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DESCRIPTORS FOR POINT PROCESSES BASED ON RUNS: THE MARKOVIAN ARRIVAL PROCESS[1]

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ABSTRACT

This paper is part of a broader investigation of properties of a point process that can be identified by imagining that the process is involved in a competition for the generation of runs of events. The general purpose of that methodology is to quantify the prevalence of gaps and bursts in realizations of the process. The Markovian arrival process (MAP) is highly versatile in qualitative behavior and its analysis is numerically tractable by matrix-analytic methods. It can therefore serve well as a benchmark process in that investigation. In this paper, we consider the MAP and a regular grid competing for runs of lengths at least r_1 and r_2, respectively. A run of length r in one of the processes is defined as a string of r successive events occurring without an intervening event in the other process.

This article is dedicated to the memory of Roland L. Dobrushin.

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1. A Comparison of Point Processes by Means of Runs

This article is part of a broader study of statistical descriptors of the behavior of point processes. One set of descriptors aims to quantify the prevalence of runs of events, and to that end, we need operational definitions of runs in point processes on the real line. One such a definition is based on the comparison of the path functions of two (independent) stationary point processes P_1 and P_2. We assume that both processes are regular and have simple events only. Positive integers r_1 and r_2 are specified. An event in the process P_2 is labeled as the end of a 1-*run* if it is preceded by at least r_1 events in the process P_1 for which there are no intervening events in P_2. We similarly define the end of a 2-*run* as an event in the process P_1 which is preceded by at least r_2 events in the process P_2 without intervening events in the process P_1.

We assign a time interval between successive ends of runs to the process which produces the final run of that interval. For example, if an interval terminates with a 1-*run*, we assign that interval to Process 1. We imagine a "competition" between P_1 and P_2 to produce respectively runs of lengths at least r_1 or r_2. We think of whichever process wins as being more "bursty" over that interval. For the given values of r_1 and r_2, the fraction of time assigned to each process in

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that competition is a global measure of their relative burstiness. For several values of r_1 and r_2, that fraction and several related quantities are easily estimated from actual or simulated traffic data. This is a clear advantage of such descriptors of burstiness. However, to explore the mathematical properties of these descriptors for familiar point processes and thereby to inspire confidence in their use, we examine the cases for which computable, tractable expressions for the descriptors can be derived.

The interest of the proposed approach can best be demonstrated by simulated examples and by using computer graphical displays. A verbal description of these examples is unavoidably lengthy and this is not the appropriate place for it. Concrete applications are to be the subject of future papers, but a brief description of the competition is given in Section 5.

This article deals with the case where P_1 is a Markovian arrival process (MAP), a broad generalization of the Poisson process that can be thoroughly analyzed by matrix-analytic techniques. The second process, which we think of as the *reference process*, is just the stationary renewal process with events that are a distance d apart. By a simple rescaling of time, we may obviously choose $d = 1$. We call P_2, the *stationary unit grid. That reference process is useful in the exploration of a single* MAP, *where the competition has a natural interpretation.* For expositions of the essential properties of the MAP, see Lucantoni [2] or Neuts [3]. Some issues in the quantification of the burstiness of point processes are discussed in Neuts [4].

The analysis of the case where P_2 is the stationary unit grid is not particularly easy. As is well-known from other contexts, for continuous-time models, the equidistant events preclude a simple Markovian analysis. There are interesting special results for the MAP in the role of P_1. These are useful to calibrate the run method of comparison when applied to other pairs of point processes.

As expected, there are major analytic simplifications when the reference process is Poisson. As always, the limitation in practical versatility of Poisson assumptions is severe. That makes the MAP such an appealing benchmark process. For the modest cost of performing calculations with matrix functions, we preserve much of the tractability of the Poisson process while retaining great modeling versatility.

In subsequent papers, we shall study the MAP with two types of events. This model, of particular interest, describes among other things the input and departure processes of a finite-capacity Markovian node in a queueing network.

Even when P_1 is Poisson, the analysis of run lengths is not entirely elementary. It requires careful attention to details, but once completed, it carries over easily to the MAP. In Section 2, we introduce the method for that case and obtain some explicit results. These are generalized to the MAP in Section 3 and the formal similarity of the derivations and results to the Poisson case is noted. Additional results for the MAP are derived in Section 4. Particular forms for the Poisson case are now simple special cases. Section 5 is devoted to a brief description of a computer-graphical exploration of the competition. Auxiliary results and the essential steps of lengthy derivations are found in the Appendix.

2. The Case of the Poisson Process

Let P_1 be a Poisson process of rate λ. To visualize the competition between P_1 and the stationary grid, we superimpose the two point processes. We mark the endpoints of unit intervals during which r_1 or more Poisson events occur, and also the first Poisson events following strings of at least r_2 unit intervals devoid of Poisson events. These events mark the endpoints of 1-*runs* and 2-*runs*, respectively.

For a transparent analysis which immediately carries over to the MAP, we define an embedded Markov renewal process. That embedded process, which has an uncountably infinite state space, is conceptually easy. Its transition epochs are the marked endpoints. Those corresponding to the end of 1-*runs* (Poisson) are labeled with state 1. To those that are endpoints of 2-*runs* and occur at the time u during the corresponding unit interval, we associate the state $(2,u)$. The bivariate sequence $\{(J_n,\tau_n)\}$ of the states $\{J_n\}$ at the successive marked epochs and of the sojourn times $\{\tau_n\}$, form a Markov renewal sequence on the state space $E=\{1\}\cup\{(2,u),$ with $0\le u<1\}$.

The case $r_1=1$ is somewhat special. In order not to detract attention from the general case, we first assume that r_1 is at least two. Next, we construct the transition probability operator G of the Markov renewal process.

The Transition Probability Operator

While elementary, that construction involves mixed discrete-continuous probability mass functions and some absorption time distributions in an associated Markov chain. To prepare the way for the case of the MAP, we discuss the construction in detail. The operator G is specified by the conditional probabilities of transitions out of the state 1, and those of transitions out of a state of the type $(2,u)$. Four cases are to be considered.

We use $G_{11}(x)$ for the probability $G_{11}(x)=P\{J_{n+1}=1;\tau_{n+1}\le x\mid J_n=1\}$, for $n\ge 1$ and $x\ge 0$. In a transition from state 1 to itself, that is, a 1-*run* followed by another, the sojourn time is necessarily a positive integer ν. It equals ν if and only if the νth interval of length one contains at least r_1 Poisson events, during all earlier unit intervals there are fewer than r_1 Poisson events, and there are no strings of r_2-1 consecutive intervals devoid of Poisson arrivals. The probability of that event is related to a simple (r_2+2)-state Markov chain with two absorbing states, whose transitions probability matrix $P(r_1,r_2)$ is of the form

$$P(r_1,6)=\left|\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & * \\ 0 & b(r_1-1) & a & 0 & 0 & 0 & 0 & c(r_1-1) \\ 1 & b(r_1-1) & 0 & a & 0 & 0 & 0 & c(r_1-1) \\ 2 & b(r_1-1) & 0 & 0 & a & 0 & 0 & c(r_1-1) \\ 3 & b(r_1-1) & 0 & 0 & 0 & a & 0 & c(r_1-1) \\ 4 & b(r_1-1) & 0 & 0 & 0 & 0 & a & c(r_1-1) \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right|,$$

here displayed for the representative value $r_2=6$. The non-zero elements are given by

$$a=e^{-\lambda},\quad b(r_1-1)=\sum_{j=1}^{r_1-1}e^{-\lambda}\frac{\lambda^j}{j!},\quad c(r_1-1)=1-a-b(r_1-1).$$

At this time, we do not yet make the simplifications resulting from the special structure of the stochastic matrix $P(r_1,r_2)$. It is preferable to emphasize its structural properties. We note that the transition probability matrix is of the form

$$P(r_1,r_2)=\left|\begin{array}{ccc} T(r_1,r_2) & \mathbf{T}^{\circ}(r_2) & \mathbf{T}'(r_1) \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{array}\right|.$$

We see that, for $x \geq 1$, $G_{11}(x)$ is the probability that, starting from the state 0, absorption into the state $*$ occurs before or at time $[x]$, where $[\cdot]$ denotes the integer part. If we write $\gamma(r_2 - 1)$, for the row vector of dimension $r_2 - 1$ with components $1, 0, \ldots, 0$, then

$$G_{11}(x) = \sum_{j=1}^{[x]} \gamma(r_2 - 1)[T(r_1, r_2)]^{j-1}\mathbf{T}'(r_1).$$

The sojourn times of transitions from state 1 to a state of the form $(2, u')$ are necessarily of the form $n + u'$, where n is an integer $\geq r_2 - 1$. To derive an explicit expression for the elementary conditional probability

$$d_{u'}G_{12}(u'; x) = P\{J_{n+1}\varepsilon[(2, u'), (2, u' + du')]; \tau_{n+1} \leq x \mid J_n = 1\},$$

we again consider the Markov chain with transition probability matrix $P(r_1, r_2)$. At some time ν, that Markov chain must be absorbed in its state r_2. A run of r_2 events in P_2 has now been completed. At some later time j, where $j \leq [x - u']$, the unit interval that contains at least one Poisson event must start and the first Poisson event must occur in the interval $(u', u' + du')$ counting from that epoch. In that case, there are $j - \nu$ unit intervals not containing Poisson events, in addition to the $r_2 - 1$ required for a 2-*run*. We see that

$$d_{u'}G_{12}(u'; x) = \sum_{j=r_2-1}^{[x-u']} \sum_{\nu=r_2-1}^{j} \gamma(r_2 - 1)[T(r_1, r_2)]^{\nu-1}\mathbf{T}^{\circ}(r_2)e^{-\lambda(j-\nu)}e^{-\lambda u'}\lambda du'$$

$$= \sum_{\nu=r_2-1}^{[x-u']} \gamma(r_2 - 1)[T(r_1, r_2)]^{\nu-1}\mathbf{T}^{\circ}(r_2) \left[\frac{1 - e^{-\lambda([x-u'] - \nu + 1)}}{1 - e^{-\lambda}}\right] e^{-\lambda u'}\lambda du'.$$

Next, the transition probabilities starting from a state $(2, u)$. First an observation that is obvious when P_1 is Poisson, yet it is essential to the embedded process being a Markov renewal process. A corresponding property also holds when P_1 is a MAP. It is then less obvious so we give an explicit argument in Lemma A-1 of the Appendix. Let the endpoint of the preceding 2-*run* fall at u in the unit interval during which it occurs. For the Poisson case, the number of *additional* events in P_2 in that unit interval is then independent of the past and has Poisson distribution with parameter $\lambda(1 - u)$.

If there are at least $r_1 - 1$ Poisson events in that interval, the next event in the stationary unit grid is also the endpoint of a 1-*run*. In this case, we have a $(2, u) \rightarrow 1$ transition after a sojourn time of length $1 - u$.

If there are fewer than $r_1 - 1$ Poisson events in that interval of length $1 - u$, then the sojourn times of the transitions, either to the state 1, or to some state $(2, u')$, are generated in exactly the same way as for the preceding two cases.

The expressions for the corresponding transition probabilities $G_{21}(u; x)$ and $G_{22}(u, u'; x)$ are therefore as follows:

$$G_{21}(u; x) = 1 - \sum_{j=0}^{r_1-2} e^{-\lambda(1-u)}\frac{[\lambda(1-u)]^j}{j!}$$

$$+ \sum_{j=0}^{r_1-2} e^{-\lambda(1-u)}\frac{[\lambda(1-u)]^j}{j!} \cdot \sum_{k=1}^{[x-1+u]} \gamma(r_2 - 1)[T(r_1, r_2)]^{k-1}\mathbf{T}'(r_1),$$

since that mass-function has atoms only at points of the form $1 - u + n$, with $n \geq 0$. By a similar argument,

$$d_{u'}G_{22}(u, u'; x) = \sum_{j=0}^{r_1-2} e^{-\lambda(1-u)}\frac{[\lambda(1-u)]^j}{j!}$$

$$\sum_{\nu=r_2-1}^{[x-1+u-u']} \gamma(r_2-1)[T(r_1,r_2)]^{\nu-1}\mathbf{T}^{\circ}(r_2)\left[\frac{1-e^{-\lambda([x-u']-\nu+1)}}{1-e^{-\lambda}}\right]e^{-\lambda u'}\lambda du'.$$

Laplace-Stieljes transforms of these mass-functions are evaluated in the Appendix. By setting $s=0$ in these transforms, we get the transition operator of the embedded discrete-time Markov process on the state space E. The Markov renewal process that concerns us here is a simple example of such processes on a general state space, see Çinlar [1]. For the present case, its analysis is entirely analogous to that of Markov renewal processes with finitely many states. The following statements are the salient results of that analysis. We note that

$$\phi(r_1,r_2)=\gamma(r_2-1)[I-T(r_1,r_2)]^{-1}\mathbf{T}'(r_1),$$

is the probability that the Markov chain with transition probability matrix $P(r_1,r_2)$ is absorbed in the state $*$. It is also clear that

$$\gamma(r_2-1)[I-T(r_1,r_2)]^{-1}\mathbf{T}^{\circ}(r_1=1-\phi(r_1,r_2),$$

is the absorption probability into the state r_2.

The quantity $\phi(r_1,r_2)$ is the first component of the vector $\mathbf{V}$ which is calculated in the Appendix. It is explicitly given by

$$\phi(r_1,r_2)=\frac{(1-a^{r_2-1})c(r_1-1)}{1-a-(1-a^{r_2-1})b(r_1-1)}.$$

For notational convenience, we set

$$\sum_{j=0}^{r_1-2} e^{-\lambda(1-u)}\frac{[\lambda(1-u)]^j}{j!}=S(u).$$

Theorem 1: *The transition probability operator of the embedded Markov process is given by:*

$$G_{11}^*(0)=\phi(r_1,r_2), \tag{1}$$

$$d_{u'}G_{12}^*(u';0)=[1-\phi(r_1,r_2)]\frac{e^{-\lambda u'}}{1-e^{-\lambda}}\lambda\, du',$$

$$G_{21}^*(u;0)=1-S(u)+S(u)\phi(r_1,r_2),$$

and

$$d_{u'}G_{22}^*(u,u';0)=S(u)[1-\phi(r_1,r_2)]\frac{e^{-\lambda u'}}{1-e^{-\lambda}}\lambda\, du'.$$

Clearly,

$$G_{11}^*(0)+\int_0^1 d_{u'}G_{12}^*(u';0)=1,$$

and

$$G_{21}^*(u;0)+\int_0^1 d_{u'}G_{22}^*(u,u';0)=1,$$

so that G is a proper Markovian transition operator. Its invariant probability measure consists of an atom g_1 at the state 1 and a density $g_2(u')$ on $[0,1]$. These are explicitly given by

$$g_1=\frac{c(r_1-1)}{(1-a)a^{r_2-1}+c(r_1-1)},\ \text{and}\ g_2(u)=K(r_1,r_2)\lambda e^{-\lambda u}, \tag{2}$$

for $0\le u<1$, where the multiplicative constant is given by

$$K(r_1,r_2)=\frac{a^{r_2-1}}{(1-a)a^{r_2-1}+c(r_1-1)}. \tag{3}$$

Proof: Formulas (1) are obtained by setting $s=0$ in the Laplace-Stieltjes transforms given in

the Appendix. The invariant probability measure on the state space E should satisfy the equations

$$g_1 = g_1\phi(r_1, r_2) + \int_0^1 g_2(\nu)[1 - [1 - \phi(r_1, r_2)]S(v)]dv, \tag{4}$$

$$g_2(u') = g_1[1 - \phi(r_1, r_2)]\frac{e^{-\lambda u'}}{1 - e^{-\lambda}}\lambda + [1 - \phi(r_1, r_2)]\int_0^1 g_2(v)S(v)dv\frac{e^{-\lambda u'}}{1 - e^{-\lambda}}\lambda, \tag{5}$$

and the normalization

$$g_1 + \int_0^1 g_2(v)dv = 1. \tag{6}$$

Equation (5) shows that $g_2(u) = K\lambda e^{-\lambda u}$, for some constant K. We substitute into equations (4-6) and notice that

$$\int_0^1 e^{-\lambda v}S(v)dv = e^{-\lambda}\sum_{j=0}^{r_1-2} \int_0^1 \frac{[\lambda(1-v)]^j}{j!}dv = \lambda^{-1}b(r_1 - 1). \tag{7}$$

Equations (4) and (5) reduce to

$$[1 - \phi(r_1, r_2)]g_1 = K\{1 - a - 1[1 - \phi(r_1, r_2)]b(r_1 - 1)\}.$$

Upon substitution of the expression for $\phi(r_1, r_2)$, the last equation further reduces to $a^{r_2-1}g_1 = K(r_1, r_2)c(r_1 - 1)$. The latter and the normalizing equation (6) lead to the stated formulas for g_1 and $K(r_1; r_2)$. □

In the Appendix, it is shown that the *fundamental mean* of the Markov renewal process is given by

$$E^*(r_1, r_2) = \frac{1}{(1-a)a^{r_2-1} + c(r_1 - 1)}. \tag{8}$$

This is an important quantity. Its inverse $[E^*(r_1, r_2)]^{-1}$ is the *rate of transitions* in the stationary version of the Markov renewal process. The ratio

$$\Psi_1(r_1, r_2) = \frac{g_1}{E^*(r_1, r_2)} = c(r_1 - 1).$$

is the *fraction of time spent in state* 1 in the stationary version of the Markov renewal process. Informally, it is the fraction of time in steady-state that the Poisson process is winning the competition to produce runs of its specified kind. Its inverse is the mean time between visits to State 1.

For completeness, we record the minor modifications needed when $r_1 = 1$. There is no need to consider the absorption time in a Markov chain. The endpoints of unit intervals containing at least one Poisson event mark the ends of 1-*runs*, and the first Poisson event after a string of at least $r_2 - 1$ empty unit intervals marks the end of a 2-*run*. The Laplace-Stieltjes transforms of the transition operator for that case are:

$$G_{11}^*(s) = (1 - e^{-\lambda})e^{-s}\frac{1 - e^{-(s+\lambda)(r_2-1)}}{1 - e^{-(s+\lambda)}}.$$

$$d_{u'}G_{12}^*(u'; s) = \frac{e^{-(s+\lambda)(r_2-1)}}{1 - e^{-(s+\lambda)}}e^{-(s+\lambda)u'}\lambda du'.$$

$$G_{21}^*(u; s) = e^{-(1-u)s}.$$

Direct transitions between states of the form $(2, u)$ and $(2, u')$ are impossible. The calculation of the invariant measure is as in the general case, but simpler. It yields the simple explicit expres-

sions $g_1 = (1 + a^{r_2 - 1})^{-1}$ and $g_2(u) = K\lambda e^{-\lambda u}$, for $0 \leq u < 1$, with $K(1, r_2) = a^{r_2 - 1}[(1-a)(1+a^{r_2-1})]^{-1}$. The row sum mean m_1 is given by

$$m_1 = \frac{a^{r_2-1}}{\lambda} + \frac{1 - a^{r_2-1}}{1-a},$$

and, clearly, $m_2(u) = 1 - u$. A direct calculation yields the fundamental mean, given by

$$E^*(1, r_2) = \frac{1}{(1-a)(1+a^{r_2-1})},$$

which is what we get when we set $b(r_1 - 1) = 0$ in the general formula. The ratio $\Psi_1(1, r_2)$ equals $1 - a = 1 - e^{-\lambda}$.

Remark: The fact that, in all cases, $\Psi_1(r_1, r_2)$ equals $c(r_1 - 1)$ is to be expected. The visits to State 1 are the ends of 1-*runs*. These are precisely the endpoints of unit intervals during which at least r_1 Poisson events occur. The times between such points have a geometric distribution with mean $[c(r_1 - 1)]^{-1}$.

3. The Case of the Markovian Arrival Process

We now generalize the preceding results to the case where P_1 is a continuous-time Markovian arrival process with parameter matrices D_0 and D_1. We assume some familiarity with the basic properties of the MAP as discussed, e.g., in Lucantoni [2] and Neuts [3]. The sum D of the parameter matrices is an $m \times m$ irreducible generator with stationary probability vector $\boldsymbol{\theta}$. The matrices $P(n;t)$, $n \geq 0$, are the familiar matrices associated with the counting process. They satisfy the system of differential equations

$$\begin{aligned} P'(0;t) &= P(0;t)D_0, \\ P'(n;t) &= P(n;t)D_0 + P(n-1;t)D_1, \quad n \geq 1. \end{aligned} \tag{9}$$

Their matrix generating function is given by

$$P^*(z;t) = \exp[(D_0 + zD_1)t], \quad t \geq 0.$$

The sequence of matrices $\{P(n;t)\}$ are a natural matrix generalization of the Poisson probability density. Methods for their numerical computation are discussed in Neuts and Li [5]. To emphasize the similarity to the Poisson probabilities and its role in the present analysis, we define the analogues of the quantities $a, b(r_1 - 1)$, and $c(r_1 - 1)$, but we use capital letters to emphasize that these objects are now $m \times m$ matrices

$$A = P(0;1) = \exp(D_0), \quad B(r_1 - 1) = \sum_{j=1}^{r_1 - 1} P(j;1),$$

$$C(r_1 - 1) = \sum_{j=r_1}^{\infty} P(j;1) = \exp(D) - A - B(r_1 - 1).$$

The states of the continuous-parameter Markov chain with generator D are called the *phases*. In modeling the competition for runs of lengths r_1 in P_1 or r_2 in the reference process P_2, the stationary unit grid, we construct a Markov renewal process analogous so that in Section 2. The only essential difference is that we must now also keep track of the phase at transition epochs. The state space is therefore

$$E = \{(1, j), 1 \leq j \leq m\} \cup \{(2, h, u), \text{ with } 1 \leq h \leq m, 0 \leq u < 1\}.$$

For notational convenience, we denote the first set of states by **1**, and a set of states with fixed u in the second by $(\mathbf{2}, u)$.

Lemma A-1 implies that the epochs labeled at the ends of runs, the future of the process is conditionally independent of its past, given the state in **1** or $(\mathbf{2}, u)$. The sequence of states and corresponding sojourn times therefore defines a Markov renewal process with the given states. We study the quantities associated with its stationary version as was done for the Poisson case.

The analogue of the stochastic matrix $P(r_1, r_2)$ is now the matrix displayed here for the representative value $r_2 = 6$.

$$P(r_1,6) = \begin{array}{c|ccccccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & * \\ \hline \mathbf{0} & B(r_1-1) & A & 0 & 0 & 0 & 0 & C(r_1-1) \\ \mathbf{1} & B(r_1-1) & 0 & A & 0 & 0 & 0 & C(r_1-1) \\ \mathbf{2} & B(r_1-1) & 0 & 0 & A & 0 & 0 & C(r_1-1) \\ \mathbf{3} & B(r_1-1) & 0 & 0 & 0 & A & 0 & C(r_1-1) \\ \mathbf{4} & B(r_1-1) & 0 & 0 & 0 & 0 & A & C(r_1-1) \\ \mathbf{5} & 0 & 0 & 0 & 0 & 0 & I & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}.$$

This matrix has the same structural form as in the Poisson case:

$$P(r_1, r_2) = \begin{vmatrix} T(r_1, r_2) & T^{\circ}(r_2) & T'(r_1) \\ 0 & I & 0 \\ 0 & 0 & I \end{vmatrix}.$$

As we need to keep track of the phase at the absorption time, the sets $\mathbf{r}_2 - 1$ and $*$ now consist of m states. The entries $T'(r_1)$ and $T^{\circ}(r_2)$ are therefore matrices of dimensions $m(r_2-1) \times m$. By $\Gamma(r_2-1)$, we denote a matrix of dimensions $m \times m(r_2-1)$ of the form $I, 0, \ldots, 0$, where I is the identity matrix of order m. That matrix enables us to calculate simultaneously the absorption probabilities from each of the initial states $(0, h)$, where h denotes the phase of the MAP.

The Transition Operator of the Markov Renewal Process

For convenience, we give the expressions for the transition probabilities of the Markov renewal process embedded at the ends of the runs immediately in transform forms. The derivation of the most complicated of these expressions, that in formula (12), is given in the Appendix; the others are similar but simpler. The analysis that follows applies when $r_1 \geq 2$ and $r_2 \geq 1$. The simple special case where $r_1 = 1$ is treated at the end.

The matrix analogues of the transforms for the transition operator in the Poisson case are as follows:

$$G_{11}^{*}(s) = e^{-s}\Gamma(r_2-1)[I - e^{-s}T(r_1, r_2)]^{-1}T'(r_1). \tag{10}$$

$$d_{u'}G_{12}^{*}(u'; s) = \sum_{n=1}^{\infty} e^{-s(n+u')} \sum_{\nu=1}^{n} \Gamma(r_2-1)[T(r_1, r_2)]^{\nu-1}T^{\circ}(r_2) \cdot \exp[(n-\nu)D_0]\exp(D_0 u')D_1 du' \tag{11}$$

$$= e^{-s}\Gamma(r_2-1)[I - e^{-s}T(r_1, r_2)]^{-1}T^{\circ}(r_2)[I - e^{-su'}\exp(D_0)]^{-1}e^{-s}\exp(D_0 u')D_1\, du' \cdot$$

$$G_{21}^{*}(u; s) = [\exp[D(1-u)] - S(u)]e^{-s(1-u)} \tag{12}$$

$$+ S(u)e^{-s(1-u)}e^{-s}\Gamma(r_2-1)[I-e^{-s}T(r_1,r_2)]^{-1}T'(r_1)$$

and

$$d_{u'}G_{22}^{*}(u,u';s) = S(u)e^{-s(1-u)}e^{-s}\Gamma(r_2-1)[I-e^{-s}T(r_1,r_2)]^{-1}T^{\circ}(r_1) \tag{13}$$

$$\cdot[I-e^{-s}\exp(D_0)]^{-1}e^{-su'}\exp(D_0u')D_1du'.$$

Setting $s=0$ in the preceding formulas, we obtain the transition probability operator of the embedded Markov chain. We shall next derive its invariant probability measure explicitly. First, we introduce some matrices that are the analogues of the absorption probabilities in the finite Markov chain of Section 2.

We define the matrices:

$$\Phi'(r_1,r_2) = \Gamma(r_2-1)[I-T(r_1,r_2)]^{-1}T'(r_1), \tag{14}$$

$$\Phi^{\circ}(r_1,r_2) = \Gamma(r_2-1)[I-T(r_1,r_2)]^{-1}T^{\circ}(r_1). \tag{15}$$

By solving the system of matrix equations

$$V_i = C(r_1-1) + B(r_1-1)V_0 + AV_{i+1},$$

for $0 \le i \le r_2-2$, with $V_{r_2-1}=0$, we find that

$$\Phi'(r_1,r_2) = [I-A-(I-A^{r_2-1})B(r_1-1)]^{-1}(I-A^{r_2-1})C(r_1-1). \tag{16}$$

Similarly,

$$\Phi^{\circ}(r_1,r_2) = [I-A-(I-A^{r_2-1})B(r_1-1)]^{-1}(I-A)A^{r_2-1}. \tag{17}$$

The Invariant Probability Measure

The invariant probability measure of the embedded Markov chain of the Markov renewal process consists of a positive row vector $\mathbf{g}_1$ and a vector $\mathbf{g}_2(u)$ of density functions on $(0,1)$. By virtue of equations (10-13), these should satisfy the steady-state equations

$$\mathbf{g}_1 = \mathbf{g}_1\Phi'(r_1,r_2) + \int_0^1 \mathbf{g}_2(u)\{\exp[D(1-u)] - S(u)\}du \tag{18}$$

$$+ \int_0^1 \mathbf{g}_2(u)S(u)du\Phi'(r_1,r_2),$$

and

$$\mathbf{g}_2(v) = \mathbf{g}_1\Phi^{\circ}(r_1,r_2)[I-\exp(D_0)]^{-1}\exp(D_0v)D_1 \tag{19}$$

$$+ \int_0^1 \mathbf{g}_2(u)S(u)du\Phi^{\circ}(r_1,r_2)[I=\exp(D_0)]^{-1}\exp(D_0v)D_1,$$

for $0 \le v < 1$, and also the normalizing equation

$$\mathbf{g}_1\mathbf{e} + \int_0^1 \mathbf{g}_2(u)\mathbf{e}du = 1. \tag{20}$$

Theorem 2: *The invariant probability measure is explicitly determined by*

$$\boldsymbol{g}_1 = k^*\boldsymbol{\theta}C(r_1-1), \text{ and } \boldsymbol{K}(r_1,r_2) = k^*\boldsymbol{\theta}A^{r_2-1}, \tag{21}$$

where $k^* = [\boldsymbol{\theta}[(I-A)A^{r_2-1}+C(r_1-1)]\mathbf{e}]^{-1}$. *By direct calculation, we shall subsequently verify that* k^* *is also the fundamental mean of the Markov renewal process. We recall that* $\boldsymbol{\theta}$ *is the stationary probability vector of the infinitesimal generator* D.

Proof: While these matrix formulas are completely analogous to the scalar expressions for Poisson case, their validation requires somewhat subtle matrix calculations. We present the details of the major steps. Equation (19) shows that $\mathbf{g}_2(v) = \mathbf{K}(r_1, r_2)\exp(D_0 v)D_1$, for some m-vector of constants $\mathbf{K}(r_1, r_2)$. Substitution of that form results in the equations

$$\mathbf{g}_1[I - \Phi'(r_1, r_2)] = \mathbf{K}(r_1, r_2)[C(r_1 - 1) + B(r_1 - 1)\Phi'(r_1, r_2)], \tag{22}$$

$$\mathbf{g}_1\Phi^{\circ}(r_1, r_2) = \mathbf{K}(r_1, r_2)[I - A - B(r_1 - 1)\Phi^{\circ}(r_1, r_2)], \tag{23}$$

and

$$\mathbf{g}_1\mathbf{e} + \mathbf{K}(r_1, r_2)(I - A)\mathbf{e} = 1. \tag{24}$$

We enter the expressions for $\Phi'(r_1, r_2)$ and $\Phi^{\circ}(r_1, r_2)$ into (22) and (23). That yields the respective equations

$$\mathbf{g}_1[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}\{I - A - (I - A^{r_2 - 1})[\exp(D) - A]\} \tag{25}$$
$$= \mathbf{K}(r_1, r_2)\{C(r_1 - 1) + B(r_1 - 1)[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}(I - A^{r_2 - 1})C(r_1 - 1)\},$$

and

$$\mathbf{g}_1[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}(I - A)A^{r_2 - 1} \tag{26}$$
$$= \mathbf{K}(r_1, r_2)\{I - A - B(r_1 - 1)[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}(I - A)A^{r_2 - 1}\}.$$

We notice that a factor $I - A$ cancels in both sides of (26). In simplifying in the following equations, we use the equality $I - A - B(r_1 - 1) - C(r_1 - 1) = \exp(D)$. Equation (25) becomes

$$\mathbf{g}_1[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}A^{r_2 - 1} \tag{27}$$
$$= \mathbf{K}(r_1, r_2)\{I - B(r_1 - 1)[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}A^{r_2 - 1}\},$$

and we replace the left-hand side of that expression in the corresponding term on the left-hand side of (26). After some calculation, that equation yields that

$$[\mathbf{g}_1 + \mathbf{K}(r_1, r_2)B(r_1 - 1)][I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}[I - \exp(D)\,] = \mathbf{0}. \tag{28}$$

This implies that the pre-factor of $I - \exp(D)$ must be a scalar multiple of the stationary probability vector $\boldsymbol{\theta}$, or

$$\mathbf{g}_1 + \mathbf{K}(r_1, r_2)B(r_1 - 1) = k^*\boldsymbol{\theta}[I - A - (I - A^{r_2 - 1})B(r_1 - 1)]. \tag{29}$$

After using that equality to eliminate $\mathbf{g}_1$ from (25), we find that $\mathbf{K}(r_1, r_2) = k^*\boldsymbol{\theta}A^{r_2 - 1}$. This further entails $\mathbf{g}_1 = k^*\boldsymbol{\theta}C(r_1 - 1)$. The explicit formula for the constant k^* is readily deduced from the normalizing equation (20). □

Calculation of the Row Sum Means

As in the scalar case where P_1 is the Poisson process, we need to calculate the row sum means of the semi-Markovian transition probability operator G. We proceed directly from the transforms in formulas (10-13). In each, we differentiate with respect to s, set $s = 0$, and change sign. Next, we "sum" over the indices of the second state. For the transitions to the macro-state **1**, that amounts to postmultiplication by a column vector $\mathbf{e}$ of ones. The results are column vectors $\mathbf{m}_{11}$ and $\mathbf{m}_{21}(u)$, $0 \le u < 1$, all the dimension m. For transitions into a macro-state of the form $(\mathbf{2}, u')$, we need to integrate between 0 and 1 with respect to u', followed by postmultiplication by $\mathbf{e}$. These operations lead to column vectors $\mathbf{m}_{12}$ and $\mathbf{m}_{22}(u)$, $0 \le u < 1$, of dimension m. While the calculations are straightforward, they are substantially more complicated than in the Poisson case. They require thorough facility with the matrix formalism of the MAP. We state the final expressions in the form of a lemma. The salient calculational steps are given in the Appendix.

Lemma 1: *The mean sojourn times and the row sum means of the transition operator G are given by the formulas:*

$$\mathbf{m}_{11} = \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T'(r_1)\mathbf{e} + \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-2} T'(r_1)\mathbf{e}. \tag{30}$$

$$\mathbf{m}_{12} = \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-2} T^{\circ}(r_1)\mathbf{e} + \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^{\circ}(r_1)(-D_0)^{-1}\mathbf{e}. \tag{31}$$

$$\mathbf{m}_{21}(u) = (1 - u)\mathbf{e} - (1 - u)S(u)\mathbf{e} \tag{32}$$
$$+ (1 - u)S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T'(r_1)\mathbf{e} + S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-2} T'(r_1)\mathbf{e}.$$

$$\mathbf{m}_{22}(u) = (1 - u)S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^{\circ}(r_1)\mathbf{e} \tag{33}$$
$$+ S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-2} T^{\circ}(r_1)\mathbf{e} + S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^{\circ}(r_1)(-D_0)^{-1}\mathbf{e}.$$

As the matrix $P(r_1, r_2)$ is stochastic, we have that

$$[I - T(r_1, r_2)]^{-1}[T'(r_1) + T^{\circ}(r_1)]\mathbf{e} = \mathbf{e},$$

where $\mathbf{e}$ is a column of ones of dimension $m(r_2 - 1)$.

That serves to simplify the sum $\mathbf{m}_{11} + \mathbf{m}_{12} = \mathbf{m}_1$ to

$$\mathbf{m}_1 = \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e} \tag{34}$$
$$+ \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^{\circ}(r_1)(-D_0)^{-1}\mathbf{e},$$

and the sum $\mathbf{m}_{21}(u) + \mathbf{m}_{22}(u) = \mathbf{m}_2(u)$ to

$$\mathbf{m}_2(u) = (1 - u)\mathbf{e} + S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e} \tag{35}$$
$$+ S(u)\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^{\circ}(r_1)(-D_0)^{-1}\mathbf{e}.$$

The vector

$$\mathbf{R}_0 = \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e},$$

is found by solving a simply structured set of linear equations. It is given by

$$\mathbf{R}_0 = [I - A - (I - A^{r_2 - 1})B(r_1 - 1)]^{-1}(I - A^{r_2 - 1})\mathbf{e}. \tag{36}$$

We now have explicit expressions for all the ingredients in the formulas

$$\mathbf{m}_1 = \mathbf{R}_0 + \Phi^{\circ}(r_1, r_2)(-D_0)^{-1}\mathbf{e},$$

and

$$\mathbf{m}_2(u) = (1 - u)\mathbf{e} + S(u)\mathbf{R}_0 + S(u)\Phi^{\circ}(r_1, r_2)(-D_0)^{-1}\mathbf{e}.$$

Upon substituting these into the formula

$$E^*(r_1, r_2) = \mathbf{g}_1\mathbf{m}_1 + \int_0^1 \mathbf{g}_2(u)\mathbf{m}_2(u)du,$$

for the *fundamental mean*, a fairly, lengthy calculation, summarized in Appendix, leads to the equality $E^*(r_1, r_2) = k^*$. In that calculation, we repeatedly use the equality $A + B(r_1 - 1) + C(r_1 - 1) = \exp(D)$.

4. More Complex Descriptors

The descriptors of the MAP obtained so far, such as the mean sojourn times of the transitions between the states, are only moderately informative. Next, we study more complex descriptors based on a detailed analysis of the recurrence times of the states in the set **1**. The time period between two successive visits to the set **1** will be called a 1-*cycle*.

The formal similarity of the general MAP to the special case of the Poisson process is now clear. Any special formulas for the Poisson case are obtained by using the scalar coefficient matrices $D_1 = -D_0 = \lambda$. Henceforth, the main discussion deals with the MAP only.

The expression for the fraction of time $\Psi_1(r_1, r_2)$ that the MAP wins the competition is entirely analogous to that of the Poisson case. It is given by

$$\Psi_1(r_1, r_2) = \frac{\mathbf{g}_1\mathbf{e}}{E^*(r_1, r_2)} = \boldsymbol{\theta} c(r_1 - 1)\mathbf{e}. \tag{37}$$

This formula also has an intuitive explanation in terms of the mean of an arbitrary waiting time between unit intervals during which, in the MAP, at least r_1 events occur.

1-Cycles

The bivariate sequence of the phases at the starts of these recurrence times and their durations is a Markov renewal sequence, which we shall call the process of 1-*cycles*. A typical 1-*cycle* consists either of a single transition from **1** to **1**, or may be made up of a string of intervals that are the successive transitions from **1** to some state in $(\mathbf{2}, u_1)$, thence to $(\mathbf{2}, u_2)$, and so on to a state in $(\mathbf{2}, u_n)$. From that state, there is a final interval in which the MAP wins and the end of the cycle is marked by a return to the set **1**.

We shall study such descriptors as the number of intervals terminating in a state in $(\mathbf{2}, u)$, the fraction of time during the 1-*cycle* that the process P_2 wins, and others. To that end, we define a number of matrices to enable us to write the transform formulas in concise forms that clearly bring out their structure.

For the transform of transition probability matrix of the process of 1-*cycles*, we use the notation $\Xi(\xi, s)$. The element $\Xi_{jj'}(\xi, s)$ is the joint Laplace-Stieltjes transform of the conditional probability that the 1-*cycle* ends with the MAP in the phase j', that during the 1-*cycle*, the process P_2 wins during a time at most x, the MAP wins during a time at most y, given that the 1-*cycle* starts with the MAP in the phase j.

The matrix $\Xi(\xi, s)$ is obtained as the sum of a series of matrices $\{\Xi_n(\xi, s)\}$. Its first term $\Xi_0(\xi, s)$ is the matrix $G^*_{11}(s)$; it corresponds to a transition from **1** to **1**. The subsequent terms each correspond to the different numbers of intervals during which the process P_2 wins. We next show how these terms are calculated, but first we define some useful matrices:

$$\Phi'(r_1, r_2; s) = e^{-s}\Gamma(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}T'(r_1), \tag{38}$$

$$\Phi^\circ(r_1, r_2; s) = e^{-s}\Gamma(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}T^\circ(r_2). \tag{39}$$

Formulas (10-13) may be concisely rewritten as

$$G^*_{11}(s) = \Phi'(r_1, r_2; s), \tag{40}$$

$$d_{u'}G^*_{12}(u'; s) = \Phi^\circ(r_1, r_2; s)[I - e^{-s}\exp(D_0)]^{-1}e^{-su'}\exp(D_0u')D_1du', \tag{41}$$

$$G^*_{21}(u; s) = [\exp[D(1-u)] - S(u)]e^{-s(1-u)} + S(u)e^{-s(1-u)}\Phi'(r_1, r_2; s), \tag{42}$$

and

$$d_{u'}G^*_{22}(u, u'; s) = S(u)e^{-s(1-u)}\Phi^\circ(r_1, r_2; s) \cdot [I - e^{-s}\exp(D_0)]^{-1}e^{-su'}\exp(D_0u')D_1du'. \tag{43}$$

In what follows, we shall also need the matrices

$$Q(\nu; \xi, s) = \int_0^1 e^{-\xi u}\exp(D_0u)D_1P(\nu; 1-u)e^{-s(1-u)}du,$$

for $\nu \geq 0$, and where $P(v; 1-u)$ is as defined in (9). Write $B^*(r_1 - 1; \xi, s)$ for the matrix

$$B^*(r_1 - 1; \xi, s) = \sum_{\nu=0}^{r_1-2} Q(\nu; \xi, s), \tag{44}$$

and note that $B^*(r_1 - 1; \xi, \xi) = e^{-\xi}B(r_1 - 1)$, in terms of the matrix $B(r_1 - 1)$. We shall also

write $B^*(\infty, \xi, s)$ for the matrix

$$B^*(\infty; \xi, s) = \int_0^1 e^{-\xi u} \exp(D_0 u) D_1 \exp[D(1-u)] e^{-s(1-u)} du, \tag{45}$$

and we note that $B^*(\infty, \xi, \xi) = e^{-\xi}[\exp(D) - \exp(D_0)]$.

Theorem 3: *The matrices* $\{\Xi_n(\xi, s)\}$ *are given by*

$$\Xi_0(\xi, s) = \Phi'(r_1, r_2; s), \tag{46}$$

and for $n \geq 1$,

$$\Xi_n(\xi, s) = \left[\Phi^\circ(r_1, r_2; \xi)[I - e^{-s}\exp(D_0)]^{-1} e^{-\xi} B(r_1 - 1)\right]^{n-1} \tag{47}$$

$$\cdot \Phi^\circ(r_1, r_2; \xi)[I - e^{-s}\exp(D_0)]^{-1}\{B^*(\infty; \xi, s) - B^*(r_1 - 1; \xi, s)[I - \Phi'(r_1, r_2; s)]\}.$$

Proof: For $n \geq 1$, we consider the elementary conditional probability $d_u L_{jj'}(n; x, u)$ that in the 1-*cycle*, the process P_2 wins the nth time in succession before time $x > u$ and that the event in the MAP which concludes the nth 2-*run* occurs in the subinterval $(u, u + du)$ of the next unit interval with the MAP in phase j', given that the MAP is in the phase j at the start of the 1-*cycle*.

The Laplace-Stieltjes transform (for which we use the variable ξ) of the corresponding matrix $d_u L^*(n; \xi, u)$ is given by $d_u G_{12}^*(u, \xi)$, for $n = 1$. We show by induction that, for $n \geq 2$, the trans form is given by

$$d_u L^*(n; \xi, u) = \left[\Phi^\circ(r_1, r_2; \xi)[I - e^{-s}\exp(D_0)]^{-1} e^{-\xi} B(r_1 - 1)\right]^{n-1}$$
$$\cdot \Phi^\circ(r_1, r_2; \xi)[I - e^{-s}\exp(D_0)]^{-1} e^{-\xi u} \exp(D_0 u) D_1 du.$$

To carry out the induction, it suffices to calculate the integral

$$\int_0^1 d_v L^*(n-1; \xi, v) d_u G_{22}^*(v, u; \xi),$$

which, apart from matrices that can be factored out, reduces to the integral we have defined as the matrix $B^*(r_1 - 1; \xi, \xi)$.

To obtain $\Xi_n(\xi, s)$, we perform the final integration

$$\int_0^1 d_v L^*(n; \xi, v) G_{21}^*(v; s),$$

which yields the state formula for $\Xi_n(\xi, s)$. □

From this result, we can now derive explicit matrix formulas for several interesting descriptors of the MAP. As a matter of verification, we check that the matrix $\Xi(0, 0)$ is stochastic. By substitution the explicit formulas for the various matrices in the expression

$$\Xi(0,0) = \Phi'(r_1, r_2) + [I - \Phi^\circ(r_1, r_2)(I - A)^{-1} B(r_1 - 1)]^{-1}$$
$$\cdot \{\exp(D) - A - B(r_1 - 1)[I - \Phi'(r_1, r_2)]\},$$

that matrix simplifies to

$$\Xi(0,0) = [I - A - B(r_1 - 1)]^{-1} C(r_1 - 1), \tag{48}$$

which is clearly a (positive) stochastic matrix. Its invariant probability vector is $[\boldsymbol{\theta} C(r_1 - 1)\mathbf{e}]^{-1} \boldsymbol{\theta} C(r_1 - 1)$, which, as expected, is also $(\mathbf{g}_1 \mathbf{e})^{-1} \mathbf{g}_1$. $\Xi(0,0)$ is the transition probability matrix of the phases of the MAP at the successive visits to set **1**.

The Number of 2-Runs during a 1-Cycle

The sequence of sub-stochastic matrices $\{\Xi_n = \Xi_n(0,0)\}$ define a discrete semi-Markov matrix. They are given by

$$\Xi_0 = \Phi'(r_1, r_2), \tag{49}$$

$$\Xi_n = \left[\Phi^\circ(r_1,r_2)(I-A)^{-1}B(r_1-1)\right]^{n-1}\Phi^\circ(r_1,r_2)(I-A)^{-1}[C(r_1-1)+B(r_1-1)\Phi'(r_1,r_2)],$$

for $n \geq 1$. There are no appreciable simplifications upon substitution of the explicit expressions for the Φ-matrices.

The probability density $\Psi(n; r_1, r_2)$ of the number of times the process P_2 wins during an *arbitrary* 1-*cycle*, given by

$$\Psi(n; r_1, r_2) = \frac{\boldsymbol{\theta} C(r_1-1)\Xi_n \mathbf{e}}{\boldsymbol{\theta} C(r_1-1)\mathbf{e}}, \tag{50}$$

for $n \geq 0$, is a useful, tractable descriptor. The quantity $\Psi(n; r_1, r_2)$ is the probability that there are n gaps of length at least $r_2 - 1$ in a typical 1-*cycle* in the MAP.

Moments and Dependence

Clearly, the index n in $\Xi_n(\xi, s)$ is just a way of keeping track of another useful random variable, the number of 2-*runs* during the 1-*cycle*. The transform

$$\Xi(\xi, s, z) = \sum_{n=0}^{\infty} \Xi_n(\xi, s) z^n,$$

is immediately obtained in a closed form by noting that

$$\sum_{n=0}^{\infty}\left[\Phi^\circ(r_1,r_2;\xi)[I-e^{-s}\exp(D_0)]^{-1}e^{-\xi}B(r_1-1)\right]^n z^n$$
$$= \left\{I - z\left[\Phi^\circ(r_1,r_2;\xi)[I-e^{-s}\exp(D_0)]^{-1}e^{-\xi}B(r_1-1)\right]\right\}^{-1}.$$

The calculation of various lower order moments from the transform is a matter of a routine (if involved) computation. It is possible, in principle, to express these quantities in terms of the coefficient matrices of the MAP. However, in doing so there are few analytic simplifications so that we prefer not to record the resulting formulas here. Without numerical algorithms, little insight can be gained from them, but the standard procedures for calculating these can be readily translated into algorithms for numerical computation. To compute the moment matrices, we differentiate twice in the matrix expression for $\Xi(\xi, s, z)$ and set $\xi = s = 0$, and $z = 1$ to obtain systems of linear equations. Their coefficients involve only vectors and matrices encountered earlier and are in a sufficiently modular form for numerical work.

In the cases where the 1-*cycles* can be expected to be long, say, because r_1 is chosen fairly large, we can also expect that the behavior of the MAP during its successive 1-*cycles* will be only weakly dependent. The most readily available analytic tool to examine that dependence is the serial correlation of sequences such as the durations of successive 1-*cycles* and of the numbers of 2-*runs* during these.

Computationally useful expressions for these correlations are again routinely derived from the joint transform

$$[\boldsymbol{\theta} C(r_1-1)\mathbf{e}]^{-1}\boldsymbol{\theta} C(r_1-1)\Xi(\xi_1, s_1, z_1)\ldots\Xi(\xi_k, s_k, z_k)\mathbf{e},$$

of k successive 1-*cycles*. Well-known results for Markov renewal processes show that a readily available indication of weak correlation is how rapidly the successive powers of the stochastic matrix $\Xi(0,0,1) = \Xi(0,0)$ converge to the limit matrix $[\boldsymbol{\theta} C(r_1-1)\mathbf{e}]^{-1}\mathbf{e}\boldsymbol{\theta} C(r_1-1)$. As that matrix is easily computed, that observation is useful in an initial numerical exploration of this issue.

5. Motivation and Applications

Before we undertook the present analytic study of the competition for runs we examined for some familiar MAPs by computer graphical displays. These suggested that this approach offers a conceptually simple way of bringing out important differences between MAPs. The classical descriptors, based on second order properties, of these did not differ significantly.

For easily simulated stationary point processes, such as the MAP, the data needed to highlight the competition are easily collected. It is sufficient to attach the value of a counter to each event in the given process and in the reference process. That counter, which can conveniently take positive integer values for the first and negative ones for the second, keeps track of how many uninterrupted events of each type have currently occurred. For these counts, the 1-*runs* and 2-*runs* corresponding to various values of r_1 and r_2 are easily identified. The effect of these parameters on a long simulation record can be examined without having to repeat the simulation and without making several passes through a long record of data.

To simplify interpretation of the results, we generated streams of several thousand events for various stationary MAPs of unit rate. The streams of interest and the reference process (the grid, in the present case) all had a common rate. We fixed r_1 at some integer value of modest size, such as eight or ten. For each event stream, we graphically recorded the ends of the successive 1-*runs* and 2-*runs* by printing a symbol 1 or 2. That was done for several increasing values of r_2.

The resulting graphs and elementary summary statistics provided clear pictures that were remarkably consistent for different simulations with the same parameter matrices. In contrast, differences between moderately distinct MAPs stood out vividly. The successive 1-*cycles* typically involved several hundreds of events and took on a small number of typical visual forms. The prevalence of gaps (as measured by the occurrence of 2-*runs* during 1-*cycles*) stood out clearly and, as is to be expected, decreased with increasing values of r_2. These, and the occurrence of bursts in affecting the frequency of 1-*runs*, clearly distinguished between the MAPs.

We concluded that the competition for runs brings out salient local properties of the realizations of these point processes, features that are missed by the averaging inherent in descriptors based on second order properties or on spectral methods. While that conclusion is based on computer experiments, we are confident that numerical computations of the descriptors in Section 4 will substantiate these findings. The present results, and others to be included in subsequent articles, should serve in a formal quantification of these experimental perceptions.

The results in Section 4 show that analytic use of the competition for runs as a tool in the investigation of MAPs requires considerable numerical work. The transform formulas obtained there clearly indicate which embedded Markov renewal processes need to be studied in detail. However, their complex forms show that this is best done by modular numerical algorithms than by explicit formulas. Computer implementation remains to be done, but the resulting, well-computable descriptors should prove very useful.

The input and departure processes of a finite-capacity Markovian node in a queueing network, to which we alluded in Section 1, offer examples of two dependent processes for which the competition for runs is worth examining. The features of the competition bring out how the number of jobs within the node "pulsates" over time. Extensive numerical computations for small nodes, based on the MAP with two event types, should serve to quantify these features for benchmark examples. For less structured stationary point processes, it appears that only detailed simulation studies can indicate whether the qualitative findings for MAPs have wider applicability.

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Appendix

1. Notes of Section 2

Absorption Probabilities: The absorption into the state $*$ for the Markov chain $P(r_1, r_2)$ are found by solving the equations

$$\mathbf{V} - T(r_1, r_2)\mathbf{V} = \mathbf{T}'(r_1 - 1)$$

which may be written as

$$V_i = c(r_1 - 1) + b(r_1 - 1)V_0 + aV_{i+1},$$

for $0 \leq i \leq r_2 - 2$, with $V_{r_2 - 1} = 0$. Upon backward substitution, we find that

$$V_i = \frac{(1 - a^{r_2 - i - 1})c(r_1 - 1)}{1 - a - (1 - a^{r_2 - 1})b(r_1 - 1)}.$$

Transforms: For calculational convenience, we evaluate the Laplace-Stieltjes transform of the transition probability operator G in Section 2. Since $G_{11}(x)$ is a mass-function with increments at the positive integers, its Laplace-Stieltjes transform is readily given by

$$G_{11}^*(s) = \sum_{n=1}^{\infty} e^{-ns}\boldsymbol{\gamma}(r_2 - 1)[T(r_1, r_2)]^{n-1}\mathbf{T}'(r_1) \tag{A-1}$$

$$= e^{-s}\boldsymbol{\gamma}(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}\mathbf{T}'(r_1).$$

Since the sojourn times of transitions from state 1 to a state of the form $(2, u')$ are of the form $n + u'$, where n is an integer $\geq r_2 - 1$, the transform $d_{u'}G_{12}^*(u'; s)$ is given by

$$d_{u'}G_{12}^*(u'; s) = \sum_{n=1}^{\infty} e^{-sn - su'} \sum_{\nu=1}^{n} \boldsymbol{\gamma}(r_2 - 1)[T(r_1, r_2)]^{\nu-1}\mathbf{T}^0(r_2)e^{-\lambda(n-\nu)}e^{-\lambda u'}\lambda du' \tag{A-2}$$

$$= e^{-s}\boldsymbol{\gamma}(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}\mathbf{T}^0(r_2)\frac{e^{-(s+\lambda)u'}}{1 - e^{-(s+\lambda)}}\lambda du'.$$

Similarly,

$$G^*_{21}(u;s) = [1 - S(u)]e^{-s(1-u)} + S(u)e^{-s(1-u)}e^{-s}\boldsymbol{\gamma}(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}\mathbf{T}'(r_1). \tag{A-3}$$

and

$$d_{u'}G^*_{22}(u, u'; s) = S(u)e^{-s(1-u)}e^{-s}\boldsymbol{\gamma}(r_2 - 1)[I - e^{-s}T(r_1, r_2)]^{-1}\mathbf{T}'(r_1)\frac{e^{-(s+\lambda)u'}}{1 - e^{-(s+\lambda)}}\lambda du'. \tag{A-4}$$

Means: As in the theory of finite-state Markov renewal processes, here also we need to calculate the mean sojourn times and the "row sum" means of the transition probability operator. To that end, we evaluate minus the derivative with respect to s, at $s = 0$, in each of the preceding transforms. We leave these derivatives in a form that avoids cumbersome calculations in the sequel. The results are denoted by the functions m_{11}, $m_{12}(u')$, $m_{21}(u)$, and $m_{22}(u, u')$. We obtain:

$$m_{12} = \boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-2}\mathbf{T}'(r_1) \cdot \tag{A-5}$$

$$d_{u'}m_{12}(u') = \left[\boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-2}\mathbf{T}^{\circ}(r_2) + [1 - \phi(r_1, r_2)] \cdot \frac{e^{-\lambda} + (1 - e^{-\lambda})u'}{1 - e^{-\lambda}}\right]\frac{e^{-\lambda u'}}{1 - e^{-\lambda}}\lambda\, du' \cdot \tag{A-6}$$

$$m^*_{21}(u) = 1 - u - S(u)(1 - u)[1 - \phi(r_1, r_2)] + S(u)\boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-2}\mathbf{T}'(r_1). \tag{A-7}$$

$$d_{u'}m_{22}(u, u') = S(u)\left[[1 - \phi(r_1, r_2)][1 - u + \frac{e^{-\lambda} + (1 - e^{-\lambda})u'}{1 - e^{-\lambda}}] + \boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-2}\mathbf{T}^{\circ}(r_2)\right]\frac{e^{-\lambda u'}}{1 - e^{-\lambda}}\lambda\, du'. \tag{A-8}$$

We also evaluate the "row sum" means, which are given by

$$m_1 = m_{11} + \int_0^1 m_{12}(u')du',$$

and

$$m_2(u) = m_{21}(u) + \int_0^1 d_{u'}m_{22}(u, u').$$

In calculating these, notice that, since $P(r_1, r_2)$ is stochastic, we have the equality

$$[I - T(r_1, r_2)]^{-1}[\mathbf{T}^{\circ}(r_2) + \mathbf{T}'(r_1)] = \mathbf{e},$$

where $\mathbf{e}$ is a column vector with all its components equal to one. That results in major simplifications. We obtain:

$$m_1 = \boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e} + [1 - \phi(r_1, r_2)]\lambda^{-1}, \tag{A-9}$$

and

$$m_2(u) = S(u)\boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e} + 1 - u + S(u)[1 - \phi(r_1, r_2)]\lambda^{-1}, \tag{A-10}$$

for $0 \leq u < 1$. The quantity $R_0 = \boldsymbol{\gamma}(r_2 - 1)[I - T(r_1, r_2)]^{-1}\mathbf{e}$ is the mean absorption time (to either of the absorbing states) in the Markov chain $P(r_1, r_2)$, conditional on starting in the state 0. We readily obtain that

$$R_0 = \frac{1 - a^{r_2 - 1}}{1 - a - (1 - a^{r_2 - 1})b(r_1 - 1)}. \tag{A-11}$$

The quantity $E^*(r_1, r_2)$ defined by

$$E^*(r_1, r_2) = g_1 m_1 + \int_0^1 g_2(u)m_2(u)du, \tag{A-12}$$

is the *fundamental mean* of the Markov renewal process. By substituting the expressions for $g_1, g_2(u)$, m_1 and $m_2(u)$, a direct calculation yields that

$$E^*(r_1, r_2) = \frac{1}{(1-a)a^{r_2-1} + c(r_1 - 1)}. \tag{A-13}$$

2. Notes on Section 3

A Conditional Independence Property of the MAP: Consider the MAP with coefficient matrices D_0 and D_1. The random variable τ is the time of the first event, $\tau^* = [\tau]$, and $\tau' = \tau - \tau^*$. By M, we denote the number of events in $(\tau, \tau^* + 1]$.

Lemma A-1: *Given that $\tau' = v$ and $J(\tau) = h$, the pair of random variables M and $J(\tau^* + 1)$ is (conditionally) independent of the random variables $J(0)$, τ^*, and $J(\tau^*)$.*

Moreover, for $0 \leq h, j \leq m$, and $0 \leq v < 1$,

$$P\{M = k, J(\tau^* + 1) = j \mid J(\tau) = h, \tau' = v\} = [P(k; 1 - v)]_{hj}.$$

Proof: By using the basic properties of the MAP, the elementary conditional probability $P\{M = k,\ J(\tau^* + 1)]j,\ J(\tau) = h,\ \tau' \varepsilon (v + dv)\} J(0) = i\}$ is given by

$$\sum_{n=0}^{\infty} \{[P(0;1)]^n P(0; v) D_1 dv]\}_i [P(m; 1 - v)]_{hj}$$

$$= \{[I - P(0;1)]^{-1} P(0; v) D_1 dv]\}_{ih} [P(m; 1 - v)]_{hj}.$$

This implies that

$$P\{J(\tau) = h, \tau' \varepsilon (v + dv) \mid J(0) = i\} = \{[I - P(0;1)]^{-1} P(0; v) D_1 dv]\}_{ih},$$

so that

$$P\{M = k, J(\tau^* + 1) = j \mid J(\tau) = h, \tau' = v\} = [P(k; 1 - v)]_{hj}. \qquad \square$$

The matrix $S(u)$, defined by

$$S(u) = \sum_{j=0}^{r_1 - 2} P(j; 1 - u). \tag{A-14}$$

is the analogue of the function $S(u)$ in the Poisson case.

Some Integration Formulas: We prove a few simple integration formulas for matrix functions. These are repeatedly needed in the calculations of the means and in various other simplifications. Clearly, we have that:

$$\int_0^1 \exp(D_0 u) du = [I - \exp(D_0)](-D_0)^{-1}. \tag{A-15}$$

Also:

$$\int_0^1 u \exp(D_0 u) du = [I - \exp(D_0) + \exp(D_0) D_0](-D_0)^{-2}. \tag{A-16}$$

Proof: Write the integral as $\int_0^1 u d[\exp(D_0 u)] \cdot (-D_0)^{-1}$,

and perform integration by parts as for scalar functions.

$$\int_0^1 \exp(D_0 u) D_1 \exp[D(1 - u)] du = \exp(D) - \exp(D_0). \tag{A-17}$$

$\square$

Proof: Since $\sum_{j=0}^{\infty} P(j;t) = \exp(Dt)$, the integral equals

$$\sum_{j=0}^{\infty} \int_0^1 \exp(D_0 u) D_1 du = \sum_{j=0}^{\infty} P(j+1;1) = \exp(D) - \exp(D_0). \qquad \square$$

By similar calculation, we establish that

$$\int_0^1 \exp(D_0 u) D_1 S(u) du = \sum_{j=0}^{r_1 - 1} P(j+1;1) = B(r_1 - 1), \tag{A-18}$$

which is the analogue of the corresponding formula for the Poisson case.

Verification of a Matrix Transform Formula: We verify the formula

$$d_{u'} G_{22}^*(u, u'; s) = S(u) e^{-s(1-u)} e^{-s} \Gamma(r_2 - 1)[I - e^{-s} T(r_1, r_2)]^{-1} T^\circ(r_1)$$
$$\cdot [I - e^{-s} \exp(D_0)]^{-1} e^{-su'} \exp(D_0 u') D_1 du',$$

which bears the number (13) in the paper. It gives the transform of the transition probability from a state in the set $(\mathbf{2}, u)$ to a state in the set $(\mathbf{2}, u')$. Rather than writing a complicated general formula, we explain how the various modular parts of the transform arise. At the start of the sojourn time, a length of time $1 - u$ remains of the unit interval in P_2. During that time, there must be fewer than r_2 events in the MAP because, otherwise, a transition to the set $\mathbf{1}$ would occur at the end of the unit interval. That accounts for the matrix factor $S(u)e^{-s(1-u)}$.

To generate a 2-*run*, starting at the end of the initial interval, the Markov chain $P(r_1, r_2)$ must be absorbed in the macro-state $\mathbf{r_2 - 1}$. Let that occur after exactly n transitions, each corresponding to a unit interval. That absorption can be followed by some number $v \geq 0$ of unit intervals during which no arrival occurs. The matrix transform of that transition time is given by

$$\sum_{n=0}^{\infty} \sum_{v=0}^{\infty} e^{-s(n+v)} \Gamma(r_2 - 1)[T(r_1, r_2)]^n T^\circ(r_1)[\exp(D_0)]^v$$
$$= \Gamma(r_2 - 1)[I - e^{-s} T(r_1, r_2)]^{-1} T^\circ(r_1)[1 - e^{-s} \exp(D_0)]^{-1}.$$

Finally, there is a first event in the MAP in the subinterval $(u', u' + du')$ of the subsequent unit interval. That accounts for the matrix factor $e^{-su'} \exp(D_0 u') D_1 du'$ in the stated transform.

Derivation of the Mean Sojourn Times: We give the essential steps in the calculation of the mean sojourn times and the row sum means of the transition probability operator G. In each transform formula, we differentiate with respect to s, set $s = 0$ and change the sign. For the transforms with subscripts 11 and 21, we simply postmultply by $\mathbf{e}$ and we readily obtain the formulas (30) and (32). For those with subscripts 12 and 22, we integrate with respect to u' from 0 to 1 and then postmultiply by $\mathbf{e}$. To carry out the integration step, we use the formulas (A-16), (A-17), and (A-18).

For example, for the transform with subscript 12, the integration yields

$$\Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-2} T^\circ(r_1)(-D_0)^{-1} D_1 + \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^\circ(r_1)(-D_0)^{-2} D_1$$
$$+ \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^\circ(r_1)\{[I - \exp(D_0)]^{-1} - I]\}(-D_0)^{-1} D_1$$
$$- \Gamma(r_2 - 1)[I - T(r_1, r_2)]^{-1} T^\circ(r_1)[I - \exp(D_0)]^{-1} \exp(D_0)(-D_0)^{-1} D_1.$$

In the postmultiplication by $\mathbf{e}$, we note that, since $D = D_0 + D_1$ is a generator, $(-D_0)^{-1}D_1\mathbf{e} = \mathbf{e}$. That results in the cancellation of several terms and formula (31) follows after routine simplifications. The verification of (33) proceeds along the same lines.

In calculating the sums $\mathbf{m}_{11} + \mathbf{m}_{12}$ and $\mathbf{m}_{21}(u) + \mathbf{m}_{22}(u)$, we notice that

$$[I - T(r_1, r_2)]^{-1}[T'(r_1) + T^{\circ}(r_1)]\mathbf{e} = \mathbf{e},$$

because the matrix $P(r_1, r_2)$ is stochastic.

Calculation of the Fundamental Mean: We shall verify the equality $(k^*)^{-1}E^*(r_1, r_2) = 1$. By using the integration formulas (A-15) and (A-18), we find that

$$\frac{1}{k^*}\int_0^1 \mathbf{g}_2(u)\mathbf{m}_2(u)du = \boldsymbol{\theta}A^{r_2-1}\mathbf{e} - \boldsymbol{\theta}A^{r_2-1}(I-A)(-D_0)^{-1}\mathbf{e}$$

$$+ \boldsymbol{\theta}A^{r_2-1}B(r_1-1)\mathbf{R}_0 + \boldsymbol{\theta}A^{r_2-1}B(r_1-1)\Phi^{\circ}(r_1, r_2)(-D_0)^{-1}\mathbf{e}.$$

Adding the expression for $(k^*)^{-1}\mathbf{g}_1\mathbf{m}_1$, we obtain that

$$\frac{1}{k^*}E^*(r_1, r_2) = \boldsymbol{\theta}A^{r_2-1}\mathbf{e} - \boldsymbol{\theta}A^{r_2-1}(I-A)(-D_0)^{-1}\mathbf{e}$$

$$+ \boldsymbol{\theta}[C(r_1-1) + A^{r_2-1}B(r_1-1)][I - A - (I - A^{r_2-1})B(r_1-1)]^{-1}(I - A^{r_2-1})\mathbf{e}$$

$$+ \boldsymbol{\theta}[C(r_1-1) + A^{r_2-1}B(r_1-1)][I - A - (I - A^{r_2-1})B(r_1-1)]^{-1}(I-A)A^{r_2-1}\mathbf{e}.$$

Now, upon replacing $C(r_1-1)$ by $\exp(D) - A - B(r_1-1)$, we notice that

$$\boldsymbol{\theta}[C(r_1-1) + A^{r_2-1}B(r_1-1)] = \boldsymbol{\theta}[I - A - (I - A^{r_2-1})B(r_1-1)],$$

so that the terms involving the factor $(-D_0)^{-1}\mathbf{e}$ cancel and the remaining terms simplify to 1. It follows that $E^*(r_1, r_2) = k^*$.