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$\text{GI}^X/\text{M}^Y/1$ SYSTEMS WITH RESIDENT SERVER AND GENERALLY DISTRIBUTED ARRIVAL AND SERVICE GROUPS

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ABSTRACT

Considered are bulk systems of GI/M/1 type in which the server stands by when it is idle, waits for the first group to arrive if the queue is empty, takes customers up to its capacity and is not available when busy. Distributions of arrival group size and server's capacity are not restricted. The queueing process is analyzed via an augmented imbedded Markov chain. In the general case, the generating function of the steady-state probabilities of the chain is found as a solution of a Riemann boundary value problem. This function is proven to be rational when the generating function of the arrival group size is rational, in which case the solution is given in terms of roots of a characteristic equation. A necessary and sufficient condition of ergodicity is proven in the general case. Several special cases are studied in detail: single arrivals, geometric arrivals, bounded arrivals, and an arrival group with a geometric tail.

Key words: Bulk Queues, Riemann Problems.

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1. Introduction

Bulk GI/M/1 systems have been studied by many authors using many methods, with the stationary queue-length probabilities and ergodicity conditions often being subjects of main interest. The results obtained usually depend on two major factors: the nature of distributions of arrival and service group sizes and the service discipline. Three types of service discipline appear most often in the literature. The first is a never idle transportation-type ("visiting") server. This server begins a new service act immediately upon completion of the previous act, regardless of the current number in the queue; the server becomes unavailable (leaves) once the group is formed. The second type of service discipline is the "open" server. It stands by if the queue is empty and arrivals are allowed to join the service act in progress until the server's capacity is met. Thirdly, there is the "closed" resident server. It stands by if the queue is empty, waits for arrivals, and takes arriving customers up to its capacity, but it is unavailable when busy. In all three cases, at the beginning of a new service act, the server takes in the minimum of the current queue-size and the server's capacity. A common feature of the first two disciplines is that in both cases there is an imbedded Markov chain $\{Q_k\}$ such that

$$Q_{k+1} = \max\{0, Q_k + \chi\}. \tag{1}$$

(Q_k is the pre-arrival queue in the visiting server case or the pre-arrival number in the system in the "open" server case.) It is easy to see that in the resident server case neither of these sequences is a Markov chain (unless the server's capacity is exactly 1).

The first two cases have been studied by several authors and methods. Under the assumption of bounded arrival groups, the chain $\{Q_k\}$ defined by (1) can be efficiently studied by the matrix-geometric technique (see, e.g., Neuts [7]). In Bhat [1], the open server case was studied by methods of fluctuation theory. For generally distributed arrival groups, the results for the steady-state probabilities were given in terms of probabilistic factorization components of $1 - E[z^{\chi}]$. For bounded arrival groups, these components were expressed explicitly in terms of roots of characteristic equations. In Dukhovny [4], using methods of the theory of Riemann boundary value problems (RBVPs), similar results for the visiting server case were obtained in terms of complex-analytic factorization components of the regularized function $[1 - E[z^{\chi}]](1 - z^{-1})^{-1}$. The methods used allowed us to obtain explicit results for arrival groups with rational generating functions.

The resident server case was studied in Cohen [2] under assumptions of single arrivals and a special distribution of the service group via a standard pre-arrival imbedded Markov chain augmented by an additional state for the idle server. The stationary probabilities for the busy-server states of the chain were shown to form a geometric sequence, the ratio of which was a root of a characteristic equation.

In the present paper, we follow the approach of Cohen [2] and study the augmented chain. Its steady-state probabilities are analyzed by the method of RBVP, which allows us to avoid any restrictions on the distributions of the arrival group size and the server's capacity. In Sections 2 and 3, in order to make this paper self-contained, along with basic definitions and assumptions we provide some information on RBVPs, related operators, and ways to find complex-analytic factorization components. (This information can be found in greater detail in Dukhovny [3, 5]). In Section 4, we introduce the augmented Markov chain and derive its transition probabilities and their generating functions. In Section 5, we prove that the chain is ergodic if and only if the familiar condition of ergodicity holds, that is, if the expected arrival group size is less then the expected number of customers served during an inter-arrival period. The generating function of the steady-state probabilities of the chain in the general case is found in Section 6 as a solution of a special RBVP. In Section 7, under the assumption that the generating function of the arrival group size is rational, we provide an explicit expression for the solution in terms of roots of a certain characteristic equation. Several important special cases are completely solved in this section: single arrivals, geometric arrivals, bounded arrivals, and the arrival group with a geometric tail.

2. Definitions, Assumptions and Notations

We assume that customers arrive at the service station in groups of random size α, with $E(z^{\alpha}) = a(z)$, and $E(\alpha) = a$. The inter-arrival times are i.i.d. random variables (RV's), each distributed as a RV γ with the density function $g(t)$ and the Laplace-Stieltjes Transform (LST) $G(s)$, where $E(\gamma) = g$. The server is always at the station and it becomes available immediately upon completion of the previous service act or, if the system is empty, upon the next arrival. The service group size is the minimum of the queue-size at the beginning of the service and the server's capacity β, where $E(z^{\beta}) = b(z)$ and $E(\beta) = b$. The service time is exponentially distributed with parameter μ. To avoid unnecessary complications (that can be studied by the same method), we additionally assume that α and β are mutually prime (that is, they may assume mutually prime values with a nonzero probability). This assumption holds most often in applications; it is guar-anteed, for example, if either α or β may assume 1 with positive probability.

Following Dukhovny [5], we introduce projections T^+ and T^- on the Wiener algebra W of the Laurent series of the complex variable z with absolutely summable coefficients: if $f(z) = \sum_{-\infty}^{\infty} f_i z^i$, then

$$T^+ f(z) = \sum_{1}^{\infty} D_i f(z), T^- f(z) = \sum_{-\infty}^{0} D_i f(z), \text{ where } D_i f(z) = f_i. \tag{2}$$

Denote $W^{\pm} = T^{\pm}(W)$ and $\bar{W}^+ = W^+ \oplus \{\text{const}\}$. While $f(z)$ converges absolutely on Γ: $|z| = 1$, $T^+ f(z) \in W^+$ converges absolutely in $\bar{\Gamma}^+: |z| \leq 1$ and $T^- f(z) = W^-$ converges absolutely in $\bar{\Gamma}^-: |z| \geq 1$. The substitution $z = 1$, where possible, will be indicated by the operator S. By definition

$$T^+T^- = T^-T^+ = 0, \; T^+T^+ = T^+, \; T^-T^- = T^-, \tag{3}$$

$$T^-S = S, \; T^+S = 0. \tag{4}$$

The following relations show the probabilistic meaning of these operators. Suppose χ is an integer random variable with $H(z) = \sum_{-\infty}^{\infty} h_j z^j = E(z^\chi)$. Then,

$$T^+ H(z) = E\{z^\chi \wedge (\chi > 0)\}, \tag{5}$$

$$T^- H(z) = E\{z^\chi \wedge (\chi \leq 0)\}, \tag{6}$$

$$S \sum_{j \in M} h_j z^j = P\{\chi \in M\}. \tag{7}$$

Let $\{\tau_i\}$ be a sequence of i.i.d. exponential random variables with parameter μ. We define an integer random variable $\nu = 0, 1, 2, \ldots$ by the inequality

$$\sum_{i=1}^{\nu} \tau_i \leq \gamma < \sum_{i=1}^{\nu+1} \tau_i.$$

In the visiting server system, ν is the number of service completions during the inter-arrival time. Its generating function is known to be

$$E(w^\nu) = K(w) = \sum_{s=0}^{\infty} k_s w^s = G(\mu - \mu w).$$

Let $\{\beta_j\}$ be a sequence of i.i.d. random variables, each distributed as β, and let $B_\nu = \beta_1 + \ldots + \beta_\nu$. In the visiting server system, B_ν is the total number of customers that can be potentially withdrawn from the queue between successive arrivals. It follows that

$$E(z^{B_\nu}) = K(b(z)) = G(\mu - \mu b(z)).$$

If we define $\chi = \alpha - B_\nu$ and $H(z) = E(z^\chi)$, then

$$H(z) = E\{z^{\alpha - B_\nu}\} = a(z)K(b(1/z)) = a(z)G(\mu - \mu b(1/z)). \tag{8}$$

3. Calculating Complex-Analytic Factorization Components

Let $f(z) = [1 - H(z)](1 - z^{-1})^{-1}$. It was proven in Dukhovny [3] that, under the assumption that a, b and g are finite, $f(z) \in W$. Also, if and only if $a < \mu bg$ do we have

$$\mathrm{Ind}_{\Gamma} f(z) = 0. \tag{9}$$

By (9), functions

$$R^{\pm}(z) = \exp\{-T^{\pm} \ln f(z)\} \tag{10}$$

satisfy the factorization identity

$$f(z)^{-1} = R^{+}(z)R^{-}(z), \tag{11}$$

and the normalizing condition

$$R^{+}(0) = 1. \tag{12}$$

Functions given by (10) are the only functions that satisfy (11) and (12) such that $R^{+}(z)$ and $[R^{+}(z)]^{-1}$ belong to $\bar{W}^{+}$, while $R^{-}(z)$ and $[R^{-}(z)]^{-1}$ belong to W^{-}.

Remark: It was proven in Dukhovny [3] that the GF $P(z)$ of the stationary queue-length probabilities in the visiting server case is given by

$$P(z) = \frac{R^{+}(z)}{R^{+}(1)}. \tag{13}$$

Lemma 1: *Suppose $a(z) = E(z^{\alpha})$ is a rational function. Denote by κ_r^{-1} is r-th pole in Γ^{-}, with multiplicity m_r, where $\sum_r m_r = N$. Then*

$$R^{+} = \prod_r (1 - \kappa_r z)^{m_r} \prod_s (1 - \lambda_s z)^{-n_s}, \tag{14}$$

where λ_s is the s-th root in Γ^{+}, with multiplicity n_s, $\sum_s n_s = N$, of the characteristic equation

$$1 - a(1/z)G(\mu - \mu b(z)) = 0. \tag{15}$$

Proof: By (9), the total number of roots of $f(z)$ inside Γ^{-} (counting with multiplicities) should be equal to the total number of its poles, which are the poles of $a(z)$. At the same time, the roots of $f(z)$ in Γ^{-} are reciprocals of the roots of (15) in Γ^{+}. Set

$$R^{+}(z) = \prod_r (1 - \kappa_r z)^{m_r} \prod_s (1 - \lambda_s z)^{-n_s}, \tag{16}$$

$$R^{-}(z) = [R^{+}(z)f(z)]^{-1}. \tag{17}$$

By construction, the function given by (16) and its reciprocal belong to $\bar{W}^{+}$, while the function given by (17) and its reciprocal belong to W^{-}. Furthermore, (11) and (12) are also satisfied. Therefore, the functions given by (16) and (17) must be equal to those given by (10). □

Corollary 1: *If $a(z)$ is a polynomial of degree N (the arrival group size is at most N), then*

$$R^{+}(z) = \prod_s (1 - \lambda_s z)^{-n_s}, \tag{18}$$

where $\sum_s n_s = N$.

Proof: In this case, the only pole of $a(z)$ in Γ^{-} is $z = \infty$ ($\kappa_1 = 0$) with multiplicity N. So, (16) yields (18). □

Corollary 2: *If $a(z) = (1-q)z(1-qz)^{-1}$ (geometric arrivals), then*

$$R^{+}(z) = \frac{1 - qz}{1 - \lambda z}, \tag{19}$$

where λ *is the only root in* Γ^+ *of the equation*

$$z = q + (1-q)G(\mu - \mu b(z)). \tag{20}$$

Proof: The only pole of $a(z)$ is $z = q^{-1}$; so (15) and (16) reduce to (20) and (19), respectively. □

Corollary 3: *If* $a(z) = \sum_{i=1}^{N-1} a_i z^i + a_N z^N (1-qz)^{-1}$ *(geometric tail), then*

$$R^+(z) = (1-qz)\prod_s (1-\lambda_s z)^{-n_s}, \tag{21}$$

where $\sum_s n_s = N$.

Proof: Here, the poles of $a(z)$ in Γ^- are $z = \infty$ ($\kappa_1 = 0$) with multiplicity $N-1$ and $z = q^{-1}$ ($\kappa_2 = q$) with multiplicity 1. So, (21) follows from (16). □

Remark: Using the geometric approximation for the tail of $a(z)$ allows us to select N much lower than the actual upper bound of the group size, so the number of roots involved in (21) will be much smaller than the number of roots involved in (18), which is equal to the upper bound.

4. The Augmented Markov Chain and its Transition Probabilities

Let $\{Q_k\}$ be a sequence of random variables such that Q_k is either the queue length immediately before the moment of the kth arrival, if the server is busy, or "e" (empty), if the server is idle. Clearly, $\{Q_k\}$ is a Markov chain. The set of its possible states is $\{e, 0, 1, \ldots\}$; its stationary probabilities will be denoted by p_i, $i = e, 0, 1, \ldots$; its transition probabilities will be denoted by a_i^j.

Lemma 2: *The transition probabilities of the chain* $\{Q_k\}$ *and their generating functions are given by the following formulas:*

$$a_e^e = ST^- H(z), \tag{22}$$

$$a_e^0 = ST^-[b(1/z) - 1]H(z), \tag{23}$$

$$A_e(z) = \sum_{j=1}^{\infty} a_e^j z^j = T^+ b(1/z)H(z), \tag{24}$$

$$a_i^e = ST^- z^i a(z) K^*(b(1/z)),\ i = \overline{0,\infty}, \tag{25}$$

$$a_i^0 = ST^- z^i a(z)[b(1/z) - 1]K^*(b(1/z)),\ i = \overline{0,\infty}, \tag{26}$$

$$A_i(z) = \sum_{j=1}^{\infty} a_i^j z^j = T^+ z^i H(z), \quad i = \overline{0,\infty}, \tag{27}$$

where $H(z)$ *is given by* (8), *and*

$$K^*(w) = \sum_{s=1}^{\infty} k_s w^{s-1} = [G(\mu - \mu w) - G(\mu)]/w. \tag{28}$$

Proof: If the queue length before an arrival is i, then immediately after the arrival, it becomes $i + \alpha$, the GF of which is $z^i a(z)$. Transitions after that occur by one of the following scenarios.

1) Suppose at an arrival, the system is empty. For the system to become empty again before the next arrival, there must be $\nu > 0$ service completions during the inter-arrival period and the total offered withdrawal B_ν from the queue during these service acts should be at least α. Using (5) through (7), we obtain from (8) that

$$a_e^e = P\{\alpha - B_\nu \leq 0 \wedge \nu > 0\} = ST^-a(z)K(b(1/z)) - ST^-a(z)k_0; \tag{29}$$

and since, obviously, $T^-a(z) = 0$, (29) reduces to (22).

2) The scenario for the transition $(e) \to (0)$ is the following: there are $\nu + 1$ withdrawals; the first ν withdrawals do not exhaust the queue, but the $(\nu+1)$-st (immediately following the ν-th completion) does. Using (5) through (7), we obtain from (8) that

$$a_e^0 = P\{\alpha - B_\nu > 0 \geq \alpha - B_\nu = \beta_{\nu+1}\} = ST^-b(1/z)T^+a(z)K(b(1/z)). \tag{30}$$

Using projection properties (3) and (4), we transform (30) into (23).

3) The scenario for transitions $(e) \to (j)$, for any $j > 0$, is that ν completions take place during an inter-arrival period and the remainder of the queue-length after the corresponding $(\nu+1)$ withdrawals, is positive. Using (5), we obtain from (8) that

$$A_e(z) = E\{z^{\alpha - B_\nu - \beta_{\nu+1}} \wedge \alpha - B_\nu - \beta_{n+1} > 0\} = T^+a(z)b(1/z)K(b(1/z)), \tag{31}$$

which yields (24).

While analyzing transitions from the busy-server states, note that there is no immediate withdrawal upon an arrival. Now it takes ν completions to make ν withdrawals from the queue during the inter-arrival period.

4) Each transition $(i) \to (e)$ takes $\nu > 0$ completions during the inter-arrival period, and $\nu - 1$ previous withdrawals must exhaust the queue of length $i + \alpha$. As

$$E\{w^{\nu-1} \wedge \nu > 0\} = \sum_{s=1}^{\infty} k_s w^{s-1} = K^*(w), \tag{32}$$

using (5) through (7), we obtain that

$$P\{i + \alpha - B_{\nu-1} \geq 0\} = ST^-z^i a(z)K^*(b(1/z)), \tag{33}$$

which proves (25).

5) Each transition $(i) \to (0)$ occurs with the following scenario. There have to be $\nu > 0$ completions; the first $\nu - 1$ withdrawals do not exhaust the queue, but the next one does. On the strength of (32) and by use of (5) through (7), we have that

$$a_i^0 = P\{i + \alpha - B_{\nu-1} > 0 \geq i + \alpha - B_{\nu-1} - \beta_\nu\} = ST^-b(1/z)T^+z^i a(z)K^*(b(1/z)), \tag{34}$$

which, on the strength of (3) and (4), yields (26).

6) The scenario for the transitions $(i) \to (j)$, for any $j > 0$, is that ν completions take place during the inter-arrival period and the remainder of the queue-length, after the corresponding ν withdrawals, is positive. Using (5), we obtain from (8) that

$$A_i(z) = E\{z^{i + \alpha - B_\nu} \wedge i + \alpha - B_\nu > 0\} = T^+z^i a(z)K(b(1/z)), \tag{35}$$

which proves (27). □

5. The Necessary and Sufficient Condition of Ergodicity

Theorem 1: *The Markov chain* $\{Q_k\}$ *is ergodic if and only if* $a < \mu bg$.

Proof: Suppose $a < \mu bg$. Under the assumption that α and β are mutually prime, all states of $\{Q_k\}$ are obviously connected. Let $x_j = j$, for $j = \overline{0,\infty}$, and consider

$$\Delta_i = \sum_{j=0}^{\infty} a_i^j x_j - x_i = A_i'(1) - i, \quad i = 0, 1, 2, \ldots. \tag{36}$$

Under the assumption that the expectations a, b and g exist, we have $\sum_{-\infty}^{\infty} | jh_j | < \infty$. So, by differentiating (27), we obtain

$$\lim_{i \to \infty} \Delta_i = \lim_{i \to \infty} [H'(1) - i] = \delta < 0.$$

By Foster's theorem (Foster [6]), the chain $\{Q_k\}$ is ergodic.

Now suppose that the chain $\{Q_k\}$ is ergodic. The stationary probabilities p_i, $i = e, 0, 1, \ldots$, comprise the only absolutely summable solution of the system of equilibrium equations:

$$p_j = \sum_i p_i a_i^j, \tag{37}$$

$$\sum_i p_i = 1. \tag{38}$$

Denote $P(z) = \sum_{i=0}^{\infty} p_i z^i$. From (24), (27), and (37) for $j = \overline{1,\infty}$, we obtain that

$$P(z) - p_0 = T^+ P(z)H(z) + p_e T^+ b(1/z)H(z). \tag{39}$$

By the definitions of T^+ and T^-, we can rewrite (39) as

$$P(z)[1 - H(z)] = p_0 - T^- P(z)H(z) + p_e T^+ b(1/z)H(z). \tag{40}$$

Applying the operator S to both sides of (40), we obtain (as $H(1) = 1$) that

$$0 = p_0 - ST^- P(z)H(z) + p_e ST^+ b(1/z)H(z). \tag{41}$$

On the strength of (41), we multiply (40) by $(1 - z^{-1})^{-1}$ and find that

$$P(z)f(z) = \psi(z), \tag{42}$$

where

$$\psi(z) = \{[ST^- - T^-]P(z)H(z) + p_e[T^+ - ST^+]b(1/z)H(z)\}(1 - z^{-1})^{-1}. \tag{43}$$

Since $P(z)f(z) \in W$, by (42), $\psi(z) \in W$ as well. By (43), all Laurent coefficients of $\psi(z)$ have to be nonnegative. If they were all zeros, then it would follow from (43) that all steady-state probabilities are zeros, in contradiction to the assumption of ergodicity. Hence $S\psi(z) = \psi(1) > 0$. Also,

$$SP(z)f(z) = P(1)f(1) = P(1)(\mu bg - a);$$

so, applying operator S to (42), we conclude that $\mu bg - a > 0$. □

6. Resident Server: Stationary Probabilities

Let us denote $w = b(1/z)$.

Theorem 2: *If* $a < \mu bg$, *then*

$$P(z) = R^+(z)\left\{\frac{1}{R^+(1)} - p_e[T^+ + ST^-]\frac{w}{R^+(z)}\right\}, \tag{44}$$

where

$$p_e = \frac{1}{R^+(1)} \frac{ST^- R^+(z)a(z)K^*(w)}{\{1 - ST^-[R^+(z)a(z)K^*(w)(T^- - ST^-)\frac{w}{R^+(z)}]\}}. \tag{45}$$

Proof: Using the definitions of T^+ and T^-, we rewrite (39) in the form

$$[P(z) + p_e w][1 - H(z)] = p_0 + p_e w - T^- P(z)H(z) - p_e T^- wH(z). \tag{46}$$

By construction, the right-hand side of (46) belongs to W^-. We multiply both sides of (46) by $(1-z)^{-1}$, denote the result on the right-hand side by $\phi^-(z)$, and obtain

$$P(z)f(z) + p_e wf(z) = \phi^-(z). \tag{47}$$

By construction and by (47), $\phi^-(z) \in W^-$. Thus, as $P(z) \in \bar{W}^+$ by construction, (47) is a Riemann boundary value problem on Γ for $P(z)$ and $\phi^-(z)$ in the class of functions from W. Under the assumptions of the theorem, relations (9) through (12) hold. We multiply both sides of (47) by $R^-(z)$, use (11), and apply T^+ to both sides. We find that

$$T^+\frac{P(z)}{R^+(z)} + T^+ p_e \frac{w}{R^+(z)} = T^+\phi^-(z)R^-(z). \tag{48}$$

The right-hand side of (48) is 0, as $\phi^-(z)R^-(z) \in W^-$ by construction. On the left-hand side,

$$T^+\frac{P(z)}{R^+(z)} = \frac{P(z)}{R^+(z)} - p_0,$$

as $P(z)/R^+(z) \in \bar{W}^+$ by construction and because of (12). Now (48) yields

$$P(z) = R^+(z)\{p_0 - p_e T^+ \frac{w}{R^+(z)}\}. \tag{49}$$

We now apply S to (49), use the normalizing condition:

$$P(1) + p_e = 1, \tag{50}$$

which follows from (38), and find that

$$p_0 = \frac{1}{R^+(1)} - p_e ST^- \frac{w}{R^+(z)}. \tag{51}$$

Using (51) in (49), we obtain (44).

Consider (37) for $(i) = (e)$ and use (22) and (25). Then

$$p_e = p_e ST^- H(z) + ST^- P(z)a(z)K^*(w).$$

Applying (44) here, after using some algebraic transformations based on identities (3) and (4), and using formulas (8) and (28), we obtain (45). □

Corollary 1: *The GF of the stationary distribution of the pre-arrival number in the system with a resident server, in the case of single service and generally distributed arrival group size, is*

$$zP(z) + p_e = \frac{R^+(z)}{R^+(1)}.$$

Proof: In this case, $w = 1/z$. Since $R^+(z)^{-1} \in \bar{W}^+$, its Laurent expansion is the same as its MacLaurin expansion. By (12),

$$D_0 R^+(z)^{-1} = R^+(0)^{-1} = 1.$$

Denote $r_1 = D_1 R^+(z)^{-1}$. We now have

$$T^- \frac{w}{R^+(z)} = \tfrac{1}{z} + r_1,\ (T^- - ST^-)\frac{w}{R^+(z)} = \tfrac{1}{z} - 1,$$

and

$$(T^+ + ST^-)\frac{w}{R^+(z)} = \frac{w}{R^+(z)} - \tfrac{1}{z} + 1. \tag{52}$$

By (28), (8), and (11),

$$R^+(z)a(z)K^*(w)(\tfrac{1}{z} - 1) = R^+(z) - \frac{z-1}{zR^-(z)} - R^+(z)a(z)[k_0 + K^*(w)].$$

So, (45) yields $p_e = R^+(1)^{-1}$, as $ST^- R^+(z) = D_0 R^+(z) = 1$. Now, the statement of the corollary follows from (44) on the strength of (52). □

7. Arrivals with a Rational Generating Function of the Group Size

Theorem 3: *Suppose $a(z)$ is a rational function. Denote by κ_r^{-1} its r-th pole in Γ^- with multiplicity m_r, $\sum_r m_r = N$. Then*

$$P(z) = \frac{p_0 \prod_r (1 - \kappa_r z)^{m_r} - p_e z^N W(1/z)}{\prod_s (1 - \lambda_s z)^{n_s}}, \tag{53}$$

where each λ_s is a root of (15) in Γ^+ with multiplicity n_s, such that $\sum_s n_s = N$, and where $W(z)$ is the $(N-1)$st degree polynomial whose value and whose derivatives at each $z = \kappa_r$, up to the order of $m_r - 1$, are equal to the value and respective derivatives of $b(z)\prod_s (z - \lambda_s)^{n_s}$.

Proof: On the strength of Lemma 1, we use (16) to represent on Γ

$$\frac{b(1/z)}{R^+(z)} = \frac{z^N W(1/z)}{\prod_r (1 - \kappa_r z)^{m_r}} + \frac{[b(1/z)\prod_s (z^{-1} - \lambda_s)^{n_s} - W(1/z)]}{\prod_r (z^{-1} - \kappa_r)^{m_r}}. \tag{54}$$

By construction, the first part of the right-hand side is analytic in Γ^+ and continuous in Γ. Also, it vanishes at $z = 0$. The reason for this is that the degree of $W(z)$ is $N - 1$ and that $\kappa_r \in \Gamma^+$, $\forall r$. The second part is analytic in Γ^- and continuous on Γ by the definition of $W(z)$. By Liouville's theorem,

$$T^+ \frac{b(1/z)}{R^+(z)} = \frac{z^N W(1/z)}{\prod_r (1 - \kappa_r z)^{m_r}}, \tag{55}$$

$$T^- \frac{b(1/z)}{R^+(z)} = \frac{[b(1/z)\prod_s (z^{-1} - \lambda_s)^{n_s} - W(1/z)]}{\prod_r (z^{-1} - \kappa_r)^{m_r}}. \tag{56}$$

Applying (55) and (16) in (49), we obtain (53). □

It follows from (53) that if $a(z)$ is a rational function, the generating function of the steady-state probabilities is also rational. To complete the formula for $P(z)$ given in (53), one needs to specify p_0 and p_e. From (51) and (56), it follows that, under the assumptions of Theorem 3,

$$p_0 = \frac{\prod_s (1-\lambda_s)^{n_s}(1-p_e) + P_e W(1)}{\prod_r (1-\kappa_r)^{m_r}}. \tag{57}$$

The formulas that emerge when one finds p_e from (45) are generally very cumbersome. We shall look at only some special cases below.

Case 1: If the arrival group size is bounded by N, then

$$P(z) = \frac{\prod_s (1-\lambda_s)^{n_s}(1-p_e) + p_e W(1) - p_e z^N W(1/z)}{\prod_s (1-\lambda_s z)^{n_s}}. \tag{58}$$

Proof: Here the only pole of $a(z)$ is $z=\infty$ ($\kappa_1 = 0$) of multiplicity N. Applying (57) in (53), we obtain (58). □

Case 2: If $a(z) = \sum_{i=1}^{N-1} a_i z^i + a_N z^N (1-qz)^{-1}$ (geometric tail), then

$$P(z) = \frac{(1-q)^{-1}[\prod_s (1-\lambda_s)^{n_s}(1-p_e) + p_e W(1)] - p_e z^N W(1/z)}{\prod_s (1-\lambda_s z)^{n_s}}. \tag{59}$$

Proof: Under the assumption of the case, the poles of $a(z)$ are $z=\infty$ ($\kappa_1 = 0$) with multiplicity $N-1$ and $z = q^{-1}$ ($\kappa_2 = q$) with multiplicity 1. So (59) follows from (53) and (57). □

In special cases 3, 4 and 5 discussed below, it is possible to utilize analytic properties of $a(z)$ and actually calculate p_e using (45). To facilitate these calculations we transform (45) into

$$p_e = \frac{K^*(1) - \frac{1}{R^+(1)} ST^+ R^+(z)a(z)K^*(w)}{\{1 - \frac{1}{R^+(1)} ST^+ R^+(z)a(z)K^*(w) + ST^+ \frac{w}{R^+(z)} T^+ R^+(z)a(z)K^*(w)\}}. \tag{60}$$

The proof of (60) is based on projection properties of operators T^+, T^- and S, and is similar to the proof of (45).

Another tool needed for calculations in (60) is the following lemma.

Lemma 3: *Let $\phi(z) \in W^-$ and $|a| < 1$. Then*

$$T^+ \frac{\phi(z)}{z^{-1}-a} = \frac{\phi(1/a)z}{1-az} \text{ and } T^- \frac{\phi(z)}{z^{-1}-a} = \frac{\phi(z)-\phi(1/a)}{z^{-1}-a}. \tag{61}$$

Proof: Consider the following partition:

$$\frac{\phi(z)}{z^{-1}-a} = \frac{\phi(1/a)z}{1-az} + \frac{\phi(z)-\phi(1/a)}{z^{-1}-a}. \tag{62}$$

The first part of the right-hand side of (62) is obviously analytic in Γ^+ and vanishes at $z=0$; the second part is analytic in Γ^- by construction. By Liouville's theorem, (62) yields (61). □

Case 3: If $a(z) = (1-q)z(1-qz)^{-1}$ (geometric arrivals), then

$$P(z) = \frac{p_0(1-qz) + p_e z b(q)(\lambda - q)}{1-\lambda z}, \tag{63}$$

$$p_0 = \frac{(1-\lambda)(1-p_e) + p_e b(q)(q-\lambda)}{(1-q)}, \tag{64}$$

$$p_e = \frac{K^*(1) - K^*(b(\lambda))}{1 - [1 - b(q)]K^*(b(\lambda))}, \tag{65}$$

where λ is the only root of (20) in Γ^+.

Proof: In this case, $a(z)$ has one simple pole, $z = 1/q$ $(\kappa_1 = q)$. So $W(z) = b(q)(q - \lambda)$, (63) follows from (53), and (64) follows from (57). To derive (65) from (60), we use (19) and apply Lemma 3 with $\phi(z) = (1-q)K(w)$:

$$T^+ R^+(z)a(z)K^*(w) = \frac{(1-q)zK^*(b(\lambda))}{1 - \lambda z}. \tag{66}$$

On the strength of (66), we have

$$ST^+ \frac{w}{R^+(z)} T^+ R^+(z)a(z)K^*(w) = ST^+ \frac{w(1-q)K^*(b(\lambda))}{z^{-1} - q} = b(q)K^*(b(\lambda)). \tag{67}$$

The second equality in (67) was obtained by using Lemma 3 with $\phi(z) = w$. Now we use (19) at $z = 1$, apply S to (66), and obtain (65). □

Remark: By (28), $K^*(1) = 1 - G(\mu)$, $K^*(b(\lambda)) = \frac{G(\mu - \mu(\lambda)) - G(\mu)}{b(\lambda)}$. Also, on the strength of (20), $G(\mu - \mu b(\lambda)) = \frac{\lambda - q}{1 - q}$.

Case 4: In the case of single arrivals,

$$P(z) = \frac{1-\lambda}{1-\lambda z}(1 - p_e), \tag{68}$$

$$p_e = \frac{b(\lambda)[1 - G(\mu)] - \lambda + G(\mu)}{b(\lambda) - \lambda + G(\mu)}. \tag{69}$$

Proof: Set $q = 0$ in formulas (63) through (65). As $b(0) = 0$, (63) and (64) yield (68); (69) follows from (65) as $G(\mu - \mu b(\lambda)) = \lambda$ (see the Remark above). □

Formula (68) shows that in the case of single arrivals, regardless of the distribution of the server's capacity, the sequence $\{p_i, i = \overline{0,\infty}\}$ is geometric with the ratio λ. In the case of geometric arrivals, it follows from (63) that the same is true for the sequence $\{p_i, i = \overline{1,\infty}\}$.

In the following case we use partial fractions to show what is involved in finding p_e in more complicated situations.

Case 5: Let $a(z) = a_1 z + a_2 z^2(1 - qz)^{-1}$ (geometric tail, $N = 2$). Here $a(z)$ has two simple poles: $z = \infty$ and $z = q^{-1}$ ($\kappa_1 = 0$ and $\kappa_2 = q$, respectively). Accordingly, (15) has two roots in Γ^+, λ_1 and λ_2. (It can easily be shown that here the two roots have to be distinct.) As $b(0) = 0$, the polynomial $W(z)$ of Theorem 3 is $W(z) = zq^{-1}b(q)(q - \lambda_1)(q - \lambda_2)$. Now, formulas (53) and (57) yield

$$P(z) = \frac{(1-p_e)(1-\lambda_1)(1-\lambda_2)(1-qz) + p_e q(q-\lambda_1)(q-\lambda_2)(1-z)}{(1-q)(1-\lambda_1 z)(1-\lambda_2 z)}.$$

By (16),

$$R^+(z) = \frac{1 - qz}{(1-\lambda_1 z)(1-\lambda_2 z)}, \tag{70}$$

$$R^+(z)a(z) = \frac{z[a_1(1-qz) + a_2 z]}{(1-\lambda_1 z)(1-\lambda_2 z)} = \sum_{r=1}^{2} \frac{\tau_r}{z^{-1} - \lambda_r}, \tag{71}$$

where

$$\tau_1 = \frac{a_1(\lambda_1 - q) + a_2}{(\lambda_1 - \lambda_2)} \text{ and } \tau_2 = \frac{a_1(\lambda_2 - q) + a_2}{(\lambda_2 - \lambda_1)}. \tag{72}$$

In the following equation, we use (71) and apply Lemma 3 with $\phi(z) = K^*(w)$ to find that

$$T^+ R^+(z)a(z)K^*(w) = \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r))}{z^{-1} - \lambda_r}. \tag{73}$$

As $zb(1/z) \in W^-$, we also have

$$\phi(z) = zw(z^{-1} - \lambda_1)(z^{-1} - \lambda_2) \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r))}{z^{-1} - \lambda_r} \in W^-.$$

With $\phi(z)$ so defined, we apply Lemma 3 to the following equation and use (73) to find that

$$T^+ \frac{w}{R^+(z)} T^+ R^+(z)a(z)K^*(w) = T^+ \frac{\phi(z)}{(z^{-1} - q)}$$

$$= \frac{b(q)(q - \lambda_1)(q - \lambda_2)}{q(z^{-1} - q)} \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r))}{q - \lambda_r}.$$

Finally, we obtain from (60) that

$$p_e = \frac{K^*(1) - \frac{(1-\lambda_1)(1-\lambda_2)}{1-q} \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r))}{1-\lambda_r}}{1 - \frac{(1-\lambda_1)(1-\lambda_2)}{1-q} \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r^2))}{1-\lambda_r} + \frac{b(q)(q-\lambda_1)(q-\lambda_2)}{q(1-q)} \sum_{r=1}^{2} \frac{\tau_r K^*(b(\lambda_r))}{q-\lambda_r}},$$

where τ_1 and τ_2 are given by (72).

References

[1] Bhat, U.N., *A Study of the Queueing Systems M/G/1 and GI/M/1*, Springer-Verlag, Berlin 1968.

[2] Cohen, J.W., *The Single Server Queue*, 2nd ed., North-Holland, Amsterdam 1982.

[3] Dukhovny, A.M., Markov chains with quasitoeplitz transition matrix, *J. Appl. Math. Sim.*, **2** (1989), 71-82.

[4] Dukhovny, A.M., Markov chains with quasitoeplitz transition matrix: Applications, *J. Appl. Math. Sim.* **2** (1990), 141-152.

[5] Dukhovny, A.M., Applications of vector Riemann boundary value problems to analysis of Markov chains, In: *Advances in Queueing: Theory, Methods and Open Problems*, ed. by J.H. Dshalalow, CRC Press, Boca Raton, Florida 1995, 353-376.

[6] Foster, F.G., On the stochastic matrix associated with certain queueing processes, *Ann. Math. Stat.* **24** (1953), 355-360.

[7] Neuts, M.F., *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The John Hopkins University Press, Baltimore, Maryland 1981.