MARKOV AND BERNSTEIN INEQUALITIES IN L^P FOR SOME WEIGHTED ALGEBRAIC AND TRIGONOMETRIC POLYNOMIALS

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Bernstein inequalities are given for polynomials of degree at most $2m$ (where $m \le n$), weighted by $(1+x^2)^{-n}$, in L^p norms on $(-\infty,\infty)$, and also in related spaces of weighted trigonometric polynomials. Also, Bernstein and Markov inequalities valid on $[0,\infty)$ are derived for polynomials of degree m weighted by $(1+t)^{-n}$.

1. Introduction

Let $\mathbf{Q}_{m,n}$ (with $m \le n$) denote the space of polynomials of degree $2m$ or less on $(-\infty,\infty)$, weighted by $(1+x^2)^{-n}$. The elements $\mathbf{Q}_{m,n}$ are thus rational functions with denominator $(1+x^2)^m$ and numerator of degree at most $2m$ (if $m = n$, we can write, more briefly, $\mathbf{Q}_n$ for $\mathbf{Q}_{n,n}$). The spaces $\mathbf{Q}_{m,n}$ form a nested sequence as n increases and $r = n - m$ is held to some given value of weighted polynomial spaces, with the weight depending upon n. As these spaces can obviously be used for approximation on the real line, their approximation-theoretic properties are worthy of a systematic investigation, of which this article is a part.

Briefly describing the previous work, the properties of Lagrange interpolation in these spaces were investigated in Kilgore [2]. The main result there was that the Bernstein-Erdös conditions characterize interpolation of minimal norm into these spaces, as was already known for spaces of ordinarily polynomials, for trigonometric polynomials, and for several other classes of polynomial spaces with weighted norm.

Inside of any of these rational function spaces $\mathbf{Q}_{m,n}$, there is the subspace of even functions. That subspace is isometrically isomorphic to a space defined upon the half-line $[0,\infty)$, which has been denoted by $\mathbf{R}_{m,n}$ in Kilgore [3]. Specifically, a typical function in $\mathbf{R}_{m,n}$ is a rational function with denominator $(1+t)^n$ and numerator $P_m(t)$, where P_m is a polynomial of degree at most m. The natural isometry between $\mathbf{R}_{m,n}$ and the even part of $\mathbf{Q}_{m,n}$ is induced by $t \leftrightarrow x^2$. This space had not been discussed in Kilgore [2], since the corresponding results about interpolation in the spaces $\mathbf{R}_{m,n}$ follow as a special case from Kilgore [1].

In the article [3], analogues of the Markov and Bernstein inequalities were shown to hold, both in the spaces $\mathbf{Q}_{m,n}$ and in the spaces $\mathbf{R}_{m,n}$, under the uniform norm. Here, we show Markov and Bernstein inequalities in L^p norms on the same spaces.

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It was also noted in Kilgore [2] that $\mathbf{Q}_n$ is isometrically isomorphic to $\mathbf{T}_n$, the space of trigonometric polynomials of degree at most n, via the mapping $x \leftrightarrow \tan\theta/2$. By the same mapping, there is an isometric isomorphism between the spaces $\mathbf{Q}_{m,n}$ and the weighted spaces of trigonometric polynomials $\mathbf{T}_{m,n}$, where $\mathbf{T}_{m,n}$ consists of the space $\mathbf{T}_m$ with weight $\cos^{2r}\theta/2$, where $r = n - m$ as previously mentioned. Thus, in [2] similar results for interpolation were shown for these weighted trigonometric polynomial spaces, too, and in [3] a weighted Bernstein inequality with uniform norm was shown to hold. Using the same induced isometry, we will also give a Bernstein inequality in $\mathbf{T}_{m,n}$ in L^p norm.

2. Norms

For $1 < p < \infty$, the p-norm used for the space $\mathbf{T}_{m,n}$ will be the usual, unweighted L^p norm

$$\|T_{m,n}\|_p = \left(\int_{-\pi}^{\pi} |T_{m,n}(\theta)|^p d\theta\right)^{1/p}. \tag{2.1}$$

Every function $T_{m,n}$ in $\mathbf{T}_{m,n}$ is a trigonometric polynomial of degree at most n, which can be written in the form $T_m(\theta)\cos^{2r}\theta/2$, where T_m is a trigonometric polynomial of degree at most m and $m + r = n$. Thus, this norm can equally be viewed as a weighted norm of the polynomial T_m, with weight function $\cos^{2r}\theta/2$.

The norm in $\mathbf{Q}_{m,n}$ which naturally corresponds to the norm (2.1) in $\mathbf{T}_{m,n}$ may be constructed via the mapping $x \leftrightarrow \tan\theta/2$. First, every function in $\mathbf{Q}_{m,n}$ may be represented as the product of a rational function

$$Q_m(x) = P_{2m}(x)(1 + x^2)^{-m}, \tag{2.2}$$

in which P_{2m} is an algebraic polynomial of degree at most m, with the additional factor $(1 + x^2)^{-r}$, in which $m + r = n$, as before. That is, a typical element f in $\mathbf{Q}_{m,n}$ is representable in the form

$$f(x) = Q_m(x)(1 + x^2)^{-r}, \tag{2.3}$$

and, via the transformation $x \leftrightarrow \tan\theta/2$, we have a correspondence

$$Q_m(x)(1 + x^2)^{-r} = T_m(\theta)\cos^{-2r}\frac{\theta}{2}. \tag{2.4}$$

The usual rules of substitution in integrals now give us the norm in $\mathbf{Q}_{m,n}$ which naturally corresponds to the norm (2.1). Specifically, for $f \in \mathbf{Q}_{m,n}$, we define

$$\|f\|_{p,w} = \left(\int_{-\infty}^{\infty} |f(x)|^p \frac{2}{(1 + x^2)} dx\right)^{1/p}. \tag{2.5}$$

This in other words is a weighted norm, with weight function $w(x) = 2/(1 + x^2)$. Similar considerations come into play for definition of the appropriate p-norm in $\mathbf{R}_{m,n}$. A typical function f in $\mathbf{R}_{m,n}$ may be identified with an even function g in $\mathbf{Q}_{m,n}$, by $f(t) = f(x^2) = g(x)$, with $t \leftrightarrow x^2$, $x \geq 0$. Therefore, for $f \in \mathbf{R}_{m,n}$, we define

$$\|f\|_{p,W} = \left(\int_0^\infty |f(t)|^p \frac{1}{(1+t)\sqrt{t}}dt\right)^{1/p}. \tag{2.6}$$

This is a weighted norm, too, with weight function $W(t) = 1/(1+t)\sqrt{t}$.

3. Results

We show a form of the Bernstein inequality for trigonometric polynomials in $\mathbf{T}_{m,n}$ and, respectively, for the rational functions in $\mathbf{Q}_{m,n}$. Specifically, we have the following.

Theorem 3.1. *Let T_m be a trigonometric polynomial of degree at most m. Then for a fixed nonnegative integer r, we have in L^p norm, for $1 < p < \infty$,*

$$\left\|T'_m(\theta)\cos^{2r}\frac{\theta}{2}\right\|_p \leq \left(1 + \frac{2pr}{p-1}\right)(m+r)\left\|T_m(\theta)\cos^{2r}\frac{\theta}{2}\right\|_p. \tag{3.1}$$

Theorem 3.2. *Let Q_m be a rational function of the form*

$$Q_m(x) = \frac{P(x)}{(1+x^2)^m}, \tag{3.2}$$

where P is a polynomial of degree at most $2m$. Then for any integer $r \geq -1$,

$$||Q'_m(x)(1+x^2)^{-r}||_{p,w} \leq 2\left(1 + \frac{2p(r+1)}{p-1}\right)(m+r+1)||Q_m(x)(1+x^2)^{-r-1}||_{p,w}. \tag{3.3}$$

For the spaces $\mathbf{R}_{m,n}$, the result corresponding to Theorem 3.2 is the following.

Theorem 3.3. *Let R_m be a rational function of the form*

$$R_m(t) = \frac{P(t)}{(1+t)^m}, \tag{3.4}$$

where P is a polynomial of degree at most m, and let $1 < p < \infty$. Then for any integer $r \geq -1$,

$$||\sqrt{t}R'_m(t)(1+t)^{-r}||_{p,W} \leq \left(1 + \frac{2p(r+1)}{p-1}\right)(m+r+1)||R_m(t)(1+t)^{-r-1}||_{p,W}. \tag{3.5}$$

There is also a result of Markov type for the spaces $\mathbf{R}_{m,n}$, which eliminates the factor of $\sqrt{t}$ on the left and gives a global weighted estimate instead.

Theorem 3.4. *Let R_m be a rational function of the form*

$$R_m(t) = \frac{P(t)}{(1+t)^m}, \tag{3.6}$$

where P is a polynomial of degree at most m and $1 < p < \infty$. Then for any integer $r \geq -1$,

$$||(1+t)^{-r}R'_m(t)||_{p,W} \leq \left(1+\frac{2p(r+1)}{p-1}\right)\left(\frac{2p}{p-1}\right)(m+r+1)^2||(1+t)^{-r-1}R_m(t)||_{p,W}. \tag{3.7}$$

Remark 3.5. As $p \to \infty$, the constants obtained in Theorems 3.1 through 3.4, which depend upon p, converge to the constants obtained in [3] for the corresponding uniform norm inequalities.

4. Proof of Theorem 3.1

A typical element of $\mathbf{T}_{m,n}$ may be represented as $T_m(\theta)\cos^{2r}\theta/2$, which is a trigonometric polynomial of degree at most $n = m + r$.

Bernstein's inequality in L^p for trigonometric polynomials states that for any trigonometric polynomial T_N of degree at most N,

$$||T'_N||_p \leq N||T_N||_p. \tag{4.1}$$

The inequality follows in L^p as well as in uniform norm from the trigonometric identity of Riesz [4].

Applying (4.1) in the present situation, we have

$$\left|\left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|\right|_p \leq (m+r)\left|\left|T_m(\theta)\cos^{2r}\frac{\theta}{2}\right|\right|_p. \tag{4.2}$$

An explicit calculation gives

$$\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)' = T'_m(\theta)\cos^{2r}\frac{\theta}{2} - rT_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}, \tag{4.3}$$

whence

$$\left|\left|T'_m(\theta)\cos^{2r}\frac{\theta}{2}\right|\right|_p \leq \left|\left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|\right|_p + r\left|\left|T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right|\right|_p. \tag{4.4}$$

The first term on the right in (4.4) may be estimated using (4.2). Estimation of the second term requires the use of Hardy's inequality (cf. Zygmund [5, Chapter 1, (9.17)]). We have first

$$\begin{aligned}
&\int_0^\pi \left| T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right|^p d\theta \\
&\quad\le \int_0^\pi \left| \frac{T_m(\theta)\cos^{2r}\theta/2}{\cos\theta/2}\sin\frac{\theta}{2}\right|^p d\theta \\
&\quad\le \int_0^\pi \left| \frac{T_m(\theta)\cos^{2r}\theta/2}{\theta-\pi}\left(\frac{\theta-\pi}{\cos\theta/2}\right)\sin\frac{\theta}{2}\right|^p d\theta \\
&\quad\le \left\|\left(\frac{\theta-\pi}{\cos\theta/2}\right)\sin\frac{\theta}{2}\right\|_\infty^p \int_0^\pi \left| \frac{T_m(\theta)\cos^{2r}\theta/2}{\theta-\pi}\right|^p d\theta \\
&\quad\le \left(\frac{p}{p-1}\right)^p \left\|\left(\frac{\theta-\pi}{\cos\theta/2}\right)\sin\frac{\theta}{2}\right\|_\infty^p \int_0^\pi \left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|^p d\theta.
\end{aligned}\tag{4.5}$$

Noting now that for $0 \le \theta < \pi$,

$$\left\|\left(\frac{\theta-\pi}{\cos\theta/2}\right)\sin\frac{\theta}{2}\right\|_\infty^p \le 2^p, \tag{4.6}$$

we get

$$\int_0^\pi \left| T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right|^p d\theta \le 2^p\left(\frac{p}{p-1}\right)^p \int_0^\pi \left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|^p d\theta, \tag{4.7}$$

and, in like fashion, it is seen that

$$\int_{-\pi}^0 \left| T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right|^p d\theta \le 2^p\left(\frac{p}{p-1}\right)^p \int_{-\pi}^0 \left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|^p d\theta. \tag{4.8}$$

Adding the left-hand side of (4.8) to the left-hand side of (4.7) and the right-hand side of (4.8) to the right-hand side of (4.7) now gives

$$\int_{-\pi}^\pi \left| T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right|^p d\theta \le 2^p\left(\frac{p}{p-1}\right)^p \int_{-\pi}^\pi \left|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right|^p d\theta, \tag{4.9}$$

and taking the pth root of both sides gives

$$\left\| T_m(\theta)\cos^{2r-1}\frac{\theta}{2}\sin\frac{\theta}{2}\right\|_p \le \left(\frac{2p}{p-1}\right)\left\|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right\|_p. \tag{4.10}$$

Using this estimate in (4.4), we obtain

$$\left\|T'_m(\theta)\cos^{2r}\frac{\theta}{2}\right\|_p \le \left\|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right\|_p + \left(\frac{2pr}{p-1}\right)\left\|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right\|_p, \tag{4.11}$$

and, applying (4.2), we get

$$\left\|T'_m(\theta)\cos^{2r}\frac{\theta}{2}\right\|_p \le (m+r)\left\|T_m(\theta)\cos^{2r}\frac{\theta}{2}\right\|_p + \left(\frac{2pr}{p-1}\right)\left\|\left(T_m(\theta)\cos^{2r}\frac{\theta}{2}\right)'\right\|_p, \tag{4.12}$$

The estimate (3.1) follows, concluding the proof.

5. Proof of Theorem 3.2

It has already been seen that the correspondence $x \leftrightarrow \tan\theta/2$ gives an isometry between the rational function space $\mathbf{Q}_{m,n}$, with underlying domain $(-\infty,\infty)$, and the space $\mathbf{T}_{m,n}$ of trigonometric polynomials, on the underlying domain $(-\pi,\pi)$. This mapping causes in particular the correspondences

$$\begin{aligned}\frac{1}{1+x^2} &= \cos^2\frac{\theta}{2},\\ \frac{1-x^2}{1+x^2} &= \cos\theta,\\ \frac{2x}{1+x^2} &= \sin\theta.\end{aligned} \tag{5.1}$$

However, differentiation gives

$$\frac{dx}{d\theta} = \frac{1}{2\cos^2\theta/2} = \frac{1}{2}(1+x^2), \tag{5.2}$$

and the problem here is to deal with the consequences.

A given function in $\mathbf{Q}_{m,n}$, can be written in the form $Q_m(x)(1+x^2)^{-r}$, where $r = m-n$. Then, via $x \leftrightarrow \tan\theta/2$, we have

$$Q_m(x) = T_m(\theta), \tag{5.3}$$

in which T_m is a trigonometric polynomial of degree at most $2m$, and

$$Q'_m(x) = 2T'_m(\theta)\cos^2\frac{\theta}{2}, \tag{5.4}$$

or equivalently

$$Q'_m(x)(1+x^2) = 2T'_m(\theta). \tag{5.5}$$

In any event, we get for $r \ge -1$ that

$$Q'_m(x)(1+x^2)^{-r} = 2T'_m(\theta)\cos^{2r+2}\frac{\theta}{2}, \tag{5.6}$$

and therefore we have, using (3.1),

$$\begin{aligned} \|Q_m'(x)(1+x^2)^{-r}\|_{p,w} &= 2\left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p \\ &\le 2\left(1+\frac{2p(r+1)}{p-1}\right)(m+r+1)\left\|T_m(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p \\ &= 2\left(1+\frac{2p(r+1)}{p-1}\right)(m+r+1)\left\|Q_m(x)(1+x^2)^{-r-1}\right\|_p, \end{aligned} \tag{5.7}$$

which is what was to be proved.

6. Proof of Theorem 3.3

We let $t \leftrightarrow x^2$, and we can assume that $x \ge 0$. We have then

$$\frac{dt}{dx} = 2x = 2\sqrt{t}. \tag{6.1}$$

Using the substitution $t = x^2$, we further notice that the function $R_m(t)$ in the theorem may be represented as

$$R_m(t) = \frac{P(t)}{(1+t)^m} = Q_m(x) \tag{6.2}$$

in which the numerator on the right is thus $P(x^2)$, which is an even polynomial. Then,

$$\sqrt{t}R_m'(t) = \frac{1}{2}Q_m'(x). \tag{6.3}$$

Applying (3.3) from Theorem 3.2 now gives the result immediately.

7. Proof of Theorem 3.4

We have for $t \ge 0$, $x \ge 0$, and $\theta \ge 0$,

$$t^{1/2} = x = \tan\frac{\theta}{2}, \tag{7.1}$$

and thus

$$\frac{d\theta}{dt} = \cos^2\frac{\theta}{2}\left(\frac{\cos\theta/2}{\sin\theta/2}\right) = t^{-1/2}(1+t). \tag{7.2}$$

Furthermore, we may identify

$$R_m(t) = \frac{P(t)}{(1+t)^m} = T_m(\theta), \tag{7.3}$$

in which P is an algebraic polynomial of degree at most m and T_m is an *even* trigonometric polynomial of degree at most m. Thus,

$$\begin{aligned} R_m'(t) &= T_m'(\theta)\frac{d\theta}{dt} \\ &= T_m'(\theta)\cos^2\frac{\theta}{2}\left(\frac{\cos\theta/2}{\sin\theta/2}\right), \end{aligned} \tag{7.4}$$

whence

$$(1+t)^{-r}R_m'(t) = T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\left(\frac{\cos\theta/2}{\sin\theta/2}\right). \tag{7.5}$$

Starting with (7.3) and (7.5), we now have

$$||(1+t)^{-r}R_m'(t)||_{p,W} = \left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\left(\frac{\cos\theta/2}{\sin\theta/2}\right)\right\|_p, \tag{7.6}$$

and we must estimate what is on the right. Noting that $T_m'(\theta)$ is odd and therefore is zero when $\theta = 0$, and that for $0 < |\theta| < |\pi|$, one has $0 < |\theta/2| < |\tan\theta/2|$, we apply Hardy's inequality, obtaining

$$\begin{aligned} \left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\left(\frac{\cos\theta/2}{\sin\theta/2}\right)\right\|_p &\le \left\|\frac{2}{\theta}\left(T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right)\right\|_p \\ &\le \frac{2p}{p-1}\left\|\left(T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right)'\right\|_p. \end{aligned} \tag{7.7}$$

The Bernstein inequality in L^p (4.2) gives

$$\left\|\left(T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right)'\right\|_p \le (m+r+1)\left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p, \tag{7.8}$$

whence

$$||(1+t)^{-r}R_m'(t)||_{p,W} \le \left(\frac{2p}{p-1}\right)(m+r+1)\left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p. \tag{7.9}$$

Then Theorem 3.1 gives

$$\left\|T_m'(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p \le \left(1+\frac{2p(r+1)r}{p-1}\right)(m+r+1)\left\|T_m(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p. \tag{7.10}$$

Combining this estimate with (7.7) provides

$$||(1+t)^{-r}R_m'(t)||_{p,W} \le \left(1+\frac{2p(r+1)}{p-1}\right)\left(\frac{2p}{p-1}\right)(m+r+1)^2\left\|T_m(\theta)\cos^{2r+2}\frac{\theta}{2}\right\|_p. \tag{7.11}$$

Then, application of (7.3) gives

$$\|(1+t)^{-r}R_m'(t)\|_{p,W} \le \left(1+\frac{2p(r+1)}{p-1}\right)\left(\frac{2p}{p-1}\right)(m+r+1)^2\|(1+t)^{-r-1}R_m(t)\|_{p,W}, \tag{7.12}$$

which proves the theorem.

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