ABSTRACT DIFFERENTIAL INEQUALITIES AND THE CAUCHY PROBLEM FOR INFINITE-DIMENSIONAL LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

ANDREI RONTÓ AND JIŘÍ ŠREMR

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We establish optimal, in a sense, unique solvability conditions of the Cauchy problem for a wide class of linear functional differential equations in a Banach space with a solid wedge. The conditions are formulated in terms of certain abstract functional differential inequalities.

1. Introduction

It is well known that, in the theory of functional differential equations, the study of the Cauchy problem requires much more effort than in the case of an ordinary differential equation. One can easily show that even the simplest scalar initial value problem

$$u'(t) = p(t)u(\theta), \quad t \in [a,b], \tag{1.1}$$

$$u(\tau) = 0, \tag{1.2}$$

where the function $p : [a,b] \to \mathbb{R}$ is integrable and $\theta \in [a,b]$ is a fixed number, may have infinitely many solutions. From the theoretical viewpoint, the Cauchy problem for functional differential equations, therefore, should be put amongst the other boundary value problems because the question on its solvability is almost as far from being obvious as is that of any other problem for this extremely general kind of equations.

At present, unfortunately, there are not but a few fruitful, leading one to sharp and easy-to-verify conditions, approaches to the Cauchy problem for general functional differential equations, the most powerful and efficient one being based on the use of differential inequalities and developed most extensively for ordinary differential equations (see, e.g., [3, 5, 6, 7]). It should be noted, however, that the techniques used in the works cited are essentially finite-dimensional, often even one-dimensional, in which circumstance excludes any opportunity to study, for example, countable systems of differential equations. Moreover, the majority of significant results on the solvability of the general Cauchy problem are currently available for the scalar equations only [2, 3].

In this paper, we suggest a new approach to the Cauchy problem, which is based on the use of order-theoretical methods, and establish considerably more general versions of

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the related results of [2, 3]. The solvability conditions obtained here involve abstract functional differential inequalities understood in a rather broad sense; they are constructed on the base of a certain preordering of the given Banach space. The approach based on the study of operators preserving a certain preordering in the given Banach space, firstly, is equally applicable in finite- and infinite-dimensional cases, without any loss in the sharpness of estimates, and, secondly, provides a unified way to obtain solvability conditions for various equations with apparently different properties.

Due to the use of rather general preorderings, which may not be, and often are not orderings, the theorems that we prove here allow one to establish the unique solvability of the Cauchy problem for (finite- or infinite-dimensional) linear functional differential equations also in the cases where the operator determining the equation may not be positive in any natural sense. In the "positive" cases, that is, if the preorderings are generated by cones, we obtain a statement (*namely* Theorem 4.4) containing the corresponding results of [2, 3].

In the proofs of the main Theorems 4.1 and 4.4, we use our previous results on the estimates of spectra of certain classes of linear operators [11] which may not necessarily be isotone with respect to any proper cone.

We do not give applications of our general theorems to any concrete classes of equations here. To demonstrate the practical realisation of the ideas on an example, we only obtain a generalised version of a theorem from [2] concerning a scalar linear equation with a single transformation of argument.

2. Notation and definitions

The following notation is used in the sequel.

(i) $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (-\infty,0]$.

(ii) $\langle X, \|\cdot\|\rangle$ is a Banach space.

(iii) $C([a,b],X)$ is the Banach space of continuous functions $u:[a,b]\to X$ endowed by the norm

$$C([a,b],X) \ni u \longmapsto \max_{t\in[a,b]} \|u(t)\|. \tag{2.1}$$

(iv) $L([a,b],X)$ is the Banach space of Bochner integrable functions $u:[a,b]\to X$ endowed by the norm

$$L([a,b],X) \ni u \longmapsto \int_a^b \|u(t)\|dt. \tag{2.2}$$

(v) $\operatorname{mes}\Omega$ is the Lebesgue measure of a set Ω.

(vi) $r(A)$ is the spectral radius of a linear operator A.

(vii) $A(H) := \{Ax \mid x\in H\}$ is the image of a set $H\subset X$ under the mapping A.

(viii) $\operatorname{Int} B$ is the set of interior points of a set B.

(ix) $\geqslant_K$ and $\gg_K$: see Definitions 2.4 and 2.6.

(x) $\operatorname{blade} K$: see Definition 2.2 and formula (2.3).

(xi) $\mathcal{B}_K(\tau,\Omega;[a,b],X)$: see Definition 2.9.

(xii) $C_{K,\Omega}([a,b],X)$: see formula (5.5).

The two subsections below contain a number of definitions used in the sequel.

2.1. Wedges. We recall some definitions from the theory of linear semigroups in Banach spaces (see, e.g., [8, 9]).

Definition 2.1. A nonempty closed set K in a Banach space X is called a *wedge* (see, e.g., [8]) if the following conditions are satisfied:
(i) $K+K \subset K$,
(ii) $\lambda K \subset K$ for an arbitrary $\lambda \in [0,\infty)$.

Here, by definition, we set $K+K := \{x_1+x_2 \mid \{x_1,x_2\} \subset K\}$ and, similarly, $\lambda K := \{\lambda x \mid x \subset K\}$.

Definition 2.2. The set $K \cap (-K)$ is referred to as the *blade* [8] of the wedge K.

We use the following notation for the blade:

$$K \cap (-K) =: \operatorname{blade} K. \tag{2.3}$$

Remark 2.3. In the original terminology introduced by Kreĭn and Rutman [9], a set K satisfying conditions (i) and (ii) of Definition 2.1 is called a *linear semigroup*.

The presence of a wedge in a Banach space X allows one to introduce a natural preordering there. More precisely, we introduce the following standard.

Definition 2.4. Two elements $\{x_1,x_2\} \subset X$ are said to be in relation $x_1 \leqslant_K x_2$ if and only if they satisfy the relation $x_2 - x_1 \in K$.

In a similar way, the relation $\geqslant_K$ is introduced: $x_1 \geqslant_K x_2$ if and only if $x_2 \leqslant_K x_1$. Thus, we have

$$\begin{aligned} K &= \{x \in X \mid x \geqslant_K 0\}, \\ \operatorname{blade} K &= \{x \in X \mid 0 \leqslant_K x \leqslant_K 0\}. \end{aligned} \tag{2.4}$$

Definition 2.5. A wedge $K \subset X$ will be called *proper* if it does not coincide with the entire X and is different from the zero-dimensional subspace $\{0\}$.

Definition 2.6. A wedge $K \subset X$ is said to be *solid* [9] if its interior is nonempty.

In the case of a solid wedge K, following [9], we write $x \gg_K 0$ if and only if $x \in \operatorname{Int} K$.

Definition 2.7. The wedge K is called a *cone* [8, 9] if it has trivial blade, that is, when

$$\operatorname{blade} K = \{0\}. \tag{2.5}$$

2.2. Definition of a (τ,Ω)-positive operator. The set $\mathcal{B}_K(\tau,\Omega;[a,b],X)$. Here, we introduce the classes of operators frequently used in the sequel.

Let τ be a point in $[a,b]$, Ω a subset of $[a,b]$, and $K \subset X$ a wedge.

Definition 2.8. An operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ is said to be (τ,Ω)-*positive with respect to the wedge* $K \subset X$ if

$$\int_\tau^t (\mathbf{l}u)(s)ds \geqslant_K 0 \quad \forall t \in [a,b] \setminus \Omega \tag{2.6}$$

whenever the function u from $C([a,b],X)$ is such that

$$u(t) \geqslant_K 0 \quad \forall t \in [a,b] \setminus \Omega. \tag{2.7}$$

Recall that the sign $\geqslant_K$ is introduced by Definition 2.4.

Definition 2.9. By the symbol $\mathcal{B}_K(\tau,\Omega;[a,b],X)$, denote the set of all operators $\mathbf{l} : C([a,b], X) \to L([a,b],X)$ that are (τ,Ω)-positive with respect to the wedge K and possess the following additional property: the fulfilment of the relation $u([a,b] \setminus \Omega) \subset \operatorname{blade} K$ for a function u from $C([a,b],X)$ always implies that

$$\int_\tau^t (\mathbf{l}u)(s)ds = 0 \quad \forall t \in [a,b]. \tag{2.8}$$

Definition 2.10. A linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ will be called *regular* if there exists a nonnegative Lebesgue integrable function $\omega : [a,b] \to \mathbb{R}$ such that, for an arbitrary u from $C([a,b],X)$,

$$||(\mathbf{l}u)(t)|| \leq \omega(t) \max_{s\in[a,b]} ||u(s)|| \quad \forall t \in [a,b]. \tag{2.9}$$

The regularity of a linear mapping from $C([a,b],X)$ to $L([a,b],X)$, obviously, implies its continuity.

Remark 2.11. The property described by Definition 2.10 is similar to a notion from the theory of K-spaces (see, e.g., [4, Chapter VII, Section 1.2]). The context, however, is quite different here because the wedge K generating the preordering in the space X is not assumed to be a minihedral [9] cone (even more, K is not assumed to be a cone at all), and, therefore, X may not be a lattice.

3. Statement of problem

We consider the linear inhomogeneous Cauchy problem

$$u'(t) = (\mathbf{l}u)(t) + f(t), \quad t \in [a,b], \tag{3.1}$$

$$u(\tau) = c. \tag{3.2}$$

Here, $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ is a continuous linear operator which is (τ,Ω)-positive with respect to a certain proper wedge K in X. By a solution of problem (3.1), (3.2), we mean an absolutely continuous function $u : [a,b] \to X$ possessing property (3.2) and satisfying (3.1) almost everywhere on $[a,b]$.

4. Conditions sufficient for the unique solvability of problem (3.1), (3.2)

We consider problem (3.1), (3.2) in a Banach space X, in which we fix a certain solid and proper wedge K.

THEOREM 4.1. *Assume that, for some solid proper wedge $K \subset X$ and some set $\Omega \subset [a,b]$ such that $[a,b] \setminus \Omega$ is closed, the linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ is regular and belongs to the set $\mathcal{B}_K(\tau,\Omega;[a,b],X)$.*

Then the existence of a constant $\alpha \in [0,1)$ and a continuous abstract function $y : [a,b] \to X$ satisfying the conditions

$$y(t) \gg_K 0 \quad \forall t \in [a,b] \setminus \Omega, \tag{4.1}$$

$$\alpha y(t) \geqslant_K \int_\tau^t (\mathbf{l}y)(s)ds \quad \text{for every } t \in [a,b] \setminus \Omega \tag{4.2}$$

ensures the unique solvability of the Cauchy problem (3.1), (3.2) for arbitrary $f \in L([a,b],X)$ and $c \in X$. Under these assumptions, the unique solution $u(\cdot)$ of the Cauchy problem (3.1), (3.2) is representable in the form of the uniformly convergent functional series

$$u(t) = f_c(t) + \int_\tau^t (\mathbf{l}f_c)(s)ds + \int_\tau^t \left(\mathbf{l}\int_\tau^{\cdot} (\mathbf{l}f_c)(\xi)d\xi\right)(s)ds + \cdots, \quad t \in [a,b], \tag{4.3}$$

where

$$f_c(t) := c + \int_\tau^t f(s)ds, \quad t \in [a,b]. \tag{4.4}$$

If, furthermore, the function $f \in L([a,b],X)$ and vector $c \in X$ are such that

$$\int_\tau^t f(s)ds \geqslant_K -c \quad \forall t \in [a,b] \setminus \Omega, \tag{4.5}$$

then the solution $u(\cdot)$ of problem (3.1), (3.2) satisfies condition (2.7).

Remark 4.2. Condition (4.5) of Theorem 4.1 is satisfied, in particular, when c and f in problem (3.1), (3.2) are such that

$$\begin{gathered} c \geqslant_K 0, \\ f(t)\operatorname{sign}(t-\tau) \geqslant_K 0 \quad \text{for a.e. } t \in [a,b]. \end{gathered} \tag{4.6}$$

Remark 4.3. One can prove that the regularity condition for the linear operator $\mathbf{l}$ in Theorem 4.1 can be replaced by the assumption on its continuity (we do not consider this problem in more detail here). Note also that, in the case where the solid wedge K is a normal cone, one can also drop the continuity assumption for $\mathbf{l}$ (cf. [10]).

In the case where the wedge K is a cone, the assumptions of Theorem 4.1 are somewhat simpler, and the corresponding assertion is formulated as follows.

THEOREM 4.4. *Assume that, for some solid cone K in X and a subset Ω of $[a,b]$ such that $[a,b] \setminus \Omega$ is closed, the linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ is regular, (τ,Ω)-positive* *with respect to K, and, moreover, for an arbitrary function u from $C([a,b],X)$ vanishing on the set $[a,b] \setminus \Omega$, relation (2.8) holds.*

Then the existence of a constant $\alpha \in [0,1)$ and a continuous abstract function $y : [a,b] \to X$ satisfying the conditions (4.1) and (4.2) ensures the unique solvability of the Cauchy problem (3.1), (3.2) for arbitrary $f \in L([a,b],X)$ and $c \in X$. Under these assumptions, the unique solution $u(\cdot)$ of the Cauchy problem (3.1), (3.2) is representable in the form of the uniformly convergent functional series (4.3), where f_c is the function given by formula (4.4).

If, furthermore, relation (4.5) holds for the function $f \in L([a,b],X)$ and vector $c \in X$, then the solution $u(\cdot)$ of problem (3.1), (3.2) satisfies condition (2.7).

Remark 4.5. A statement similar to Theorem 4.4 for the scalar (i.e., when $X = \mathbb{R}$ and $K = \mathbb{R}_+$) initial value problem (3.1), (3.2) with $\tau \in \{a,b\}$ is contained in [3, Theorem 2.1]. The former theorem generalises the result of [3] cited above to the infinite-dimensional case.

Corollary 4.6. *Assume that the linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ is regular and satisfies the following condition for some solid cone K in X: the relation*

$$(\mathbf{l}u)(t)\,\mathrm{sign}(t-\tau) \geqslant_K 0 \quad \text{for a.e. } t \in [a,b] \tag{4.7}$$

holds whenever the function u from $C([a,b],X)$ is such that

$$u([a,b]) \subset K. \tag{4.8}$$

Furthermore, assume that there exist a constant $\alpha \in [0,1)$ and a continuous abstract function $y : [a,b] \to X$ such that

$$y(t) \gg_K 0 \quad \forall t \in [a,b], \tag{4.9}$$

$$\alpha y(t) \geqslant_K \int_\tau^t (\mathbf{l}y)(s)ds \quad \text{for every } t \in [a,b]. \tag{4.10}$$

Then the Cauchy problem (3.1), (3.2) has a unique solution for arbitrary $f \in L([a,b],X)$ and $c \in X$. Under these assumptions, the unique solution $u(\cdot)$ of the Cauchy problem (3.1), (3.2) is representable in the form of the uniformly convergent functional series (4.3), where f_c is the function given by formula (4.4).

If, furthermore,

$$\int_\tau^t f(s)ds \geqslant_K -c \quad \forall t \in [a,b], \tag{4.11}$$

then the solution $u(\cdot)$ of problem (3.1), (3.2) satisfies condition (4.8).

Remark 4.7. Conditions (4.2) and (4.10) in the theorems formulated above are optimal in the sense that the inequality $\alpha < 1$ for the constant α involved therein, generally speaking, cannot be replaced by the corresponding nonstrict inequality $\alpha \leq 1$.

Indeed, we consider the simplest scalar functional differential equation (1.1), where $\theta \in [a,b] \setminus \{\tau\}$ and p is a function from $L([a,b],\mathbb{R})$ such that

$$\int_{\tau}^{\theta} p(s)ds = 1. \tag{4.12}$$

Equation (1.1), obviously, can be rewritten as (3.1) with $X = \mathbb{R}$, $f = 0$, and the operator $\mathbf{l}$ given by

$$C([a,b],\mathbb{R}) \ni u \longmapsto \mathbf{l}u := p(\cdot)u(\theta). \tag{4.13}$$

If we set, for example, $K := [0,+\infty)$, then (4.8) always implies (4.7), whenever p satisfies the condition

$$p(t)\operatorname{sign}(t-\tau) \geq 0, \quad t \in [a,b]. \tag{4.14}$$

It is easy to see that, by (4.12), the function u_λ of the form

$$u_\lambda(t) = \lambda \int_{\tau}^{t} p(s)ds, \quad t \in [a,b], \tag{4.15}$$

where λ is an arbitrary real constant, is a solution of the homogeneous Cauchy problem (1.2) for (1.1). Indeed, differentiating (4.15), we obtain

$$u_\lambda'(t) = \lambda p(t), \quad t \in [a,b]. \tag{4.16}$$

However, in view of condition (4.12), we have

$$u_\lambda(\theta) = \lambda, \tag{4.17}$$

and hence, equality (1.1) holds.

We put

$$y := u_\lambda + \lambda. \tag{4.18}$$

For positive λ, this function satisfies the corresponding condition (4.9) of Corollary 4.6. Assume that, besides relations (4.12) and (4.14), the function p satisfies also the condition

$$\max_{t\in[a,b]} \int_{\tau}^{t} p(s)ds \leq 1. \tag{4.19}$$

Note that conditions (4.12), (4.14), and (4.19) are satisfied, for example, by the function

$$p(t) := \begin{cases} 0 & \text{for } t \in [a, \min\{\tau,\theta\}] \cup [\max\{\tau,\theta\},b], \\ (\theta-\tau)^{-1} & \text{for } t \in (\min\{\theta,\tau\}, \max\{\theta,\tau\}). \end{cases} \tag{4.20}$$

Since

$$u_\lambda(t) = \int_{\tau}^{t} (\mathbf{l}u_\lambda)(s)ds, \quad t \in [a,b], \tag{4.21}$$

where $\mathbf{l}$ is the operator given by formula (4.13), it follows that, for positive λ, the function y also satisfies the relation

$$y(t) \geq \int_{\tau}^{t} (\mathbf{l}y)(s)ds, \quad t \in [a,b], \tag{4.22}$$

which, clearly, is a particular case of inequality (4.10) with $\alpha = 1$. Indeed, in view of (4.18), (4.12), (4.19), and (4.15), for $\lambda > 0$, we have

$$\begin{aligned} y(t) - \int_{\tau}^{t} (\mathbf{l}y)(s)ds &= \lambda + \lambda \int_{\tau}^{t} p(s)ds - \int_{\tau}^{t} p(s)ds \cdot y(\theta) \\ &= \lambda + \lambda \int_{\tau}^{t} p(s)ds - \int_{\tau}^{t} p(s)ds \cdot 2\lambda \\ &= \lambda\left(1 - \int_{\tau}^{t} p(s)ds\right) \geq 0 \end{aligned} \tag{4.23}$$

at every point t from $[a,b]$.

Thus, all the assumptions of Corollary 4.6 are satisfied with the function y defined by relation (4.18), except condition (4.10), instead of which inequality (4.22) holds. However, for nonzero λ, the function u_λ is a nontrivial solution of the homogeneous Cauchy problem (1.1), (1.2).

The example given above shows that, for $\alpha = 1$, assumption (4.10) in Corollary 4.6 may not guarantee the unique solvability of problem (3.1), (3.2) for arbitrary f and c, and thus, is strict in this sense and cannot be weakened. Clearly, the same is true for condition (4.2) appearing in Theorems 4.1 and 4.4.

Remark 4.8. Results similar to those stated above have been recently obtained in [10] for finite systems of linear functional differential equations determined by operators satisfying certain positivity conditions (cf. Theorem 4.4). The methods of [10] are different from those used in this paper. Furthermore, the differential inequalities here are understood in a different and more broad sense.

Note also that, similarly to [1, 10], the results presented here allow one to obtain a series of efficient conditions of the unique solvability of problem (3.2) for various kinds of (in this case, generally speaking, infinite-dimensional) functional differential equations (3.1). However, we do not obtain such results in this paper, restricting ourselves mainly to the general theorems.

The results stated above allow one to derive various efficient conditions sufficient for the unique solvability of problem (3.1), (3.2) for arbitrary values of forcing terms. However, we do not dwell on this here in more detail, trying instead to focus on general ideas. As an illustrative example, we only show a possible way to apply Theorem 4.4 to the Cauchy problem (3.2) for the following scalar functional differential equation:

$$u'(t) = p(t)u(\theta(t)) + f(t), \quad t \in [a,b]. \tag{4.24}$$

Here, $\{p, f\} \subset L([a,b],\mathbb{R})$, and θ is a measurable transformation of the interval $[a,b]$ into itself. The choice of dimension one and of this specific type of equation is motivated by a previous study of equations of such a kind made in [2].

Remark 4.9. It should be noted that in case X is a one-dimensional space, there is no any opportunity to demonstrate an "essentially nonpositive" example of application of Theorem 4.4. Indeed, in the one-dimensional space $X = \mathbb{R}$, we have only four wedges, namely, $\mathbb{R}_+$, $\mathbb{R}_-$, $\{0\}$, and $\mathbb{R}$ itself. Of these, the first two are cones, whereas the other ones are not proper wedges and, therefore, are excluded from consideration.

Corollary 4.10. *Assume that the function $p : [a,b] \to \mathbb{R}$ in (4.24) is such that*

$$p(t)\operatorname{sign}(t-\tau) \ge 0 \quad \text{for a.e. } t \in [a,b] \tag{4.25}$$

and, moreover, the inequality

$$\max\left\{ \int_a^\tau |p(s)|\,ds, \int_\tau^b p(s)ds \right\} < 1 \tag{4.26}$$

is true. Let the measurable function $\theta : [a,b] \to [a,b]$ satisfy the inclusion

$$\theta([a,b]) \subset [a,b] \setminus \Omega \tag{4.27}$$

with some subset Ω of $[a,b]$ such that $[a,b] \setminus \Omega$ is closed.

Then the inhomogeneous Cauchy problem (4.24), (3.2) has a unique solution for arbitrary $f \in L([a,b],\mathbb{R})$ and real c, and this solution is represented by the uniformly convergent series

$$\begin{aligned} u(t) = c p_{\theta,\tau}(t) + \int_\tau^t f(s)ds + \int_\tau^t p(s_1) \int_\tau^{\theta(s_1)} f(s_2)ds_2\, ds_1 \\ + \int_\tau^t p(s_1) \int_\tau^{\theta(s_1)} p(s_2) \int_\tau^{\theta(s_2)} f(s_3)ds_3\, ds_2\, ds_1 + \cdots, \end{aligned} \tag{4.28}$$

where

$$p_{\theta,\tau}(t) := 1 + \int_\tau^t p(s)ds + \int_\tau^t p(s_1) \int_\tau^{\theta(s_1)} p(s_2)ds_2\, ds_1 + \cdots. \tag{4.29}$$

If, furthermore, f and c are such that

$$\int_\tau^t f(s)ds \ge -c \tag{4.30}$$

(resp., their relation

$$\int_\tau^t f(s)ds \le -c \tag{4.31}$$

holds) for all $t \in [a,b] \setminus \Omega$, then the unique solution of problem (4.24), (3.2) is nonnegative (resp., nonpositive) on the set $[a,b] \setminus \Omega$.

Remark 4.11. Similarly to Remark 4.7 one can show that condition (4.26) in Corollary 4.10 is strict in the sense that, generally speaking, it cannot be replaced by the corresponding nonstrict inequality

$$\max\left\{ \int_a^\tau |p(s)|\,ds, \int_\tau^b p(s)ds \right\} \le 1. \tag{4.32}$$

As follows from the example below, under the conditions of Corollary 4.10 guaranteeing the nonnegativeness of the solution of problem (4.24), (3.2) on the set $[a,b]\setminus\Omega$, the solution may not be nonnegative on the entire interval $[a,b]$.

Example 4.12. We consider the Cauchy problem (3.2) for the scalar equation

$$u'(t)=p(t)u(\theta)+f(t),\quad t\in[a,b],\tag{4.33}$$

where $\theta\in[a,b]$ and $\{p,f\}\subset L([a,b],\mathbb{R})$. Assume that p satisfies conditions (4.25) and (4.26), whereas the point θ does not belong to a certain subset Ω of $[a,b]$ such that the corresponding set $[a,b]\setminus\Omega$ is closed. It follows from Corollary 4.10 that problem (4.33), (3.2) is uniquely solvable for an arbitrary integrable function $f:[a,b]\to\mathbb{R}$ and constant c and, moreover, its solution is nonnegative (resp., nonpositive) on the set $[a,b]\setminus\Omega$ if f and c satisfy condition (4.30) (resp., (4.31)) for every t from $[a,b]\setminus\Omega$.

One can verify that the unique solution u of problem (4.33), (3.2) is given by the formula (conditions (4.26) and (4.25), in particular, imply the inequality $\int_\tau^\theta p(s)ds<1$ and, hence, formula (4.34) makes sense)

$$u(t)=c+\int_\tau^t f(s)ds+\frac{c+\int_\tau^\theta f(s)ds}{1-\int_\tau^\theta p(s)ds}\int_\tau^t p(s)ds,\quad t\in[a,b].\tag{4.34}$$

If f is such that $\int_\tau^\theta f(s)ds=-c$, then the solution u of (4.33), (3.2) is negative (resp., positive) at those points t from $[a,b]$ where inequality (4.30) (resp., (4.31)) is not satisfied.

As a particular case of Corollary 4.10 for $\Omega=\varnothing$ and $\tau=a$, we obtain the following result of [2].

Corollary 4.13 [2]. *If $p:[a,b]\to\mathbb{R}$ is a nonnegative integrable function satisfying the inequality*

$$\int_a^b p(t)dt<1,\tag{4.35}$$

then the Cauchy problem (4.24), (3.2) with $\tau=a$ is uniquely solvable for arbitrary f from $L([a,b],\mathbb{R})$ and real c, and the solution of (4.24), (3.2) is nonnegative whenever f and c have this property.

5. Auxiliary statements and proofs of main results

The proofs use a number of auxiliary statements given in the subsections below.

5.1. Operators vanishing on the blade of a wedge. Let $\mathscr{X}$ be a Banach space, $\mathscr{K}$ a wedge in it, and $A:\mathscr{X}\to\mathscr{X}$ a linear operator.

Definition 5.1. The operator $A:\mathscr{X}\to\mathscr{X}$ is said *to leave invariant a set $H\subset\mathscr{X}$* if

$$A(H)\subset H.\tag{5.1}$$

The following statement is, in fact, [11, Theorem 4].

THEOREM 5.2 [11]. *Let $A : \mathcal{X} \to \mathcal{X}$ be a completely continuous linear operator leaving invariant a solid wedge $\mathcal{K} \subset \mathcal{X}$ and such that*

$$\text{blade}\,\mathcal{K} \subset \ker A. \tag{5.2}$$

Then the existence of constants $\alpha \in [0, +\infty)$, $m \in \mathbb{N}$, and an element $g \in \text{Int}\,\mathcal{K}$ such that

$$\alpha g - A^m g \in \mathcal{K} \tag{5.3}$$

implies the estimate

$$r(A) \le \sqrt[m]{\alpha}. \tag{5.4}$$

5.2. Set $C_{K,\Omega}([a,b],X)$ and operator $\mathcal{J}_{\tau,\mathrm{I}}$. Throughout this section, we fix a set $\Omega \subset [a,b]$ such that $[a,b] \setminus \Omega$ is closed.

Let $C_{K,\Omega}([a,b],X)$ be the set of all continuous abstract functions $u : [a,b] \to X$ satisfying condition (2.7):

$$C_{K,\Omega}([a,b],X) := \{u \in C([a,b],X) \mid u(t) \geqslant_K 0 \ \forall t \in [a,b] \setminus \Omega\}. \tag{5.5}$$

LEMMA 5.3. *For an arbitrary proper wedge $K \subset X$, the following assertions are true.*

(i) *The set $C_{K,\Omega}([a,b],X)$ is a proper wedge in $C([a,b],X)$.*

(ii) *The wedge $C_{K,\Omega}([a,b],X)$ is solid whenever K possesses the property indicated. In this case, $y \in \text{Int}\,C_{K,\Omega}([a,b],X)$ if and only if relation (4.1) holds.*

(iii) *The equality*

$$\text{blade}\,C_{K,\Omega}([a,b],X) = \{u \in C([a,b],X) \mid u([a,b] \setminus \Omega) \subset \text{blade}\,K\} \tag{5.6}$$

is true.

Proof. The fulfilment of conditions (i) and (ii) in Definition 2.1 is obvious. The set $C_{K,\Omega}([a,b],X)$ is nonempty because it contains, in particular, constant functions with values in K. It is also easy to see that (5.5) is a set closed with respect to the uniform norm in $C([a,b],X)$.

According to formula (2.1) for the norm in $C([a,b],X)$, a function y from $C([a,b],X)$ is an interior element of $C_{K,\Omega}([a,b],X)$ if and only if there exists a $\delta \in (0, +\infty)$ such that every $u \in C([a,b],X)$ satisfying the condition

$$\|u(t) - y(t)\| < \delta \quad \forall t \in [a,b] \tag{5.7}$$

belongs to $C_{K,\Omega}([a,b],X)$, that is,

$$u([a,b] \setminus \Omega) \subset K. \tag{5.8}$$

Indeed, if $y \in \text{Int}\,C_{K,\Omega}([a,b],X)$, then, obviously,

$$y([a,b] \setminus \Omega) \subset \text{Int}\,K. \tag{5.9}$$

If, conversely, (5.9) holds, then, for an arbitrary $t \in [a,b] \setminus \Omega$, the number

$$\delta(t) := \sup\{\delta \mid \delta > 0,\ x \in X,\ \|y(t) - x\| < \delta \text{ imply } x \in K\} \tag{5.10}$$

is finite and different from zero. The function y is continuous and the set $[a,b] \setminus \Omega$ is assumed to be closed. Therefore, it can be shown that the number $\delta := \inf_{t\in[a,b]\setminus\Omega} \delta(t)$ is strictly positive and, hence, every continuous function $u : [a,b] \to X$ for which (5.7) is true with this value of δ satisfies condition (5.8), that is, y belongs to $\operatorname{Int} C_{K,\Omega}([a,b],X)$.

Finally, we note that relation (5.6) is an immediate consequence of Definition 5.1 of the set $C_{K,\Omega}([a,b],X)$. □

Given a linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$, we introduce the mapping $\mathcal{J}_{\tau,\mathbf{l}}$ by putting

$$(\mathcal{J}_{\tau,\mathbf{l}}u)(t) := \int_\tau^t (\mathbf{l}u)(s)\,ds, \quad t \in [a,b]. \tag{5.11}$$

It is clear that $\mathcal{J}_{\tau,\mathbf{l}}$ is a linear operator transforming the space $C([a,b],X)$ into itself. Note that, for (τ,Ω)-positive $\mathbf{l}$, the inclusion

$$\mathcal{J}_{\tau,\mathbf{l}} C_{K,\Omega}([a,b],X) \subset C_{K,\Omega}([a,b],X) \tag{5.12}$$

is true.

Lemma 5.4. *For every linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$ belonging to the set $\mathcal{B}_K(\tau,\Omega;[a,b],X)$, the corresponding mapping $\mathcal{J}_{\tau,\mathbf{l}}$ satisfies the condition*

$$\operatorname{blade} C_{K,\Omega}([a,b],X) \subset \ker \mathcal{J}_{\tau,\mathbf{l}}. \tag{5.13}$$

Proof. In view of Definition 2.9, this statement is an immediate consequence of Lemma 5.3 and the inclusion $\mathbf{l} \in \mathcal{B}_K(\tau,\Omega;[a,b],X)$. □

Lemma 5.5. *For every regular linear operator $\mathbf{l} : C([a,b],X) \to L([a,b],X)$, the corresponding linear mapping $\mathcal{J}_{\tau,\mathbf{l}} : C([a,b],X) \to C([a,b],X)$ is completely continuous.*

Proof. Indeed, due to the linearity, it will be sufficient to show that the set

$$Q_{\tau,\mathbf{l}} := \Big\{\mathcal{J}_{\tau,\mathbf{l}}u \mid u \in C([a,b],X),\ \max_{t\in[a,b]} \|u(t)\| \le 1\Big\} \tag{5.14}$$

is relatively compact.

This set is, obviously, uniformly bounded because for every $v \in Q_{\tau,\mathbf{l}}$ there exists a function $u \in C([a,b],X)$ such that

$$\mathcal{J}_{\tau,\mathbf{l}}u = v, \qquad \max_{t\in[a,b]} \|u(t)\| \le 1, \tag{5.15}$$

and therefore, in view of (5.11),

$$\begin{aligned}\max_{t\in[a,b]}\|v(t)\| &= \max_{t\in[a,b]}\left\|\int_\tau^t(\mathbf{l}u)(s)ds\right\| \\ &\le \int_{\min\{\tau,t\}}^{\max\{\tau,t\}}\|(\mathbf{l}u)(s)\|ds \le \int_a^b\|(\mathbf{l}u)(s)\|ds.\end{aligned} \tag{5.16}$$

The boundedness of the operator $\mathbf{l}: C([a,b],X) \to L([a,b],X)$ means the existence of a constant $C \in (0,+\infty)$ such that

$$\int_a^b\|(\mathbf{l}u)(s)\|ds \le C\max_{t\in[a,b]}\|u(t)\|. \tag{5.17}$$

Consequently, in view of (5.16),

$$\max_{t\in[a,b]}\|v(t)\| \le C\max_{t\in[a,b]}\|u(t)\| \le C, \tag{5.18}$$

that is, the norms of all elements of set (5.14) are bounded from above by one and the same constant, C.

Set (5.14) is also equicontinuous. Indeed, we fix an arbitrary $\epsilon \in (0,+\infty)$ and consider an arbitrary element v from $Q_{\tau,\mathbf{l}}$. For all $\{t',t''\} \subset [a,b]$, we have

$$\|v(t'') - v(t')\| = \left\|\int_{t'}^{t''}(\mathbf{l}u)(s)ds\right\|, \tag{5.19}$$

where u is a function from $C([a,b],X)$ possessing properties (5.15).

Due to the properties of the Bochner integral, the right-hand side term of the last relation can be estimated as

$$\left\|\int_{t'}^{t''}(\mathbf{l}u)(s)ds\right\| \le \int_{\min\{t',t''\}}^{\max\{t',t''\}}\|(\mathbf{l}u)(s)\|ds, \tag{5.20}$$

whence, by virtue of relation (5.15) and the regularity of the operator $\mathbf{l}$ (see Definition 2.10), we obtain the inequality

$$\left\|\int_{t'}^{t''}(\mathbf{l}u)(s)ds\right\| \le \int_{\min\{t',t''\}}^{\max\{t',t''\}}\omega(s)ds\cdot\max_{t\in[a,b]}\|u(t)\| \le \int_{\min\{t',t''\}}^{\max\{t',t''\}}\omega(s)ds, \tag{5.21}$$

where ω is a nonnegative function from $L([a,b],\mathbb{R})$ appearing in Definition 2.10. It follows from the continuity of the function

$$[a,b] \ni t \longmapsto \int_\tau^t\omega(s)ds, \tag{5.22}$$

that for the given ϵ, there exists a positive δ_ϵ such that the right-hand side term in (5.21) is less than ϵ whenever $|t'-t''| < \delta_\epsilon$. The arbitrariness of a function v from $Q_{\tau,\mathbf{l}}$ proves that the latter set is equicontinuous. Applying the Arzelà-Ascoli theorem, we conclude that set (5.14) is relatively compact and, hence, the mapping $\mathcal{J}_{\tau,\mathbf{l}}$ is completely continuous. □

5.3. Proof of Theorem 4.1. We first formulate the following obvious lemma.

LEMMA 5.6. *The set of solutions of the Cauchy problem (3.1), (3.2) coincides with that of absolutely continuous solutions of the functional equation*

$$u = \mathcal{J}_{\tau,\mathbf{l}}u + f_c. \tag{5.23}$$

Thus, it will suffice to prove the unique solvability of (5.23) for arbitrary values of f from $L([a,b],X)$ and c from X. Recall that f_c in (5.23) is the function constructed from f and c by formula (4.4).

To prove the statement indicated above, it, in turn, would be sufficient to show that the spectrum of the operator $\mathcal{J}_{\tau,\mathbf{l}}$ is contained inside the unit circle with centre at zero.

We show that, under the conditions assumed, the spectral radius $r(\mathcal{J}_{\tau,\mathbf{l}})$ of the operator $\mathcal{J}_{\tau,\mathbf{l}}$ admits the estimate

$$r(\mathcal{J}_{\tau,\mathbf{l}}) \leq \alpha. \tag{5.24}$$

Indeed, condition (4.2) assumed in Theorem 4.1 for a continuous function $y : [a,b] \to X$ can be rewritten as

$$\alpha y(t) \geqslant_K (\mathcal{J}_{\tau,\mathbf{l}}y)(t) \quad \forall t \in [a,b] \setminus \Omega, \tag{5.25}$$

which means nothing but the inclusion

$$\alpha y - \mathcal{J}_{\tau,\mathbf{l}}y \in C_{K,\Omega}([a,b],X). \tag{5.26}$$

The set $C_{K,\Omega}([a,b],X)$, as we know from Lemma 5.3, forms a solid wedge in the space $C([a,b],X)$. Furthermore, assertion (ii) of the lemma mentioned guarantees that the function y, which also satisfies condition (4.1), belongs to the interior of the wedge $C_{K,\Omega}([a,b],X)$.

Note also that, in view of Lemma 5.4, the inclusion $\mathbf{l} \in \mathcal{B}_K(\tau,\Omega;[a,b],X)$ implies that the mapping $\mathcal{J}_{\tau,\mathbf{l}} : C([a,b],X) \to C([a,b],X)$ satisfies condition (5.13). According to Lemma 5.5, the regularity of $\mathbf{l}$ implies the complete continuity of $\mathcal{J}_{\tau,\mathbf{l}}$. Finally, in view of (5.12), the operator $\mathcal{J}_{\tau,\mathbf{l}}$ leaves the wedge $C_{K,\Omega}([a,b],X)$ invariant.

Applying Theorem 5.2 with $m = 1$, $\mathcal{X} = C([a,b],X)$, $\mathcal{K} = C_{K,\Omega}([a,b],X)$, $g = y$, and $A = \mathcal{J}_{\tau,\mathbf{l}}$, we establish inequality (5.24). In view of the condition $0 \leq \alpha < 1$, this, as is well known, guarantees the convergence of the series

$$u := f_c + \mathcal{J}_{\tau,\mathbf{l}}f_c + \mathcal{J}_{\tau,\mathbf{l}}^2 f_c + \cdots \tag{5.27}$$

to the unique solution u of (5.23) or, which is the same (see Lemma 5.6), of the inhomogeneous Cauchy problem (3.1), (3.2). Clearly, series (5.27) coincides with (4.3).

Property (2.7) of the solution of problem (3.1), (3.2) follows immediately from its series representation (5.27) and inclusion (5.12) because $C_{K,\Omega}([a,b],X)$ is a closed set possessing property (i) of Definition 2.1.

5.4. Proof of Theorem 4.4. Since blade$K = \{0\}$ in the case where K is a cone, it is clear that Theorem 4.4 is an immediate consequence of Theorem 4.1.

5.5. Proof of Corollary 4.6. It is sufficient to apply Theorem 4.4 with $\Omega = \varnothing$. Note that, to prove the $(\tau,\varnothing)$-positivity of the operator $\mathcal{J}_{\tau,\mathbf{l}}$, it suffices to apply the the following statement: if a Bochner integrable function $h : [a,b] \to X$ satisfies the condition

$$h(t)\operatorname{sign}(t-\tau) \geqslant_K 0 \quad \text{for a.e. } t \in [a,b], \tag{5.28}$$

then the relation $\int_\tau^t h(s)ds \geqslant_K 0$ holds for all $t \in [a,b]$.

5.6. Proof of Corollary 4.10. We apply Theorem 4.4 to the scalar Cauchy problem (4.24), (3.2). To do so, we put $X := \mathbb{R}$, $K := \mathbb{R}_+$, and

$$(\mathbf{l}u)(t) := p(t)u(\theta(t)), \quad t \in [a,b]. \tag{5.29}$$

Operator (5.29) is obviously regular because under our assumptions,

$$|p(t)u(\theta(t))| \le |p(t)| \max_{s\in[a,b]} |u(s)| \tag{5.30}$$

for almost every $t \in [a,b]$ and an arbitrary u from $C([a,b],\mathbb{R})$. We show that the operator $\mathbf{l}$ defined by formula (5.29) belongs to the set $\mathcal{B}_{\mathbb{R}_+}(\tau,\Omega;[a,b],\mathbb{R})$.

According to Definition 2.8, conditions (4.25) and (4.27) imposed on the functions p and θ imply the (τ,Ω)-positivity of the corresponding operator (5.29) with respect to the cone $\mathbb{R}_+$.

The function θ is assumed to satisfy condition (4.27) and, therefore, the fulfilment of the relation $u([a,b]\setminus\Omega) \subset \{0\}$ for u from $C([a,b],X)$ yields that the function $[a,b] \ni t \mapsto p(t)u(\theta(t))$ vanishes almost everywhere on the interval $[a,b]$. This means that, under our assumptions, $\mathbf{l} \in \mathcal{B}_{\mathbb{R}_+}(\tau,\Omega;[a,b],\mathbb{R})$. It remains to construct a continuous function $y : [a,b] \to \mathbb{R}$ satisfying conditions (4.1) and (4.2), which, by (5.29), have the form

$$y(t) > 0 \quad \forall t \in [a,b]\setminus\Omega, \tag{5.31}$$

$$\alpha y(t) \ge \int_\tau^t p(s)y(\theta(s))ds \quad \text{for every } t \in [a,b]\setminus\Omega, \tag{5.32}$$

respectively.

We set

$$y(t) := 1, \quad t \in [a,b]. \tag{5.33}$$

Then condition (5.31) is obviously satisfied, whereas relation (5.32) takes the form

$$\alpha \ge \int_\tau^t p(s)ds \quad \text{for every } t \in [a,b]\setminus\Omega. \tag{5.34}$$

Note that (5.34) is true in view of inequality (4.26) because $0 \le \alpha < 1$ and, by assumption, condition (4.25) holds for the function p.

Thus, we have shown that, in the case indicated, all the conditions of Theorem 4.4 are satisfied. Application of the theorem mentioned leads us to the conclusion required.

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References

[1] N. Z. Dil'naya and A. N. Rontó, *Some new conditions for the solvability of the Cauchy problem for systems of linear functional-differential equations*, Ukrainian Math. J. **56** (2004), no. 7, 1033–1053.

[2] R. Hakl, A. Lomtatidze, and B. Půža, *On nonnegative solutions of first order scalar functional differential equations*, Mem. Differential Equations Math. Phys. **23** (2001), 51–84.

[3] R. Hakl, A. Lomtatidze, and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations*, Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica, vol. 10, Masaryk University, Brno, 2002.

[4] L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker, *Functional Analysis in Semi-Ordered Spaces*, Gostekhizdat, Moscow, 1950.

[5] I. T. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations*, Izdat. Tbilis. Univ., Tbilisi, 1975.

[6] ———, *Initial and Boundary Value Problems for Systems of Ordinary Differential Equations. I. Linear Theory*, Metsniereba, Tbilisi, 1997.

[7] I. Kiguradze and B. Půža, *Boundary Value Problems for Systems of Linear Functional Differential Equations*, Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, vol. 12, Masaryk University, Brno, 2003.

[8] M. A. Krasnosel'skij, Je. A. Lifshits, and A. V. Sobolev, *Positive Linear Systems. The Method of Positive Operators*, Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989.

[9] M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Translation **1950** (1950), no. 26, 1–128.

[10] A. N. Rontó, *Exact solvability conditions of the Cauchy problem for systems of linear first-order functional differential equations determined by $(\sigma_1,\sigma_2,\ldots,\sigma_n;\tau)$-positive operators*, Ukrainian Math. J. **55** (2003), no. 11, 1561–1588.

[11] A. Rontó, *On substantial eigenvalues of linear operators leaving invariant a closed wedge*, Colloquium on Differential and Difference Equations, CDDE 2002 (Brno), Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., vol. 13, Masaryk Univ., Brno, 2003, pp. 235–246.

Andrei Rontó: Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkovskaya, 01601 Kiev, Ukraine

Current address: Mathematical Institute, Czech Academy of Sciences, Žižkova 22, 61662 Brno, Czech Republic

E-mail address: ronto@ipm.cz

Jiří Šremr: Mathematical Institute, Czech Academy of Sciences, Žižkova 22, 61662 Brno, Czech Republic

E-mail address: sremr@ipm.cz