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SOME EXTENSIONS OF BATEMAN'S PRODUCT FORMULAS FOR THE JACOBI POLYNOMIALS

MING-PO CHEN
Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529 Taiwan
E-mail: maapo@ccvax.sinica.tw

H.M. SRIVASTAVA
University of Victoria
Department of Mathematics and Statistics
Victoria, British Columbia V8W 3P4 Canada
E-mail: hmsri@uvvm.uvic.ca

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ABSTRACT

The authors derive generalizations of some remarkable product formulas of Harry Bateman (1882-1946) for the classical Jacobi polynomials. They also show how the results considered here would lead to various families of linear, bilinear, and bilateral generating functions for the Jacobi and related polynomials.

Key words: Product Formulas, Jacobi Polynomials, Generating Functions, Series Inversion, Linearization Formula, Gaussian Hypergeometric Function, Hypergeometric Identity, Quadratic Transformation, Appell Functions, Polynomial Expansions, Complex Sequence, Dixon's Summation Theorem, Clausenian Hypergeometric Series, Polynomial Identity, Reduction Formula.

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1. Introduction and Preliminaries

As long ago as 1905, Bateman [6] gave the remarkable *product formula*:

$$P_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\left(\frac{2}{x+y}\right)^n \cdot\sum_{k=0}^{n}\frac{(\alpha+\beta+2k+1)k!\Gamma(\alpha+\beta+k+1)}{(n-k!)\Gamma(\alpha+\beta+n+k+2)} \cdot\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y), \tag{1}$$

from which, by applying an elementary series inversion [13, p. 388, Problem 74], it is not difficult to deduce the following *linearization formula* for the classical Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)=\sum_{k=0}^{n}\frac{(-1)^{n+k}(\alpha+\beta+n+1)_k}{n!(n-k)!}\left(\frac{x+y}{2}\right)^k \tag{2}$$

$$\cdot \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} P_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right),$$

which was indeed proved directly by Bateman [7, p. 392] by showing that both sides of (2) satisfy the same partial differential equation. Here, and in what follows, $(\lambda)_\mu := \Gamma(\lambda+\mu)/\Gamma(\lambda)$, in terms of Gamma functions, and $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree n in x, defined by (cf., eg., Szegö [13, Chapter 4])

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}. \tag{3}$$

Each of Bateman's formulas (1) and (2) has been applied in the literature in a number of different directions (see, for details, Askey [2, pp. 11 and 33]). In addition, Bateman's formula (1) was applied by Al-Salam [1] in order to derive the following interesting result due to Feldheim [9]:

$$F_4\left[\gamma,\delta;\alpha+1,\beta+1;\tfrac{1}{4}(1-x)(1-y)t,\tfrac{1}{4}(1+x)(1+y)t\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n(\gamma)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+1)_{2n}} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) \tag{4}$$

$$\cdot\ {}_2F_1(\gamma+n,\delta+n;\alpha+\beta+2n+2;t)t^n,$$

where ${}_2F_1$ is the Gaussian hypergeometric function and F_4 denotes one of Appell's double hypergeometric functions defined by

$$F_4[a,b;c,d;x,y] := \sum_{p,q=0}^{\infty} \frac{(a)_{p+q}(b)_{p+q}}{(c)_p(d)_q} \frac{x^p}{p!}\frac{y^q}{q!} \tag{5}$$

$$\left(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1\right).$$

It should be noticed in passing that, in the particular case when

$$\gamma = \tfrac{1}{2}(\alpha+\beta+1) \text{ and } \delta = \tfrac{1}{2}(\alpha+\beta+2),$$

Feldheim's formula (4) would reduce to Bailey's bilinear generating function for the Jacobi polynomials [5]:

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)t^n$$

$$= (1+t)^{\alpha-\beta-1} F_r\left[\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha+\beta+2);\alpha+1,\beta+1;X,Y\right] \tag{6}$$

$$\left(X := \frac{(1-x)(1-y)t}{(1+t)^2};\quad Y := \frac{(1+x)(1+y)t}{(1+t)^2}\right),$$

in view, of course, of the familiar hypergeometric identity (cf., eg., Erdélyi et al. [8, p. 101]):

$$ {}_2F_1\left(a-\tfrac{1}{2},a;2a;z\right) = \left(\frac{2}{1+\sqrt{1-z}}\right)^{2a-1} \quad (|z|<1), \tag{7}$$

which is, in fact, a *special* case of the following quadratic transformation for the Gaussian hypergeometric function [8, p. 111, Equation 2.11(10)]:

$$_2F_1(a,b;a+b+\tfrac{1}{2};z) = {_2F_1}\left(2a,2b;a+b+\tfrac{1}{2};\frac{1-\sqrt{1-z}}{2}\right) \quad (|z|<1) \tag{8}$$

when $b = 1-\frac{1}{2}$. For several further applications of (1) in the theory of generating functions, one may refer to a recent treatise on the subject by Srivastava and Manocha [12, Chapter 2, Problem 14].

Motivated by the aforementioned potential for applications of Bateman's formulas (1) and (2), we aim here at investigating some interesting generalizations of (1). We also show how these general results can be applied in the theory of generating functions.

2. Polynomial Expansions in Several Variables

We begin by introducing the class of *multivariable* polynomials

$$\left(\Pi_n^{(\lambda)}(m_1,\ldots,m_r;z_1,\ldots,z_r)\right)_{n=0}^{\infty}$$

defined by

$$\Pi_n^{(\lambda)}(m_1,\ldots,m_r;z_1,\ldots,z_r) := \sum_{\substack{k_1,\ldots,k_r=0 \\ M\leq n}} (-n)_M(\lambda+n)_M\Lambda(k_1,\ldots,k_r)z_1^{k_1}\ldots z_r^{k_r} \tag{9}$$

$$(M := m_1k_1+\ldots+m_rk_r; \quad m_j\in\mathbb{N} := \{1,2,3,\ldots\} \ (j=1,\ldots,r); \quad \lambda\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}),$$

where $\{\Lambda(k_1,\ldots,k_r)\}$ is a (suitably bounded) multiple complex sequence. [Here we have used the parameters

$$\lambda \text{ and } m_1,\ldots,m_r$$

in order to identify the members of the class of the multivariable polynomials defined by (9) above.] In terms of these multivariable polynomials as the basis functions, Srivastava [11] gave three general families of polynomial expansions for a multivariable function

$$\Phi(z_1,\ldots,z_r) := \sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r)\Omega_M\, z_1^{k_1}\ldots z_r^{k_r}, \tag{10}$$

where M is given already with the definition (9) and $\{\Omega\}_{n=0}^{\infty}$ is a bounded sequence of essentially arbitrary complex numbers. Of our interest in the present paper is only one of these families, which we recall here in the form (cf. Srivastava [11, p. 300, Equation (1.4)]):

$$(\omega^{m_1}z_1,\ldots,\omega^{m_r}z_r) = \sum_{n=0}^{\infty}\frac{(-\omega)^n}{n!(\lambda+n)_n}\,\Xi_n(\lambda;\omega)\cdot\Pi_n^{(\lambda)}(m_1,\ldots,m_r;z_1,\ldots,z_r), \tag{11}$$

where, for convenience,

$$\Xi_n(\lambda;\omega) := \sum_{\ell=0}^{\infty}\frac{\Omega_{n+\ell}}{(\lambda+2n+1)_\ell}\,\frac{\omega^\ell}{\ell!}. \tag{12}$$

It is understood that the variables $|\omega|$ and $|z_1|,\ldots,|z_r|$ are so constrained that both sides of the polynomial expansion (11) exists.

Upon substituting from (7) into the left-hand side of (11), we readily obtain

$$\Phi(\omega^{m_1}z_1,\ldots,\omega^{m_r}z_r) = \sum_{n=0}^{\infty} S_n(m_1,\ldots,m_r;z_1,\ldots,z_r)\Omega_n\omega^n, \tag{13}$$

where

$$S_n(m_1,\ldots,m_r;z_1,\ldots,z_r):=\sum_{m_1k_1+\ldots+m_rk_r=n}\Lambda(k_1,\ldots,k_r)z_1^{k_1}\ldots z_r^{k_r}$$

$$(m_j\in\mathbb{N}(j=1,\ldots,r);\quad n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}). \tag{14}$$

On the other hand, the right-hand side of (11) can easily be rewritten as

$$\sum_{n=0}^{\infty}\Omega_n\frac{\omega^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2k)\Gamma(\lambda+k)}{\Gamma(\lambda+n+k+1)}\cdot\Pi_k^{(\lambda)}(m_1,\ldots,m_r;z_1,\ldots,z_r).$$

Thus, upon equating the coefficients of ω^n from both sides of Srivastava's expansion (11), we find that

$$S_n(m_1,\ldots,m_r;z_1,\ldots,z_r)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2k)\Gamma(\lambda+k)}{n!\Gamma(\lambda+n+k+1)}$$

$$\cdot\Pi_k^{(\lambda)}(m_1,\ldots,m_r;z_1,\ldots,z_r), \tag{15}$$

where the multivariable polynomials

$$S_n(m_1,\ldots,m_r;\ z_1,\ldots,z_r)$$

are defined by (14). Indeed, by appealing to Dixon's summation theorem for a well-poised Clausenian hypergeometric ${}_3F_2$ series (cf., eg., Erdélyi et al. [8, p. 189, Equation 4.4(5)], it is not difficult to give a *direct* proof of the polynomial identity (15).

In the *two-variable* case ($r=2$), if we further set

$$m_1=1,\quad m_2=m\quad(m\in\mathbb{N}),\quad z_1=z,\quad\text{and}\quad z_2=\zeta,$$

we find from (14) and (15) that

$$\sum_{k=0}^{[n/m]}\Lambda(n-mk,k)z^{n-mk}\zeta^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2k)\Gamma(\lambda+k)}{n!\Gamma(\lambda+n+k+1)}\cdot\Pi_k^{(\lambda)}(1,m;z,\zeta), \tag{16}$$

where $[n/m]$ denotes, as usual, the greatest integer in n/m ($n\in\mathbb{N}_0;m\in\mathbb{N}$), and $\Pi_k^{(\lambda)}(1,m;z,\zeta)$ is a two-variable polynomial given, by analogy with (9), by

$$\Pi_k^{(\lambda)}(1,m;z,\zeta):=\sum_{p,q=0}^{p+mq\le k}(-k)_{p+mq}(\lambda+k)_{p+mq}\Lambda(p,q)z^p\zeta^q\quad(m\in\mathbb{N}). \tag{17}$$

For $m=1$, the polynomials occurring on the left-hand side of (16) can be identified with the classical Jacobi polynomials if we specialize the double sequence $\{\Lambda(p,q)\}_{p,q=0}^{\infty}$ by

$$\Lambda(p,q)=\{p!q!(\alpha+1)_p(\beta+1)_q\}^{-1}\qquad(p,q\in\mathbb{N}_0).$$

In this special case, the innermost double series on the right-hand side of (16) becomes an Appell function F_4 defined by (5). Thus we obtain

$$P_n^{(\alpha,\beta)}\left(\frac{\zeta+z}{\zeta-z}\right)=\frac{n!}{(\zeta0-z)^n}\binom{\alpha+n}{n}\binom{\beta+n}{n}\cdot\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\lambda+2k)\Gamma(\lambda+k)}{\Gamma(\lambda+n+k+1)} \tag{18}$$

$$\cdot F_4[-k, \lambda + k; \alpha + 1, \beta + 1; z, \zeta] \quad (z \neq \zeta).$$

This last consequence (18) of the general result (15) may be viewed as an extension of Bateman's formula (1). In fact, in view of the familiar F_4 representation (cf., Watson [14]; see also Watson [15, p. 371]):

$$P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) = (-1)^n \binom{\alpha+n}{n}\binom{\beta+n}{n} \tag{19}$$

$$\cdot F_r\left[-n, \alpha+\beta+n+1; \alpha+1, \beta+1; \tfrac{1}{4}(1-x)(1-y), \tfrac{1}{4}(1+x)(1+y)\right],$$

which follows from a more general reduction formula for F_4 given by Bailey [3] (see also [4, Section 9.6]), (18) in the special case when

$$\lambda = \alpha + \beta + 1, \quad z = \tfrac{1}{4}(1-x)(1-y), \quad \text{and} \quad \zeta = \tfrac{1}{4}(1+x)(1+y)$$

yields Bateman's formula (1).

3. Applications Involving Generating Functions

For suitably bounded coefficients $\Omega_n (n \in \mathbb{N}_0)$, if we start from the definition (3) with α and β replaced by $\alpha + \mu n$ and $\beta + \nu n$, respectively, it is fairly straightforward to derive the following family of generating functions for the Jacobi polynomials:

$$\sum_{n=0}^{\infty} \frac{\Omega_n}{(\alpha+\mu n+1)_n(\beta+\nu n+1)_n} \; P_n^{(\alpha+\mu n, \beta+\nu n)}(x)t^n$$

$$= \sum_{p,q=0}^{\infty} \frac{\Omega_{p+q}(\alpha+1)_{\mu(p+q)}(\beta+1)_{\nu(p+q)}}{(\alpha+1)_{(\mu+1)p+\mu q}(\beta+1)_{\nu p+9\nu+1)q}} \frac{\{\frac{1}{2}(x-1)t\}^p}{p!} \frac{\{\frac{1}{2}(x+1)t\}^q}{q!}, \tag{20}$$

which, for $\mu = \nu = 0$, was given by Rahman [10] (see also Srivastava and Manocha [12, p. 168, Problem 14(*ii*)]).

By appropriately choosing the coefficients Ω_n $(n \in \mathbb{N}_0)$, and the free parameters μ and ν, one can apply (20) to deduce various families of linear, bilinear, and bilateral generating functions for the Jacobi polynomials. Furthermore, if in the generating function (20) we set

$$x = \frac{\zeta+z}{\zeta-z} \quad (z \neq \zeta)$$

and apply the formula (18), we get

$$\sum_{p,q=0}^{\infty} \frac{\Omega_{p+q}(\alpha+1)_{\mu(p+q)}(\beta+1)_{\nu(p+q)}}{(\alpha+1)_{(\mu+1)p+\mu q}} \frac{(zt)^p}{p!} \frac{(\zeta t)^q}{q!}$$

$$= \sum_{n=0}^{\infty} (\lambda+2n)\Gamma(\lambda+n)\frac{(-t)^n}{n!} \sum_{k=0}^{\infty} \frac{\Omega_{n+k}}{\Gamma(\lambda+n+k+1)} \tag{21}$$

$$F_4[-n, \lambda+n; \alpha+\mu(n+k)+1, \beta+\nu(n+k)+1; z, \zeta],$$

which may be looked upon as a family of generating functions for the F_4 polynomials involved.

In its special case when $\mu = \nu = 0$, if we further set $\lambda = \alpha + \beta + 1$,

$$\Omega_n = (\gamma)_n(\delta)_n \quad (n \in \mathbb{N}_0), \quad z = \tfrac{1}{4}(1-x)(1-y), \quad \text{and } \zeta = \tfrac{1}{4}(1+x)(1+y),$$

and make use of Watson's result (19), our generating function (21) would yield Feldheim's formula (4).

The general results (15) and (16) can also be applied similarly with a view to obtaining various families of generating functions.

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