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STABLE DISCRETIZATION METHODS WITH EXTERNAL APPROXIMATION SCHEMES

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ABSTRACT

We investigate the external approximation-solvability of nonlinear equations - an upgrade of the external approximation scheme of Schumann and Zeidler [3] in the context of the difference method for quasilinear elliptic differential equations.

Key words: External Approximation Scheme, Approximation-Solvability, Difference Method.

AMS (MOS) subject classifications: 65J15, 47H17.

1. Introduction

Based on the inner approximation schemes of Petryshyn [1, 2] for projection methods, Schumann and Zeidler [3] applied an external approximation scheme to difference method for quasilinear elliptic differential equations. Here we generalize the approximation-solvability of nonlinear operator equations corresponding to an external approximation scheme, which upgrades the external approximation of Schumann and Zeidler. Finally, we consider an application to the abstract generalization.

For details on the approximation-solvability, see [1-5].

Next, let $\pi_0 = \{X, F, X_n, X^*, X_n^*, A, W, A_n, R_n, K_n, E_n\}$ be an external approximation scheme represented by a diagram

$$\begin{array}{ccccc} F & \overset{W}{\leftarrow} & X & \overset{A}{\rightarrow} & X^* \\ \uparrow E_n & & \downarrow R_n & & \\ X_n & \overset{K_n}{\leftarrow} & X_n & \overset{A_n}{\rightarrow} & X_n^* \end{array} \tag{1}$$

where X, F, X_n are real Banach spaces with F reflexive and $dim\, X_n < \infty$ for all $n \in N$. Here $R_n: X \rightarrow X_n$ is a restriction operator, $E_n: X_n \rightarrow F$ is an extension operator, $K_n: X_n \rightarrow X_n$ is a linear continuous operator, and $W: X \rightarrow F$ is a synchronization operator. All operators R_n, K_n and E_n are linear and continuous with $\sup \| R_n \| < \infty$, $\sup \| K_n \| < \infty$ and $\sup \| E_n \| < \infty$. The operator W is linear, continuous and injective. Furthermore, all operators A_n are continuous.

The approximation scheme π_0 coincides with the following external approximation scheme of Schumann and Zeidler [3], $\pi_1 = \{X, F, X_n, X^*, X_n^*, A, W, A_n, R_n, E_n\}$ when K_n is the identity:

$$\begin{array}{ccccc} F & \overset{W}{\leftarrow} & X & \overset{A}{\rightarrow} & X^* \\ E_n\uparrow & & \downarrow R_n & & \\ & & X_n & \overset{A_n}{\rightarrow} & X_n^* \end{array} \tag{2}$$

and π_0 reduces to the inner approximation schemes of Petryshyn [1, 2] for projection methods when $F = X$, and W and K_n are the identities.

Let us recall some definitions for the sake of the completeness. In what follows, the symbols "$\rightarrow$" and "$\overset{w}{\rightarrow}$" shall denote the strong convergence and weak convergence, respectively.

D1.1 (Admissible external approximation scheme): The approximation scheme π_0 is called an *admissible external approximation scheme* if the following implications should hold:

(I1) **Compatibility condition:** For all $x \in X$, as $n \rightarrow \infty$,

$$E_n K_n R_n x \rightarrow W(x) \text{ in } F.$$

(I2) **Synchronization condition:** The weak limits in F of the sequences $\{E_n K_n x_n\}$ and their subsequences are synchronized, that is, if

$$E_{n'} K_{n'} x_{n'} \overset{w}{\rightarrow} f \text{ in } F \text{ as } n \rightarrow \infty,$$

then $f \in W(X)$.

D1.2 (Discrete convergence): For a sequence (x_n) of elements with $x_n \in X_n$ for all $n \in N$, (x_n) is said to *converge discretely* to x $\left(x_n \overset{d}{\rightarrow} x\right)$ iff

$$\lim_{n \rightarrow \infty} \| x_n - R_n x \| = 0.$$

D1.3 (Discrete* convergence): For a sequence (x_n^*) of functionals with $x_n^* \in X_n^*$ for all $n \in N$, the sequence (x_n^*) is said to *converge discretely** to $x^* \in X^*\left(x_n^* \overset{d^*}{\rightarrow} x^*\right)$ iff

$$\lim_{n \rightarrow \infty} [x_n^*, x_n]_{X_n} = [x^*, x]_X$$

holds for all sequences (x_n), $x_n \in X_n$ with $\sup \| x_n \|_{X_n} < \infty$ and

$$E_n K_n x_n \overset{w}{\rightarrow} W(x) \text{ in } F \text{ as } n \rightarrow \infty.$$

2. External Approximation-Solvability

In this section, we consider the unique approximation-solvability of the initial value problem

$$Ax = b, \quad x \in X, \tag{3}$$

and corresponding discretized problem

$$A_n x_n = b_n, \ \ x_n \in X_n, \ \ n = 1, 2, \ldots, \tag{4}$$

with respect to the approximation scheme π_0 represented by the diagram (1).

Theorem 2.1: *Suppose that the approximation scheme π_0 represents an admissible external approximation scheme, and the following assumptions hold:*

(A1) ***Weak Consistency:*** *For all $x \in X$,*

$$A_n R_n x \xrightarrow{d^*} Ax.$$

(A2) ***Stability:*** *For all $x, y \in X_n$ and $n \geq n_0$,*

$$\| A_n x - A_n y \|_{X_n^*} \geq \mu(\| x - y \|_{X_n}),$$

where μ is a suitable gauge function.

(A3) ***Approximation of the term b in* (3):** *For each $b \in X^*$, there exists a sequence (b_n) such that*

$$b_n \xrightarrow{d^*} b \text{ for } b_n \in X_n^* \text{ and for all } n \geq n_0.$$

Then the following conditions are equivalent:

(C1) **Solvability:** For each $b \in X^*$, the equation

$$Ax = b, \ \ x \in X,$$

has a solution.

(C2) **Unique approximation-solvability:** The equation $Ax = b$ is said to be *uniquely approximation-solvable* if the following implications hold:

(*i*) For $b \in x^*$, the equation $Ax = b$ has a unique solution $x \in X$.

(*ii*) For each $b_n \in X_n^*$ and all $n \geq n_0$, the approximate equation

$$A_n x_n = b_n$$

has a unique solution $x_n \in X_n$.

(*iii*) As $n \to \infty$,

$$b_n \xrightarrow{d^*} b \Rightarrow x_n \xrightarrow{d} x \text{ and } E_n K_n x_n \to W(x) \text{ in } F.$$

(C3) ***A*-properness:** The operator $A: X \to X^*$ is *A-proper with respect to the approximation scheme* π_0, that is, if the following implications hold:

$$A_{n'} x_{n'} \xrightarrow{d^*} b \text{ and } \sup \| x_{n'} \|_{X_{n'}} < \infty$$

imply the existence of a subsequence $(x_{n''})$ such that

$$x_{n''} \xrightarrow{d} x \text{ and } Ax = b.$$

More precisely, the theorem can be expressed as follows: **If the approximation scheme π_0 is an admissible external approximation scheme with weak consistency and stability, then the equation $Ax = b$, $x \in X$, is uniquely approximation-solvable iff A is A-proper.**

Remark 2.2: Let the assumptions $(A1)$-$(A3)$ hold. Then we have two different situations for using Theorem 2.1:

(S1) Abstract existence theorems imply the unique approximation-solvability, that is, if the equation $Ax = b$, $x \in X$, has a solution, i.e., $(C1)$ holds, then, by Theorem 2.1, the equation $Ax = b$ is uniquely approximation-solvable, and $A: X \to X^*$ is A-proper.

(S2) A-properness implies the unique approximation-solvability, that is, if we show the A-properness of $A: X \to X^*$ by a direct argument, then the equation $Ax = b$, $x \in X$, by Theorem 2.1, is uniquely approximation-solvable.

Corollary 2.3: *Theorem* 2.1 *reduces to the theorem of Schumann and Zeidler* [3] *when* K_n *is the identity.*

Before proving Theorem 2.1, we give a lemma, crucial to the proof.

Lemma 2.4: *Let* π_0 *be an admissible external approximation scheme. Then the following implications hold:*

(*i*) $x_n^* \xrightarrow{d^*} x^* \Rightarrow sup_n \| x_n^* \| < \infty$.

(*ii*) $x_n^* \xrightarrow{d^*} 0 \Rightarrow \lim_{n \to \infty} \| x_n^* \| = 0$.

Proof: (*i*) Let $x_n^* \xrightarrow{d^*} x^*$. Assume $\sup_n \| x_n^* \| < \infty$ does not hold. Then there is a subsequence, again denoted by (x_n^*), such that

$$\| x_n^* \| > n \text{ for all } n.$$

As $\| x_n^* \| = \sup\{[x_n^*, x_n]: \| x_n \| = 1\}$, there exists a subsequence, again denoted by (x_n), such that

$$\| x_n \| = 1 \text{ and } [x_n^*, x_n] > n \text{ for all } n. \tag{5}$$

Since $\sup_n \| E_n \| < \infty$ and $\sup_n \| K_n \| < \infty$, we have $\sup_n \| E_n K_n x_n \| < \infty$. Given that F is reflexive,

$$E_n K_n x_n \xrightarrow{w} f \text{ in } F \text{ as } n \to \infty.$$

The synchronization condition (I2) implies that

$$f = W(x).$$

Thus, $x_n^* \xrightarrow{d^*} x^*$ leads to

$$[x_n^*, x_n] \to [x^*, x] \text{ as } n \to \infty,$$

which contradicts (5).

(*ii*) Let $x_n^* \xrightarrow{d^*} 0$. Since

$$\| x_n^* \| = \sup\{[x_n^*, x_n]: \| x_n \| = 1\},$$

there exists a sequence (x_n) with $\| x_n \| = 1$ and

$$| \, \| x_n^* \| - [x_n^*, x_n] \, | < _{\iota\iota} \text{ for all } n.$$

By similar arguments as in the proof of (*i*), there is a subsequence, again denoted by (x_n), such that

$$E_n K_n x_n \xrightarrow{w} W(x) \text{ in } F \text{ as } n \to \infty.$$

Since $x_n^* \xrightarrow{d^*} 0$, this implies that

$$[x_n^*, x_n] \to 0 \text{ as } n \to \infty,$$

and hence

$$\| x_n^* \| \to 0 \text{ as } n \to \infty.$$

Proof of Theorem 2.1: To prove (C3)⇒(C2), we first show that, for fixed $b_n \in X_n^*$, the equation $A_n x_n = b_n$ has exactly one solution $x_n \in X_n$ for all $n \geq n_0$.

Since the operator $A_n: X_n \to X_n^*$ is injective, by the stability condition (A2), the set $A_n(X_n)$ is open by the Brouwer theorem on the invariance of domain ([6], Theorem 16C). Next, to show that the set $A_n(X_n)$ is closed, let $A_n x_k \to z$ as $k \to \infty$. Then $(A_n x_k)$ is a Cauchy sequence in X_n. It is easy to see that (x_k) is also a Cauchy sequence, by the stability condition (A2), in X_n. Hence, $x_k \to x$ as $k \to \infty$. Since A_n is continuous, we get $A_n x = z$, that means, $z \in A_n(X_n)$. To this end, since the nonempty set $A_n(X_n)$ is both open and closed, this implies that $A_n(X_n) = X_n^*$.

Second, we proceed to show, for fixed $b \in X^*$, that the equation $Ax = b$ has at most one solution $x \in X$. Let us assume $Ax = Ay$. Then, by the stability condition (A2), we obtain

$$\mu(\| R_n x - R_n y \|) \leq \| A_n R_n x - A_n R_n y \| \text{ for all } n.$$

By the weak consistency condition (A1), we get

$$A_n R_n x - A_n R_n y \xrightarrow{d^*} 0,$$

and by Lemma 2.4(*ii*), we have

$$\| A_n R_n x - A_n R_n y \| \to 0 \text{ as } n \to \infty.$$

Thus,

$$\mu(\| R_n x - R_n y \|) \leq \| A_n R_n x - A_n R_n y \| \to 0 \text{ as } n \to \infty.$$

This implies that

$$\| R_n x - R_n y \| \to 0 \text{ as } n \to \infty.$$

It follows that

$$\| E_n K_n R_n x - E_n K_n R_n y \| \leq (\sup \| E_n \|)(\sup \| K_n \|) \| R_n x - R_n y \| \to 0 \text{ as } n \to \infty,$$

and by the compatibility condition (I1),

$$W(x - y) = 0,$$

that is, $x = y$.

Third, we show that, for $b \in X^*$, the equation $Ax = b$ has exactly one solution $x \in X$. Let us choose a sequence (b_n) such that $b_n \xrightarrow{d^*} b$ as in (A3), $A_n x_n = b_n$ as in the first part of the proof. Then it follows from (A1) and Lemma 2.4(*i*) that

$$A_nR_n(0) \xrightarrow{d^*} A(0) \text{ and } \sup \| A_nR_n(0) \| < \infty.$$

Since $R_n(0) = 0$, we have

$$\| b_n \| = \| A_nx_n \| \geq \| A_nx_n - A_n(0) \| - \| A_nR_n(0) \|$$
$$\geq \mu(\| x_n \|) - \| A_nR_n(0) \|.$$

This implies that

$$\sup \mu(\| x_n \|) < \infty \Rightarrow \sup \| x_n \| < \infty.$$

Since A is A-proper, we obtain

$$x_{n'} \xrightarrow{d} x \text{ and } Ax = b.$$

Fourth, we show that $b_n \xrightarrow{d^*} b$ and $A_nx_n = b_n$ imply that

$$x_n \xrightarrow{d} x \text{ and } E_nK_nx_n \rightarrow W(x) \text{ in } F.$$

It follows from the preceding part that each subsequence $(x_{n'})$ of (x_n) has another subsequence $(x_{n''})$ such that

$$x_{n''} \xrightarrow{d} x \text{ and } Ax = b.$$

The limit element x remains the same for all subsequences since $Ax = b$ has exactly one solution x. Thus, the convergence of the whole sequence follows, that is,

$$x_n \xrightarrow{d} x.$$

Finally, we show that

$$x_n \xrightarrow{d} x \Rightarrow E_nK_nx_n \rightarrow W(x) \text{ in } F.$$

Since $x_n \xrightarrow{d} x$ and π_0 is an admissible external approximation scheme, we get, as $n \rightarrow \infty$,

$$\| E_nK_nx_n - W(x) \| = \| E_nK_nx_n - E_nK_nR_nx + E_nK_nR_nx - W(x) \|$$
$$\leq \| E_nK_nx_n - E_nK_nR_nx \| + \| E_nK_nR_nx - W(x) \|$$
$$\leq (\sup \| E_n \|)(\sup \| K_n \|) \| x_n - R_nx \| + \| E_nK_nR_nx - W(x) \| \rightarrow 0.$$

The proof of (C2)$\Rightarrow$(C1) is trivial.

Finally, we prove: (C1)$\Rightarrow$(C3). We denote the subsequence of a sequence (x_n), again by (x_n). Let

$$A_nx_n \xrightarrow{d^*} b \text{ with } \sup_n \| x_n \| < \infty.$$

We further choose a point $x \in X$ with $Ax = b$ as in (C1). It suffices to show that

$$x_n \xrightarrow{d} x,$$

that is, the condition (C3) holds. By (A1), we have

$$A_n R_n x \xrightarrow{d^*} Ax.$$

It follows that

$$A_n x_n - A_n R_n x \xrightarrow{d^*} 0,$$

and by Lemma 2.4(*ii*),

$$\| A_n x_n - A_n R_n x \| \to 0 \text{ as } n \to \infty.$$

Then, by the stability condition (A2), we get, as $n \to \infty$,

$$\mu(\| x_n - R_n x \|) \leq \| A_n x_n - A_n R_n x \| \to 0.$$

This implies that

$$\| x_n - R_n x \| \to 0 \text{ as } n \to \infty,$$

that is,

$$x_n \xrightarrow{d} x.$$

This completes the proof.

Theorem 2.5: *Let $\pi_0 = \{X, F, X_n, X^*, X_n^*, A, W, A_n, R_n, K_n, E_n\}$ be an admissible external approximation scheme represented by the diagram (1). If X_0 is dense in X, then*

$$E_n K_n R_n x \to W(x) \textit{ for all } x \in X_0$$

implies that

$$E_n K_n R_n x \to W(x) \textit{ for all } x \in X.$$

Proof: Let $E_n K_n R_n x \to W(x)$ as $n \to \infty$ for all $x \in X_0$, where X_0 is dense in X. We need to show that, for all $y \in X$,

$$E_n K_n R_n y \to W(y) \text{ as } n \to \infty.$$

Let $y \in X$ and $\epsilon > 0$ be fixed. Then

$$\| E_n K_n R_n y - W(y) \| \leq \| E_n K_n R_n y - E_n K_n R_n x \| + \| E_n K_n R_n x - W(x) \| + \| W(x) - W(y) \|$$

$$\leq (\sup_n \| E_n \| \; \| K_n \| \; \| R_n \|) \| y - x \| + \| E_n K_n R_n x - W(x) \| + \| W \| \; \| y - x \|$$

$$= (\sup_n \| E_n \| \; \| K_n \| \; \| R_n \| + \| W \|) \| y - x \| + \| E_n K_n R_n x - W(x) \|$$

$$< \epsilon \text{ for all } n \geq n_0(\epsilon),$$

where $x \in X_0$ is so chosen that $\| y - x \|$ is sufficiently small. This competes the proof.

3. Application

Let us consider the following external approximation scheme $\pi_2 = \{X, F, X_n, X^*, A, W, A_n, R_n, K_n, E_n\}$:

$$\begin{array}{ccccc} F & \overset{W}{\leftarrow} & X & \overset{A}{\rightarrow} & X^* \\ \uparrow E_n & & \downarrow R_n & & \\ X_n & \overset{K_n}{\leftarrow} & X_n & \overset{A_n}{\rightarrow} & X_n^* \end{array} \tag{6}$$

where $X = \overset{\circ}{W}{}^1_p(G)$, $X_n = \overset{\circ}{W}{}^1_p(g_{h_n})$, the Sobolev spaces, and $F = \prod_{i=1}^{N} L_p(G)$, $2 \leq p < \infty$. Here G is a bounded region in $\mathbb{R}^N$ $N \geq 1$, with sufficiently smooth boundary, that is, $\delta G \in C^{0,1}$. A sufficiently small positive number h_0 is chosen so that the set g_h of interior lattice points is not empty for all h, $0 \leq h \leq h_0$. Furthermore, $\overline{f}_h(P)$ represents the integral mean value of f over the cube $c_h(P)$ belonging to P, that is,

$$\overline{f}_h(P) = h^{-N} \int_{c_h(P)} f(t)dt.$$

The operators $W: X \to F, R_n: X \to X_n, K_n: X_n \to X_n$ and $E_n: X_n \to F$ are defined as follows:

$$W(x) = (x, D_1 x, \ldots, D_N x),$$

$$(R_n x)(P) = \begin{cases} k^{-N} \int_{c_h(P)} x(t)dt & \text{for } P \in g_{k,1} \\ 0 & \text{for } P \notin g_{k,1}, \end{cases}$$

and

$$E_n K_n x_n = (x_n, \nabla_1 x_n, \ldots, \nabla_N x_n).$$

Now, we can apply Theorem 2.1, for example, to the boundary value problem

$$\begin{cases} -\sum_{i=1}^{N} D_i(|D_i x|^{p-2} D_i x) + sx = f & \text{on } G, \\ x = 0 & \text{on } \delta G, \end{cases} \tag{7}$$

with corresponding difference equations

$$\begin{cases} -\sum_{i=1}^{N} \nabla_i^-(|\nabla_i x_h(P)|^{p-2} \nabla_i x_h(P)) + s x_h(P) = \overline{f}_h(P) & \text{for all } P \in g_h \\ x_h(P) = 0 & \text{for all } P \in \delta g_h. \end{cases} \tag{8}$$

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