\tau)\right|>N. \tag{2.31}$$

Without loss of generality, we assume that $y^{(n-1)}(\tau)>N$. There exists $\xi\in(a,b)$, such that

$$y^{(n-1)}(\xi)=\frac{y^{(n-2)}(b)-y^{(n-2)}(a)}{b-a}\leq\frac{2M}{b-a}<N. \tag{2.32}$$

Hence, there exists some subinterval $[c,d]$ (or $[d,c]$) $\subset[a,b]$ such that

$$\begin{gathered}y^{(n-1)}(c)=\frac{2M}{b-a},\qquad y^{(n-1)}(d)=N,\\ \frac{2M}{b-a}\leq y^{(n-1)}(t)\leq N,\quad\forall t\in[c,d]\ (\text{or } [d,c]).\end{gathered} \tag{2.33}$$

From condition (A2),

$$\left|\int_c^d \frac{y^{(n-1)}(s)y^{(n)}(s)}{\Phi(|y^{(n-1)}(s)|)}ds\right| \le \left|\int_c^d y^{(n-1)}(s)ds\right| = \left|y^{(n-2)}(d)-y^{(n-2)}(c)\right| \le 2M. \tag{2.34}$$

On the other hand, from (2.7) we know that

$$\left|\int_c^d \frac{y^{(n-1)}(s)y^{(n)}(s)}{\Phi(|y^{(n-1)}(s)|)}ds\right| = \left|\int_{2M/(b-a)}^N \frac{rdr}{\Phi(r)}\right| = \int_{2M/(b-a)}^N \frac{rdr}{\Phi(r)} > 2M. \tag{2.35}$$

This inequality contradicts the above one and Proposition 2.6 holds. □

3. Main theorem

Now, the main result of this paper is given in the following theorem.

Theorem 3.1. *Assume that the conditions (A1), (A2) in Lemma 2.3 hold and added to (A3). The function $P_i(\eta_0,\dots,\eta_{n-1})$ $(i=1,2,\dots,n)$ satisfies*

(i) *$P_i(\eta_0,\dots,\eta_{n-1})$ is increasing in η_{i-1} and decreasing in η_i, $i=1,2,\dots,n-2$;*

(ii) *$P_{n-1}(\eta_0,\dots,\eta_{n-1})$ is decreasing in η_{n-1};*

(iii) *$P_n(\eta_0,\dots,\eta_{n-1})$ is increasing in η_{n-1}.*

Then BVP (1.1) has a solution $y(t)\in C^n(I,\mathbb{R})$ such that

$$\begin{gathered}\alpha^{(i)}(t) \le y^{(i)}(t) \le \beta^{(i)}(t), \quad i=0,1,\dots,n-2,\\ |y^{(n-1)}(t)| \le N, \quad t\in I,\end{gathered} \tag{3.1}$$

where N is the positive constant given in the definition of $\overline{f}$.

Proof. From Lemma 2.3 and the definition of $\overline{f}$, the solution $y(t)$ of the modified BVP (2.11) satisfies (1.1). As soon as it is proved that $y(t)$ satisfies the boundary conditions of (1.1) under condition (A3), we may say that $y(t)$ is a solution of BVP (1.1).

First, we prove

$$P_i\left(y(a),\dots,y^{(n-1)}(a)\right) = 0, \quad i=1,2,\dots,n-2. \tag{3.2}$$

Case 1. Suppose that

$$\alpha^{(i-1)}(a) \le y^{(i-1)}(a) - P_i\left(y(a),\dots,y^{(n-1)}(a)\right) \le \beta^{(i-1)}(a). \tag{3.3}$$

Then

$$y^{(i-1)}(a) = \overline{P}_i\left(y(a),\dots,y^{(n-1)}(a)\right) = y^{(i-1)}(a) - P_i\left(y(a),\dots,y^{(n-1)}(a)\right). \tag{3.4}$$

Thus

$$P_i\left(y(a),\dots,y^{(n-1)}(a)\right) = 0. \tag{3.5}$$

Case 2. Suppose that there exists some $i \in \{1,2,\dots,n-2\}$ such that

$$\alpha^{(i-1)}(a) > y^{(i-1)}(a) - P_i\Big(y(a),y'(a),\dots,y^{(n-1)}(a)\Big). \tag{3.6}$$

Then

$$y^{(i-1)}(a) = \overline{P}_i\Big(y(a),y'(a),\dots,y^{(n-1)}(a)\Big) = \alpha^{(i-1)}(a). \tag{3.7}$$

Hence

$$P_i\Big(y(a),y'(a),\dots,y^{(n-1)}(a)\Big) > 0. \tag{3.8}$$

From Propositions 2.5 and 2.6 and condition (A3),

$$P_i\Big(\overline{y}(a),\dots,\overline{y}^{(i-2)}(a),\alpha^{(i-1)}(a),\alpha^{(i)}(a),\overline{y}^{(i+1)}(a),\dots,\overline{y}^{(n-1)}(a)\Big) > 0. \tag{3.9}$$

It is easy to see that the last inequality contradicts Definition 2.1(iii). Therefore, Case 2 is not true.
Case 3. Suppose that there exists some $i \in \{1,2,\dots,n-2\}$ such that

$$y^{(i-1)}(a) - P_i\Big(y(a),y'(a),\dots,y^{(n-1)}(a)\Big) > \beta^{(i-1)}(a). \tag{3.10}$$

Then by the analogous analysis, we have

$$P_i\Big(\overline{y}(a),\dots,\overline{y}^{(i-2)}(a),\beta^{(i-1)}(a),\beta^{(i)}(a),\overline{y}^{(i+1)}(a),\dots,\overline{y}^{(n-1)}(a)\Big) \le P_i\Big(y(a),\dots,y^{(n-1)}(a)\Big) < 0. \tag{3.11}$$

Obviously, the last inequality contradicts Definition 2.1(iii). Therefore, this case cannot hold.

To sum up, (3.2) holds.

A similar proof shows that

$$P_{n-1}\Big(y(a),y'(a),\dots,y^{(n-1)}(a)\Big) = 0, \qquad P_n\Big(y(b),y'(b),\dots,y^{(n-1)}(b)\Big) = 0. \tag{3.12}$$

The proof is completed. □

4. A generalized problem

Now, we consider the following boundary value problem with more general boundary conditions:

$$y^{(n)} = f\Big(t,y,\dots,y^{(n-1)}\Big), \qquad P_i\Big(y(a),\dots,y^{(n-1)}(a),y(b),\dots,y^{(n-1)}(b)\Big) = 0, \tag{4.1}$$

where $t \in I$, $i = 1,2,\dots,n$, f and P_i are continuous functions.

Similarly to Definition 2.1, we give the following.

Definition 4.1. Assume $\alpha(t),\beta(t) \in C^n(I,\mathbb{R})$. The pair of functions $(\alpha(t),\beta(t))$ is called a bounding function pair of BVP (4.1) in case that for all $u(t) \in C^n(I,\mathbb{R})$

(i) the same as Definition 2.1(i);

(ii) the same as Definition 2.1(ii);

(iii)′

$$\begin{aligned}
&P_i\Big(\overline{u}(a),\dots,\alpha^{(i-1)}(a),\alpha^{(i)}(a),\dots,\overline{u}^{(n-1)}(a),\overline{u}(b),\dots,\overline{u}^{(n-1)}(b)\Big)\\
&\qquad \le 0 \le P_i\Big(\overline{u}(a),\dots,\beta^{(i-1)}(a),\beta^{(i)}(a),\dots,\overline{u}^{(n-1)}(a),\overline{u}(b),\dots,\overline{u}^{(n-1)}(b)\Big),\\
&P_n\Big(\overline{u}(a),\dots,\overline{u}^{(n-1)}(a),\overline{u}(b),\dots,\overline{u}^{(n-3)}(b),\alpha^{(n-2)}(b),\alpha^{(n-1)}(b)\Big)\\
&\qquad \le 0 \le P_n\Big(\overline{u}(a),\dots,\overline{u}^{(n-1)}(a),\overline{u}(b),\dots,\overline{u}^{(n-3)}(b),\beta^{(n-2)}(b),\beta^{(n-1)}(b)\Big),
\end{aligned} \tag{4.2}$$

where $i = 1,2,\dots,n-1$.

For BVP (4.1), we have the following existence theorem.

Theorem 4.2. *Assume that*

(A1)′ *BVP (4.1) has a bounding function pair $(\alpha(t),\beta(t))$ in the interval I by Definition 4.1;*

(A2)′ *the function $f(t,y,y',\dots,y^{(n-1)})$ in BVP (4.1) satisfies the Nagumo condition with respect to $y^{(n-1)}(t)$ by Definition 2.2;*

(A3)′ *the function $P_i(\eta_0,\dots,\eta_{n-1},\zeta_0,\dots,\zeta_{n-1})$ $(i = 1,2,\dots,n)$ satisfies*

  (i) *$P_i(\eta_0,\dots,\eta_{n-1},\zeta_0,\dots,\zeta_{n-1})$ is increasing in η_{i-1} and decreasing in η_i, $i = 1,2,\dots, n-2$;*

  (ii) *$P_{n-1}(\eta_0,\dots,\eta_{n-1},\zeta_0,\dots,\zeta_{n-1})$ is decreasing in η_{n-1};*

  (iii) *$P_n(\eta_0,\dots,\eta_{n-1},\zeta_0,\dots,\zeta_{n-1})$ is increasing in ζ_{n-1}.*

Then BVP (4.1) has a solution $y(t) \in C^n(I,\mathbb{R})$ such that

$$\begin{gathered}
\alpha^{(i)}(t) \le y^{(i)}(t) \le \beta^{(i)}(t), \quad i = 0,1,\dots,n-2,\\
\big|y^{(n-1)}(t)\big| \le N, \quad t \in I,
\end{gathered} \tag{4.3}$$

where N is the positive constant given in the definition of $\overline{f}$.

Proof. Consider the modified problem

$$\begin{gathered}
y^{(n)} = \overline{f}(t,y,\dots,y^{(n-1)}), \qquad y^{(i-1)}(a) = \overline{P}_i(a), \qquad y^{(n-2)}(b) = \overline{P}_n(b),\\
i = 1,2,\dots,n-1.
\end{gathered} \tag{4.4}$$

The modified function $\overline{f}(t,y,\dots,y^{(n-1)})$ is defined as BVP (2.11), and

$$\begin{aligned}
\overline{P}_i(t) &\equiv \overline{P}_i\Big(y(t),\dots,y^{(n-1)}(t),y(b+a-t),\dots,y^{(n-1)}(b+a-t)\Big)\\
&\equiv \delta\Big(\alpha^{(i-1)}(t),y^{(i-1)}(t) - P_i\Big(y(t),\dots,y^{(n-1)}(t),\\
&\qquad y(b+a-t),\dots,y^{(n-1)}(b+a-t)\Big),\beta^{(i-1)}(t)\Big),
\end{aligned} \tag{4.5}$$

where $i = 1,2,\dots,n-1$,

$$\begin{aligned}\overline{P}_n(t) &\equiv \overline{P}_n\Big(y(b+a-t),\dots,y^{(n-1)}(b+a-t),y(t),\dots,y^{(n-1)}(t)\Big)\\ &\equiv \delta\Big(\alpha^{(n-2)}(t),y^{(n-2)}(t)-P_n\Big(y(b+a-t),\dots,y^{(n-1)}(b+a-t),\\ &\qquad y(t),\dots,y^{(n-1)}(t)\Big),\beta^{(n-2)}(t)\Big).\end{aligned} \tag{4.6}$$

Using the same argument as the proof of Lemma 2.3, it follows that under the conditions (A1)′ and (A2)′, BVP (4.4) has a solution $y(t)$ satisfying the two inequalities in the conclusions of Lemma 2.3. Furthermore, in an analogous way to the proof of Theorem 3.1, it follows that the solution $y(t)$ of BVP (4.4) is a solution of BVP (4.1). Consequently, the proof of Theorem 4.2 is completed. The details of the proof will be omitted. □

5. An example

In this section, we study an example by making use of Theorems 3.1 and 4.2.

Example 5.1. Consider the 4th-order nonlinear boundary value problem

$$\begin{gathered}y^{(iv)} = (t-y)^2 - t(1+t^2)y' + \frac{112}{\sin 2}(1+(y')^2)\sin(y'') + (t+t^2)^2(1+(y''')^2),\\ 4y(1) - \frac{1}{8}(y'(1))^3 - y''(1) + \frac{k}{6}y(2) = A,\\ 5y'(1) - \frac{1}{2}y''(1) + \frac{k}{8}(y'(2))^2 = B,\\ y(1) + 2y''(1) - y'''(1) - \frac{k}{2}y''(2) = C,\\ ky(1) - y'(2) - 4(y''(2))^2 + 4(y'''(2))^3 = D,\end{gathered} \tag{5.1}$$

where $t \in [1,2]$, k is a constant.

Let

$$\begin{gathered}f(t,\xi_0,\xi_1,\xi_2,\xi_3) = (t-\xi_0)^2 - t(1+t^2)\xi_1 + \frac{112}{\sin 2}(1+\xi_1^2)\sin\xi_2 + (t+t^2)^2(1+\xi_3^2),\\ P_1(\eta_0,\eta_1,\eta_2,\eta_3,\zeta_0,\zeta_1,\zeta_2,\zeta_3) = 4\eta_0 - \frac{1}{8}\eta_1^3 - \eta_2 + \frac{k}{6}\zeta_0 - A,\\ P_2(\eta_0,\eta_1,\eta_2,\eta_3,\zeta_0,\zeta_1,\zeta_2,\zeta_3) = 5\eta_1 - \frac{1}{2}\eta_2 + \frac{k}{8}\zeta_1^2 - B,\\ P_3(\eta_0,\eta_1,\eta_2,\eta_3,\zeta_0,\zeta_1,\zeta_2,\zeta_3) = \eta_0 + 2\eta_2 - \eta_3 - \frac{k}{2}\zeta_2 - C,\\ P_4(\eta_0,\eta_1,\eta_2,\eta_3,\zeta_0,\zeta_1,\zeta_2,\zeta_3) = k\eta_0 - \zeta_1 - 4\zeta_2^2 + 4\zeta_3^3 - D.\end{gathered} \tag{5.2}$$

Let

$$\alpha(t) = -t^2, \qquad \beta(t) = t. \tag{5.3}$$

Then, for the case of $k = 0$, $A \in [-1, 31/8]$, $B \in [-9, 5]$, $C \in [-3, -1]$, $D \in [-12, -1]$, and the case of $k = 1$, $A \in [-2/3, 77/24]$, $B \in [-7, 5]$, $C \in [-2, -1]$, $D \in [-11, -2]$, it is easy to prove that $(\alpha(t), \beta(t))$ is a bounding pair of BVP (5.1) and all assumptions of Theorems 3.1 and 4.2 are fulfilled, respectively. Hence, for any of the two cases, BVP (5.1) has at least one solution $y(t)$ satisfying

$$-t^2 \le y(t) \le t, \quad -2t \le y'(t) \le 1, \quad -2 \le y''(t) \le 0, \quad t \in [1, 2]. \tag{5.4}$$

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