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MEAN NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL. III

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ABSTRACT

If $a_1, a_2, \ldots, a_n$ are independent, normally distributed random variables with mean 0 and variance 1, and if ν_n is the mean number of zeros on the interval $(0, 2\pi)$ of the trigonometric polynomial $a_1 \cos x + 2^{1/2} a_2 \cos 2x + \ldots n^{1/2} a_n \cos nx$, then $\nu_n = 2^{-1/2}\{(2n+1) + D_1 + (2n+1)^{-1} D_2 + (2n+1)^{-2} D_3\} + O\{(2n+1)^{-3}\}$, in which $D_1 = -0.378124$, $D_2 = -\frac{1}{2}$, $D_3 = 0.5523$. After tabulation of $5D$ values of ν_n when $n = 1(1)40$, we find that the approximate formula for ν_n, obtained from the above result when the error term is neglected, produces $5D$ values that are in error by at most 10^{-5} when $n \geq 8$, and by only about 0.1% when $n = 2$.

Key words: Random Trigonometric Polynomials, Real Zeros.

AMS (MOS) subject classifications: 60G99.

1. Introduction

Suppose that n is an integer greater than 1, that a_j $(j = 1, 2, \ldots, n)$ are independent, normally distributed random variables with mean 0 and variance 1, that p is a real number greater than $-\frac{1}{2}$, and that ν_{np} is the mean value of the number of zeros on the interval $(0, 2\pi)$ of the random trigonometric polynomial

$$\sum_{j=1}^{n} j^p a_j \cos jx. \tag{1.1}$$

Das [4] has shown that, for large n,

$$\nu_{np} = 2\mu_p n + O(n^{1/2}), \ \mu_p = \{(2p+1)/(2p+3)\}^{1/2}. \tag{1.2}$$

The author ([6] when $p = 0$ and [7] when p is a positive integer) has exhibited constants $D_{0p} = 1$, D_{1p}, D_{2p} and D_{3p} such that

$$\nu_{np} = (2n+1)\mu_p \sum_{r=0}^{3} (2n+1)^{-r} D_{rp} + O\{(2n+1)^{-3}\}. \tag{1.3}$$

It follows that the error term $O(n^{1/2})$ in the Das result is actually $O(1)$ when p is a nonnegative integer. In this paper we will prove that a relation of the form (1.3) is also valid when $p = \frac{1}{2}$. This proof emulates the analysis in [7], although that analysis actually fails when p is not a positive integer.

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After a statement of the basic formulas on which our analysis rests, we devote Section 2 to the derivation of a series representation of $\nu_{n,1/2}$ that converges when n is sufficiently large. (Henceforth we will omit the subscript $\frac{1}{2}$.) Asymptotic representations of the first four coefficients in that series are derived in Section 3, and are used to deduce (1.3) when $p=\frac{1}{2}$. We tabulate in Section 4 $5D$ values of ν_n when $n=1(1)40$. We find that the approximation to ν_n, obtained from (1.3) when the $O\{(2n+1)^{-3}\}$ term is neglected, produces $5D$ values that differ from the tabulated values by at most 10^{-5} when $n \geq 8$ and by only about 0.1% when $n=2$. In Section 5 we show that the series representation of ν_n, derived in Section 2, actually converges when $n \geq 2$.

2. Preliminary Analysis

It is a consequence of the basic formulas in [7] (that were copied from [3, p. 285] or [2, p. 107]) that, when $n \geq 2$,

$$\nu_n = 4\pi^{-1} \int_0^{\pi/2} F_n(x)dx, \tag{2.1}$$

in which

$$F_n(x) = A_n^{-1}(A_n C_n - B_n^2)^{1/2}, \tag{2.2}$$

$$A_n = \sum_{j=1}^{n} j\cos^2 jx, \quad B_n = \sum_{j=1}^{n} j^2 \sin jx \cos jx, \quad C_n = \sum_{j=1}^{n} j^3 \sin^2 jx. \tag{2.3}$$

We will need the explicit representations of A_n, B_n and C_n stated in the following lemma.

Lemma 1: *It is true that*

$$8A_n = (2n+1)^2 g_0(z) + (2n+1)g_1 + g_2, \tag{2.4}$$

$$16B_n = (2n+1)^3 h_0(z) + (2n+1)^2 h_1 + (2n+1)h_2 + h_3, \tag{2.5}$$

$$32C_n = (2n+1)^4 k_0(z) + (2n+1)^3 k_1 + (2n+1)^2 k_2 + (2n+1)k_3 + k_4, \tag{2.6}$$

in which

$$z = (2n+1)x, f(x) = cscx - x^{-1}, \varphi(x) = f^2(x) + 2x^{-1}f(x) = csc^2 x - x^{-2}, \tag{2.7}$$

$$g_0(z) = \left(\frac{1}{2}\right) + z^{-1} sin\, z - z^{-2}(1 - cos\, z), g_1 = f(x)\, sin\, z, \tag{2.8}$$

$$g_2 = -\{\left(\frac{1}{2}\right) + \varphi(x) + f'(x) cos\, z\}, \tag{2.9}$$

$$h_0(z) = -g_0'(z) = -z^{-1} cos\, z + 2z^{-2} sin\, z - 2z^{-3}(1 - cos\, z), \tag{2.10}$$

$$h_1 = -f(x) cos\, z, h_2 = -2f'(x) sin\, z, h_3 = \varphi'(x) + f''(x) cos\, z, \tag{2.11}$$

$$k_0(z) = \left(\frac{1}{4}\right) - z^{-1} sin\, z - 3z^{-2} cos\, z + 6z^{-3} sin\, z - 6z^{-4}(1 - cos\, z), \tag{2.12}$$

$$k_1 = -f(x) sin\, z, k_2 = 3f'(x) cos\, z - \left(\frac{1}{2}\right), k_3 = 3f''(x) sin\, z, \tag{2.13}$$

$$k_4 = \left(\frac{1}{4}\right) - \varphi''(x) - f'''(x)\cos z. \tag{2.14}$$

With the help of a known trigonometric summation [5, p. 133, Eq. (31)], we see that

$$8A_n = 4\sum_{j=1}^{n} j + 4\sum_{j=1}^{n} j\cos 2jx$$
$$= 2n(n+1) + 2\{(n+1)\sin z - \sin x\}csc\, x + \{\cos(z+x) - \cos 2x\}csc^2 x.$$

A little trigonometric and algebraic manipulation then suffices to establish (2.4), (2.7), (2.8), and (2.9). The remaining results of the lemma are consequences of these and the inferences from (2.3) that

$$2B_n = -dA_n/dx, 2C_n = \sum_{j=1}^{n} j^3 - dB_n/dx = \{n(n+1)/2\}^2 - dB_n/dx.$$

We calculate the first factor A_n^{-1} in (2.2) in the following lemma.

Lemma 2: *There is a positive integer n_0 such that, if $0 \le x \le \pi/2$ and $n \ge n_0$,*

$$(8A_n)^{-1} = (2n+1)^{-2} g_0^{-1} \sum_{r=0}^{\infty} (2n+1)^{-r} b_r, \tag{2.15}$$

in which $b_0 = 1$, $b_1 = -g_1/g_0$, $b_2 = b_1^2 - (g_2/g_0)$, $b_3 = (2g_1g_2/g_0^2) + b_1^3, \ldots$ The series (2.15) converges absolutely and uniformly when $0 \le x \le \pi/2$ and $n \ge n_0$.

It is a consequence of (2.8) that

$$g_0 = \int_0^1 t(1 + \cos zt)dt. \tag{2.16}$$

Therefore, $g_0 > 0$ when $0 \le z < +\infty$, and $\lim_{z \to \infty} g_0 = \frac{1}{2}$, so that both g_0 and g_0^{-1} are bounded functions of z. Because the functions g_1 and g_2, defined in (2.8) and (2.9), are obviously bounded, uniformly in x and n when $0 \le x \le \pi/2$ and $n \ge 2$, it follows that there exists a positive integer n_0 so large that

$$(2n+1)^{-1} | g_1/g_0 | + (2n+1)^{-2} | g_2/g_0 | < 0.65 \tag{2.17}$$

when $n \ge n_0$ and $0 \le x \le \pi/2$. The expression (2.4) can be inverted for such values of x and n; this process yields (2.15), in which the first four coefficients are those specified in the lemma.

A straightforward calculation based on (2.4), (2.5) and (2.6), proves the following lemma.

Lemma 3: *It is true that*

$$256(A_nC_n - B_n^2) = (2n+1)^6 \sum_{r=0}^{6} (2n+1)^{-r} f_r, \tag{2.18}$$

in which

$$f_r = \sum_{m=0}^{r} (g_m k_{r-m} - h_m h_{r-m}) \quad (r = 1, 2, \ldots, 6), \tag{2.19}$$

and $g_m = 0$ if $m > 2$, $h_m = 0$ if $m > 3$ and $k_m = 0$ if $m > 4$.

We calculate the second factor $(A_nC_n - B_n^2)^{1/2}$ of (2.2) in the following lemma.

Lemma 4: *The integer n_0 of Lemma 2 may be chosen so that, if $0 \le x \le \pi/2$ and $n \ge n_0$,*

$$16(A_nC_n - B_n^2)^{1/2} = (2n+1)^3 f_0^{1/2} \sum_{r=0}^{\infty} (2n+1)^{-r} c_r, \tag{2.20}$$

in which $c_0 = 1$, $c_1 = f_1/(2f_0)$, $c_2 = f_2/(2f_0) - f_1^2/(8f_0^2)$, $c_3 = f_3/(2f_0) - f_1f_2/(4f_0^2) + f_1^3/(16f_0^3)$, ... *The series* (2.20) *converges absolutely and uniformly when* $0 \le x \le \pi/2$ *and* $n \ge n_0$.

We infer from (2.10) and (2.12) that

$$h_0 = \int_0^1 t^2 \sin zt\,dt, k_0 = \int_0^1 t^3(1 - \cos zt)dt, \tag{2.21}$$

so that (2.16) and the Schwartz inequality imply that $f_0 = g_0k_0 - h_0^2 > 0$ when $0 < z < +\infty$. Moreover, $f_0 = z^2/48 + O(z^4)$ for small z, and $f_0 = 1/8 + O(z^{-1})$ for large z.

Because $f(x)$ is an odd analytic function of x when $|x| < \pi$, and because $0 \le x = z/(2n+1) \le z/5$ when $n \ge 2$, it follows from (2.21) and (2.11) that $h_r = O(z)$ $(r = 0,1,2,3)$, uniformly in x and n, and from (2.21), (2.13) and (2.14) that $k_r = O(z^2)$ $(r = 0,1,2,3,4)$, uniformly in x and n. It is now a consequence of (2.19) and the earlier observation that $g_r = O(1)$ $(r = 0,1,2)$, uniformly in x and n, that $f_r = O(z^2)$ $(r = 1,2,3,4,5,6)$, uniformly in x and n. Because it is obvious that $h_r = O(1)$ and $k_r = O(1)$, uniformly in x and n, we conclude that $f_r/f_0 = O(1)$, uniformly in x and n (whether or not z is small). Hence we can choose n_0 so large that (2.17) holds, and

$$\sum_{r=1}^{6} (2n+1)^{-r} |f_r/f_0| < 0.92 \tag{2.22}$$

when $n \ge n_0$ and $0 \le x \le \pi/2$. For such values of x and n the square root of the expression (2.18) can be written in the form (2.20), the first four coefficients of which are those specified in the lemma.

We will show in Section 5 that the inequalities (2.17) and (2.22) hold when $n \ge 2$. Therefore, Lemmas 2 and 4 (also Lemmas 5 and 6 below) are valid when $n_0 = 2$.

If we use (2.2) and Lemmas 2 and 4, we obtain the following lemma.

Lemma 5: *It is true when* $0 \le x \le \pi/2$ *and* $n \ge n_0$ *that*

$$2F_n(x) = (2n+1)G(z) \sum_{r=0}^{\infty} (2n+1)^{-r} u_r, \tag{2.23}$$

in which

$$G(z) = f_0^{1/2}(z)/g_0(z), u_r = \sum_{m=0}^{r} b_m c_{r-m}. \tag{2.24}$$

Moreover, the series (2.23) *converges absolutely and uniformly in* x *and* n.

The final lemma in this section is a consequence of (2.1) and Lemma 5.

Lemma 6: *It is true when* $n \ge n_0$ *that*

$$\nu_n = (2n+1) \sum_{r=0}^{\infty} (2n+1)^{-r} v_r, \tag{2.25}$$

in which

$$v_r = 2\pi^{-1} \int_0^{\pi/2} G(z) u_r dx. \tag{2.26}$$

Moreover, the series (2.25) *converges absolutely and uniformly when* $n \geq n_0$.

3. Proof of (1.3) when $p = \frac{1}{2}$

In the next four lemmas we will exhibit constants $S_{rm}(0 \leq m \leq 3-r,\ r = 0,1,2,3)$ and S_r $(r = 0,1,2,3)$ such that

$$2^{1/2}v_r = \sum_{m=0}^{3-r}(2n+1)^{-m}S_{rm} + (-1)^n(2n+1)^{r-3}S_r + O\{(2n+1)^{r-4}\} \tag{3.1}$$

when $r = 0,1,2,3$. In the proofs of these lemmas, it will be convenient to use $T_q(z)$ as a generic symbol for a trigonometric sine polynomial of degree q, not necessarily the same at each occurrence.

Lemma 7: *Equation* (3.1) *is true when* $r = 0$ *if*

$$S_{00} = 1,\ S_{01} = (2/\pi)\int_0^\infty \{2^{1/2}G(z) - 1\}dz, \tag{3.2}$$

$$S_{02} = 1/\pi^2,\ S_{03} = 0,\ S_0 = -32/\pi^3. \tag{3.3}$$

It follows from (2.19), (2.8), (2.10) and (2.12) that

$$8f_0 = 1 - 2z^{-1}\sin z - 10z^{-2}(1+\cos z) + 32z^{-3}\sin z - 8z^{-4}(1-\cos z)^2 \tag{3.4}$$
$$- 32z^{-5}(1-\cos z)\sin z + 16z^{-6}(1-\cos z)^2.$$

For sufficiently large z we conclude that

$$(8f_0)^{-1} = 1 + 2z^{-1}\sin z + 2z^{-2}(6 + 5\cos z - \cos 2z) + z^{-3}T_3(z) + O(z^{-4}), \tag{3.5}$$

$$(8f_0)^{1/2} = 1 - z^{-1}\sin z - (4z)^{-2}(21 + 20\cos z - \cos 2z) \tag{3.6}$$
$$+ z^{-3}T_3(z) + O(z^{-4}).$$

Because it follows from (2.8) that

$$(2g_0)^{-1} = 1 - 2z^{-1}\sin z + 2z^{-2}(2 - \cos z - \cos 2z) + z^{-3}T_3(z) + O(z^{-4}), \tag{3.7}$$

we infer from (2.24) and (3.6) that

$$2^{1/2}G(z) = 1 - 3z^{-1}\sin z - (4z)^{-2}(1 + 28\cos z + 11\cos 2z) \tag{3.8}$$
$$+ z^{-3}T_3(z) + O(z^{-4}).$$

If we define λ to be $(2n+1)\pi/2$, it follows from (2.26) that

$$2^{1/2}v_0 = 1 + \lambda^{-1}\int_0^\infty \{2^{1/2}G(z) - 1\}dz - \lambda^{-1}\int_\lambda^\infty \{2^{1/2}G(z) - 1\}dz; \tag{3.9}$$

the improper integrals exist by virtue of (3.8). Repeated integration by parts of the last term in

(3.9) now shows, with the help of (3.8), that the lemma is true.

Lemma 8: *Equation* (3.1) *is true when* $r=1$ *if*

$$S_{10}=0,\ S_{11}=-(5/\pi)\int_0^{\pi/2} x^{-1}f(x)dx, S_1=-24/\pi^3, \tag{3.10}$$

$$S_{12}=-(3\pi)^{-1}\int_0^{\infty} z\{2^{1/2}H(z)-3\,sin\,z-(2z)^{-1}(5+11\,cos\,2z)\}dz, \tag{3.11}$$

in which

$$H(z)=[\{(2f_0)^{-1}(g_0-k_0)+g_0^{-1}\}sin\,z-f_0^{-1}h_0\,cos\,z]G(z). \tag{3.12}$$

It follows from (2.19), (2.8), (2.11) and (2.13) that

$$f_1=-\{(g_0-k_0)\sin z-2h_0\cos z\}f(x), \tag{3.13}$$

and then from (2.8), (2.10) and (2.12) that

$$f_1=-\{4^{-1}\sin z+2z^{-1}-z^{-2}\sin z-z^{-3}(5-4\cos z-\cos 2z) \tag{3.14}$$
$$+6z^{-4}(1-\cos z)\sin z\}f(x).$$

We can now deduce from (2.24), Lemmas 2 and 4, and (3.13) that

$$G(z)u_1=-f(x)H(z), \tag{3.15}$$

in which $H(z)$ is the function defined in (3.12). With the help of (3.5), (3.7), (3.8) and the definitions (2.8), (2.10) and (2.12), it is easy to see that, for large z,

$$H(z)=H^*(z)+O(z^{-3}) \tag{3.16}$$

in which

$$2^{1/2}H^*(z)=3\sin z+(2z)^{-1}(5+11\cos 2z) \tag{3.17}$$
$$+8^{-1}z^{-2}(151\sin z-60\sin 2z-69\sin 3z).$$

Moreover, $H(z)=O(1)$ and $H^*(z)=O(z^{-1})$ for small z. Hence (3.16) is true for all positive z. It now follows from (2.26) and (3.15) that

$$-\pi v_1/2=I_1+I_2+I_3, \tag{3.18}$$

in which

$$I_1=\int_0^{\pi/2}\{f(x)-(x/6)\}\{H(z)-H^*(z)\}dx, \tag{3.19}$$

$$I_2=\int_0^{\pi/2}(x/6)\{H(z)-H^*(z)\}dx, \tag{3.20}$$

$$I_3 = \int_0^{\pi/2} f(x)H^*(z)dx. \tag{3.21}$$

We recall the identity [6, Lemma 2],

$$\begin{aligned} f(x) &= \sum_{m=1}^{\infty} (-1)^{m-1}(2^{2m}-2)\beta_{2m}x^{2m-1}/(2m)! \\ &= (x/6) + (7x^3/360) + (31x^5/15120) + \ldots, \end{aligned} \tag{3.22}$$

in which β_{2m} is the Bernoulli number of order $2m$ [1, p. 804]. It is known [1, pp. 75, 805] that the power series in (3.22) converges when $|x| < \pi$, and that all of its nonzero coefficients are positive. We conclude from (3.22), (3.16), and (3.19) that

$$I_1 = \int_0^{\pi/2} O(x^3)O(z^{-3})dx = O\{(2n+1)^{-3}\}. \tag{3.23}$$

Similarly, we deduce from (3.16) that

$$6I_2 = (2n+1)^{-2}\int_0^{\lambda} z\{H(z) - H^*(z)\}dz$$

$$= (2n+1)^{-2}\int_0^{\infty} z\{H(z) - H^*(z)\}dz - (2n+1)^{-2}\int_{\lambda}^{\infty} zO(z^{-3})dz,$$

$$\begin{aligned} 6\cdot 2^{1/2}I_2 = (2n+1)^{-2}\Big[\int_0^{\infty} & z\{2^{1/2}H(z) - 3\sin z \\ & - (2z)^{-1}(5 + 11\cos 2z)\}dz - 11\pi/8\Big] + O\{(2n+1)^{-3}\}. \end{aligned} \tag{3.24}$$

In order to evaluate the integral I_3, we use the results [7, Eqs. (3.15)],

$$\int_0^{\pi/2} f(x)\sin z\,dx = (2n+1)^{-2}(4/\pi^2)(-1)^n + O\{(2n+1)^{-3}\},$$

$$\int_0^{\pi/2} f(x)z^{-1}dx = (2n+1)^{-1}\int_0^{\pi/2} f(x)x^{-1}dx,$$

$$\int_0^{\pi/2} f(x)z^{-1}\cos 2z\,dx = O\{(2n+1)^{-3}\},$$

$$\int_0^{\pi/2} f(x)z^{-2}\sin qz dx = (2n+1)^{-2}(\pi/12) + O\{(2n+1)^{-3}\},$$

when $q > 0$. Therefore,

$$2^{1/2}I_3 = (5/2)(2n+1)^{-1}\int_0^{\pi/2} f(x)x^{-1}dx + (12/\pi^2)(2n+1)^{-2}(-1)^n \tag{3.25}$$

$$+(11\pi/48)(2n+1)^{-2}+O\{(2n+1)^{-3}\}.$$

We now combine (3.23), (3.24) and (3.25) with (3.18) to conclude that Lemma 8 is true.

Lemma 9: *Equation* (3.1) *is true when* $r=2$ *if*

$$S_{20}=(2/\pi)\int_0^{\pi/2}\varphi(x)dx-(5/2\pi)\int_0^{\pi/2}f^2(x)dx-\left(\tfrac{1}{2}\right), \tag{3.26}$$

$$S_{21}=(1/3\pi)\int_0^{\infty}\{2^{1/2}J(z)+1-7\cos z\}dz,\ \ S_2=56/\pi^3, \tag{3.27}$$

in which

$$J(z)=[(1-\cos z)\{(2f_0)^{-1}(k_0-3g_0)-g_0^{-1}\}+2f_0^{-1}h_0\sin z \tag{3.28}$$
$$+6g_0^{-1}-3f_0^{-1}k_0]G(z).$$

It follow from (2.19), (2.8), (2.9), (2.11) and (2.13) that

$$f_2=\{(3g_0-k_0)\cos z+4h_0\sin z\}f'(x)-f^2(x)-k_0\varphi(x)-(g_0+k_0)/2, \tag{3.29}$$

and then from (2.24), Lemmas 2 and 4, (3.13) and (3.29) that

$$Gu_2=f'(0)J(z)+\{f'(x)-f'(0)\{K(z)-f^2(x)L(z)+\{\varphi(x)-\varphi(0)\}M(z), \tag{3.30}$$

in which $J(z)$ is defined in (3.28) and

$$K(z)=[\{(2f_0)^{-1}(3g_0-k_0)+g_0^{-1}\}\cos z+2h_0f_0^{-1}\sin z]G(z), \tag{3.31}$$

$$L(z)=[(1-\cos 2z)\{(4f_0)^{-2}[(g_0-k_0)^2-4h_0^2]-(4f_0g_0)^{-1}(g_0-k_0) \tag{3.32}$$
$$-(2g_0^2)^{-1}\}+\{g_0^{-1}-(2f_0)^{-1}(g_0-k_0)\}(2f_0)^{-1}h_0\sin 2z$$
$$+(2f_0^2)^{-1}(h_0^2+f_0)]G(z),$$

$$M(z)=\{g_0^{-1}-(2f_0)^{-1}k_0\}G(z). \tag{3.33}$$

With the help of (3.5), (3.7), (3.8) and the definitions (2.8), (2.10) and (2.12), it is easy to see that, for large z,

$$2^{1/2}J(z)=-1+7\cos z-z^{-1}(31+15\cos z)\sin z+O(z^{-2}). \tag{3.34}$$

In fact, (3.34) is valid for all positive z because $J(z)$ and $z^{-1}\sin z$ are bounded. It then follows that

$$\int_0^{\pi/2}2^{1/2}J(z)dx=-(\pi/2)+[7(-1)^n+\int_0^{\infty}\{2^{1/2}J(z)+1-7\cos z\}dz \tag{3.35}$$

$$-\int_{\lambda}^{\infty}\{2^{1/2}J(z)+1-7\cos z\}dz](2n+1)^{-1}.$$

In view of (3.34) the two improper integrals converge, and the last integral in (3.35) is $O\{(2n+1)^{-1}\}$. In a similar manner, we find that

$$2^{1/2}K(z)=7\cos z-15z^{-1}\sin z\cos z+O(z^{-2}), \tag{3.36}$$

$$2^{1/2}L(z)=(5+11\cos 2z)/4+z^{-1}(111\sin z-69\sin 3z)/8+O(z^{-2}), \tag{3.37}$$

$$2^{1/2}M(z)=1-2z^{-1}\sin z+O(z^{-2}); \tag{3.38}$$

$$2^{1/2}\int_0^{\pi/2}\{f'(x)-f'(0)\}K(z)dx=7\{(2/\pi)^2-(1/6)\}(-1)^n(2n+1)^{-1} \tag{3.39}$$
$$+O\{(2n+1)^{-2}\},$$

$$2^{1/2}\int_0^{\pi/2}f^2(x)L(z)dx=(5/4)\int_0^{\pi/2}f^2(x)dx+O\{(2n+1)^{-2}\}, \tag{3.40}$$

$$2^{1/2}\int_0^{\pi/2}\{\varphi(x)-\varphi(0)\}M(z)dx=\int_0^{\pi/2}\varphi(x)dx-(\pi/6)+O\{(2n+1)^{-2}\}. \tag{3.41}$$

It now follows from (2.26), (3.30), (3.35), (3.39), (3.40) and (3.41) that Lemma 9 is true.

Lemma 10: *Equation* (3.1) *is true when* $r=3$ *if* $S_{30}=0,\ S_3=0$.

It follows from (2.19) and (2.8) through (2.13) that

$$f_3=(3g_0\sin z-2h_0\cos z)f''(x)+f(x)\varphi(x)\sin z-2h_0\varphi'(x), \tag{3.42}$$

and then from (2.24), Lemmas 2 and 4, (3.13), (3.29) and (3.42) that

$$2^{1/2}Gu_3=f''(x)N(z)+f(x)\varphi(x)P(z)+\varphi'(x)Q(z)+f(x)\{f'(x)-f'(0)\}R(z) \tag{3.43}$$
$$+f^3(x)U(z)+f(x)V(z),$$

in which N, P, Q, R, U and V are certain explicitly definable functions of z concerning which we need to know first that, for large z,

$$N(z)=6\sin z+O(z^{-1}),\ P(z)=-5\sin z+O(z^{-1}), Q(z)=O(z^{-1}), \tag{3.44}$$

$$R(z)=-7.5\sin 2z+(4z)^{-1}(193\cos z-97\cos 3z)+O(z^{-2}), \tag{3.45}$$

$$U(z)=8^{-1}(23\sin 3z-37\sin z)-(16z)^{-1}(95+132\cos 2z \tag{3.46}$$
$$-99\cos 4z)+O(z^{-2}),$$

$$V(z)=-5\sin z-(35/6)\sin 2z+O(z^{-1}). \tag{3.47}$$

We also need to know that, for small positive z, the functions $N(z)$, $P(z)$, $Q(z)$, and $V(z)$ are bounded (they actually have limits as $z \to 0^+$), and the functions $R(z)$ and $U(z)$ are $O(z^{-2})$. Hence equations (3.44) through (3.47) are valid for all positive z. Each of the integrals with respect to x on the interval $(0, \pi/2)$ of each of the six terms on the right hand side of (3.43) is, therefore, $O\{(2n+1)^{-1}\}$. In view of (2.26), this remark suffices to prove Lemma 10.

Our principal result, stated in the following theorem, is an immediate consequence of Lemmas 6 through 10 and the observation that $S_0 + S_1 + S_2 + S_3 = 0$.

Theorem: *When $p = \frac{1}{2}$ and $n \geq n_0$, the mean value of the number of zeros on the interval $(0, 2\pi)$ of the random trigonometric polynomial* (1.1) *is*

$$\nu_n = (2n+1)2^{-1/2} \sum_{r=0}^{3} (2n+1)^{-r} D_r + O\{(2n+1)^{-3}\}, \tag{3.48}$$

in which

$$D_r = \sum_{m=0}^{r} S_{r-m,m} \quad (r = 0, 1, 2, 3). \tag{3.49}$$

If we use the explicit formulas for S_{rm} furnished in Lemmas 7 through 10, it follows that

$$D_0 = 1,\ D_1 = (2/\pi) \int_0^\infty \{2^{1/2} G(z) - 1\} dz,\ \ D_2 = -\tfrac{1}{2}, \tag{3.50}$$

$$D_3 = (1/3\pi) \int_0^\infty \{2^{1/2} J(z) + 1 - 7\cos z\} dz \tag{3.51}$$

$$- (1/3\pi) \int_0^\infty z\{2^{1/2} H(z) - 3\sin z - (2z)^{-1}(5 + 11\cos 2z)\} dz.$$

(For the calculation of D_2, it is necessary to know [6, Eq. (3.56)] that $\int_0^{\pi/2} \varphi(x) dx = 2/\pi$.)

4. Numerical Results

With the help of (3.8), (3.16), (3.17), and (3.34), we can transform the conditionally convergent integrals in (3.50) and (3.51) into absolutely convergent integrals, i.e.,

$$D_1 = (2/\pi) \int_0^\infty \{2^{1/2} G(z) - 1 + 3z^{-1} \sin z\} dz - 3, \tag{4.1}$$

$$D_3 = (1/3\pi) \int_0^\infty [2^{1/2}\{J(z) - zH(z)\} + 3z\sin z + 2^{-1}(7 - 14\cos z + 11\cos 2z) \tag{4.2}$$

$$+ (8z)^{-1}(399\sin z - 69\sin 3z)\} dz - (55/8).$$

We calculated the integrals in (4.1) and (4.2) over the interval $(0, 25\pi)$ by Simpson's rule. The integrals over the interval $(25\pi, \infty)$ were calculated using the asymptotic relations,

$$2^{1/2}G(z)-1+3z^{-1}\sin z=-(2z)^{-2}(1+28\cos z+11\cos 2z) \tag{4.3}$$

$$+(2z)^{-3}(19\sin z+60\sin 2z+23\sin 3z)-2^{-6}z^{-4}(1063+2376\cos z$$

$$+716\cos 2z-776\cos 3z-179\cos 4z)+z^{-5}T_5(z)+O(z^{-6}),$$

$$2^{1/2}H(z)-3\sin z-(2z)^{-1}(5+11\cos 2z) \tag{4.4}$$

$$-2^{-3}z^{-2}(151\sin z-60\sin 2z-69\sin 3z)$$

$$=2^{-4}z^{-3}(207+1028\cos z+612\cos 2z-388\cos 3z$$

$$-179\cos 4z)+z^{-4}T_5(z)+O(z^{-5}),$$

$$2^{1/2}J(z)+1-7\cos z+(2z)^{-1}(62\sin z+15\sin 2z) \tag{4.5}$$

$$=2^{-3}z^{-2}(46-335\cos z-478\cos 2z-97\cos 3z)+z^{-3}T_4(z)+O(z^{-4});$$

these relations contain one or two more terms than (3.8), (3.16) and (3.34). In this manner, we find that $D_1=-0.378124$, $D_3=0.5523$. Hence

$$2^{1/2}\nu_n=(2n+1)-0.378124-2^{-1}(2n+1)^{-1}+0.5523(2n+1)^{-2}+O\{(2n+1)^{-3}\}. \tag{4.6}$$

We have also used Simpson's rule to calculate numerical values of the integral (2.1) when $n=2(1)40$. The results for ν_n are recorded to $5D$ in Table 1. (The tabulated value $\nu_1=2$ is obviously correct, but is not a consequence of (2.1) which is nugatory when $n=1$.) The approximation $2^{-1/2}(2n+1+D_1)$ produces $5D$ values that exceed the values in Table 1 by about 0.00756% when $n=40$, 0.0292% when $n=20$, 0.110% when $n=10$ and only 1.61% when $n=2$. The more accurate approximation $2^{-1/2}\{2n+1+D_1-(4n+2)^{-1}\}$ produces $5D$ values that are less than the values in Table 1 by about 6×10^{-5} when $n=40$, 23×10^{-5} when $n=20$, 89×10^{-5} when $n=10$, and only 0.0189 or 0.588% when $n=2$. The most accurate approximation $2^{-1/2}\{2n+1+D_1-(4n+2)^{-1}+0.5523(2n+1)^{-2}\}$ produces $5D$ values that agree with those in Table 1 when $n\geq 8$, except for a discrepancy of one unit in the last decimal place when $n=14$. Moreover, it produces a $5D$ value when $n=2$ that is less than the Table 1 value by only 0.00328, or 0.102%.

n	ν_n	n	ν_n
1	2	21	30.13021
2	3.21635	22	31.54477
3	4.64048	23	32.95930
4	6.06232	24	34.37381
5	7.48196	25	35.78829
6	8.90016	26	37.20275
7	10.31741	27	38.61720
8	11.73400	28	40.03163
9	13.15013	29	41.44605
10	14.56592	30	42.86045
11	15.98145	31	44.27484
12	17.39678	32	45.68922
13	18.81195	33	47.10359
14	20.22699	34	48.51795
15	21.64194	35	49.93231
16	23.05679	36	51.34665
17	24.47158	37	52.76099
18	25.88631	38	54.17532
19	27.30098	39	55.58965
20	28.71561	40	57.00397

Table 1. Values of the mean number of zeros of the random trigonometric polynomial (1.1) when $p = \frac{1}{2}$.

5. The Integer n_0

Although it is not logically necessary to know a specific value for the integer n_0 in the theorem and Lemmas 2, 4, 5 and 6, it is interesting to observe that n_0 can actually be chosen as small as 2. We begin the proof of this assertion with the following lemma.

Lemma 11: *It is true that* $g_0(z) > 0.206715$ *when* $0 \le z < \infty$.

It follows from (2.16) that

$$g_0'(z) = -\int_0^1 t^2 \sin zt\, dt < 0 \text{ when } 0 < z \le \pi, \tag{5.1}$$

and from (2.10) that

$$g_0'(z) > 0 \text{ when } 3\pi/2 \le z \le 2\pi. \tag{5.2}$$

If we define $W(z)$ to be $z^4 g_0''(z)$, we infer from (5.1) and (2.10) that

$$W(z) = -\int_0^z u^3 \cos u\, du = -z^3 \sin z - 3z^2 \cos z + 6z \sin z - 6(1 - \cos z). \tag{5.3}$$

Therefore, if $\pi \le z < 3\pi/2$, $W'(z) = -z^3 \cos z > 0$, so that $W(z) \ge W(\pi) = 3(\pi^2 - 4) > 0$. Hence $g_0''(z) > 0$ when $\pi \le z \le 3\pi/2$, and there exists a unique z_1 such that $\pi < z_1 < 3\pi/2$, $g_0'(z_1) = 0$.

By virtue of (5.1) and (5.2), the minimum value of $g_0(z)$ on the interval $(0, 2\pi)$ is $g_0(z_1)$.

We use Newton's method, starting with the value $z = 4\pi/3$, to locate the point z_1 with sufficient accuracy to determine that $g_0(z_1) > 0.206715$, so that $g_0(z) > 0.206715$ when $0 \leq z \leq 2\pi$. If $z \geq 2\pi$, we conclude from (2.8) that $g_0(z) \geq 2^{-1} - z^{-1} - 2z^{-2} \geq 2^{-1} - (2\pi)^{-1} - 2(2\pi)^{-2} > 0.290184$. The proof of the lemma is now complete.

Lemma 12: *It is true when $0 \leq x \leq \pi/2$ and $n \geq 2$ that*

$$|g_1| \leq 1 - 2\pi^{-1} < 0.363381, \quad |g_2| \leq 3/2. \tag{5.4}$$

Because the nonzero coefficients in the power series (3.22) for $f(x)$ are positive, we infer from (2.8) and (2.9) that $|g_1| \leq f(x) \leq f(\pi/2) = 1 - 2\pi^{-1}$, $|g_2| \leq (1/2) + \varphi(x) + f'(x) \leq (1/2) + \varphi(\pi/2) + f'(\pi/2) = 3/2$. It is helpful to note that

$$\varphi(x) + f'(x) = (1 + \cos x)^{-1}. \tag{5.5}$$

Lemma 13: *It is true when $0 \leq x \leq \pi/2$ and $n \geq 2$ that*

$$|(2n+1)^{-1} g_1/g_0| + |(2n+1)^{-2} g_2/g_0| < 0.641832. \tag{5.6}$$

Therefore, Lemma 2 is true when $n_0 = 2$.

The lemma is an immediate consequence of Lemmas 11 and 12, and the fact that (2.17) is implied by (5.6). (Although adequate, this result is rather crude. It is possible to show, with the help of numerical calculations similar to those we will use below in the proofs of Lemmas 14 and 15, that the left hand side of (5.6) does not exceed 0.299394.)

The analysis to show that (2.22) is true when $n \geq 2$ and $0 \leq x \leq \pi/2$, so that Lemma 4 is true when $n_0 = 2$, is somewhat more recondite. We begin with the following assertion.

Lemma 14: *The left hand side of* (2.22) *does not exceed* 0.884786 *when $0 < x \leq \pi/2$, $n \geq 2$ and $z = (2n+1)x \geq 6\pi$.*

This lemma will be a consequence of the inequalities,

$$f_0 \geq 0.105264, \quad |f_1| \leq 0.128401, \quad |f_2| \leq 1.20476, \quad |f_3| \leq 1.08086, \tag{5.7}$$

$$|f_4| \leq 5.57529, \quad |f_5| \leq 4.51528, \quad |f_6| \leq 3.95661, \tag{5.8}$$

when $n \geq 2$. We now proceed to prove these inequalities when $0 \leq x \leq \pi/2$ and $z \geq 6\pi$.

If we write (3.4) in the form,

$$\begin{aligned} 8f_0 = {} & 1 - 2z^{-1}(1 - 16z^{-2})\sin z - 10z^{-2}(1 + \cos z) \\ & - 8z^{-4}(1 - 2z^{-2})(1 - \cos z)^2 - 32z^{-5}(1 - \cos z)\sin z, \end{aligned} \tag{5.9}$$

and observe that $1 - 16z^{-2}$ and $1 - 2z^{-2}$ are positive (when $z > 4$), that $\sin z \leq 1$ and $-1 \leq \cos z \leq 1$, and that $(1 - \cos z)\sin z \leq 3^{3/2}/4$, we see that

$$8f_0 \geq 1 - 2z^{-1}(1 - 16z^{-2}) - 20z^{-2} - 32z^{-4}(1 - 2z^{-2}) - 24 \cdot 3^{1/2} z^{-5}. \tag{5.10}$$

The right hand side of (5.10) is an increasing function of z when $z \geq 6\pi$, because its derivative

$$2z^{-2}(1-48z^{-2})+40z^{-3}+128z^{-5}(1-3z^{-2})+120\cdot 3^{1/2}z^{-6},$$

is positive (when $z^2>48$). If we replace z by 6π in the right hand side of (5.10), we conclude that the first inequality in (5.7) is true when $z\geq 6\pi$.

If we write (3.14) in the form

$$-4f_1=[(1-4z^{-2})\sin z+8z^{-1}\{1-z^{-2}(1-\cos z)(3+\cos z)\} \tag{5.11}$$

$$+24z^{-4}(1-\cos z)\sin z]f(x),$$

and observe that $0\leq(1-\cos z)(3+\cos z)\leq 4$, that $1-4z^{-2}>0$ (when $z>2$), and that $0\leq f(x)\leq f(\pi/2)=1-2\pi^{-1}$, we see that

$$4\,|\,f_1\,|\,\leq(1-4z^{-2}+8z^{-1}+18\cdot 3^{1/2}z^{-4})(1-2\pi^{-1}). \tag{5.12}$$

The right hand side of (5.12) is a decreasing function of z when $z\geq 6\pi$, because its derivative,

$$\{-8z^{-2}(1-z^{-1})-72\cdot 3^{1/2}z^{-5}\}(1-2\pi^{-1}),$$

is negative (when $z>1$). If we replace z by 6π in the right hand side of (5.12), we conclude that the second inequality in (5.7) is true when $z\geq 6\pi$.

With the help of (2.8), (2.10), and (2.12) we can write (3.29) in the form,

$$24f_2=6\{f'(x)-f'(0)\}P_1(z)-24f^2(x)-24\varphi(x)k_0(z)-P_2(z), \tag{5.13}$$

in which

$$P_1(z)=5\cos z+4z^{-2}(8-3\cos z-2\cos^2 z) \tag{5.14}$$

$$-8z^{-3}(4-\cos z)\sin z+24(1-\cos z)\cos z,$$

$$P_2(z)=9-5\cos z-4z^{-2}(11+3\cos z-2\cos^2 z) \tag{5.15}$$

$$+8z^{-3}(13-\cos z)\sin z-24z^{-4}(1-\cos z)(3+\cos z).$$

Because $3\leq 8-3\cos z-2\cos^2 z\leq 73/8$, $|(4-\cos z)\sin z\,|\leq 4.11667$, and $|(1-\cos z)\cos z\,|\leq 2$, we see that

$$|\,P_1(z)\,|\,<5+36.5z^{-2}+32.9334z^{-3}+48z^{-4}<5.10803 \tag{5.16}$$

when $z\geq 6\pi$. Moreover, it follows from (2.12) and (2.21) that

$$0<k_0(z)=\{1-4z^{-1}(1-6z^{-2})\sin z-12z^{-2}(1-2z^{-2})\cos z-24z^{-4}\}/4 \tag{5.17}$$

$$\leq\{1+4z^{-1}(1-6z^{-2})+12z^{-2}(1-2z^{-2})-24z^{-4}\}/4$$

$$<0.310505 \text{ when } z\geq 6\pi.$$

Because $4\leq 9-5\cos z\leq 14$, $6\leq 11+3\cos\ z-2\cos^2 z\leq 97/8$, $|(13-\cos\ z)\sin z\,|\leq 13.0382$, and $0\leq(1-\cos z)(3+\cos z)\leq 4$, we see that, when $z\geq 6\pi$,

$$P_2(z) \leq 14 - 24z^{-2} + 104.306z^{-3} < 14, \tag{5.18}$$

$$P_2(z) \geq 4 - 48.5z^{-2} - 104.306z^{-3} - 96z^{-4} > 3.84716. \tag{5.19}$$

Hence

$$|P_2(z)| < 14 \tag{5.20}$$

when $z \geq 6\pi$. It now follows from (5.13), (5.16), (5.17) and (5.20) that

$$24|f_2| \leq 5.10803\{6f'(\pi/2) - 1\} + 24f^2(\pi/2) + 7.452124\varphi(\pi/2) + 14. \tag{5.21}$$

If we observe that $f'(\pi/2) = 4\pi^{-2}$, $f(\pi/2) = 1 - 2\pi^{-1}$, and $\varphi(\pi/2) = 1 - 4\pi^{-2}$, the third inequality in (5.7) is a consequence of (5.21).

It follows from (3.42), (2.8) and (2.10) that

$$f_3 = f''(x)P_3(z) - 2\varphi'(x)h_0(z) + f(x)\varphi(x)\sin z, \tag{5.22}$$

in which

$$P_3(z) = \{1.5 - z^{-2}(3 + \cos z)\}\sin z + z^{-1}(3 - \cos^2 z) \tag{5.23}$$

$$+ 4z^{-3}(1 - \cos z)\cos z.$$

Because $0 < 1.5 - 4z^{-2} < 1.5 - z^{-2}(3 + \cos z) < 1.5$ (when $z^2 > 8/3$), $|3 - \cos^2 z| \leq 3$, and $|(1 - \cos z)\cos z| \leq 2$, we see that

$$P_3(z)| \leq 1.5 + 3z^{-1} + 8z^{-3} < 1.66035 \tag{5.24}$$

when $z \geq 6\pi$. Moreover, we infer from (2.10) that

$$|h_0(z)| \leq z^{-1} + 2z^{-2} + 4z^{-3} < 0.059278 \tag{5.25}$$

when $z \geq 6\pi$. It now follows from (5.22), (5.24) and (5.25) that

$$|f_3| \leq 1.66035f''(\pi/2) + 0.118556\varphi'(\pi/2) + f(\pi/2)\varphi(\pi/2). \tag{5.26}$$

Because $f''(\pi/2) = 1 - 16\pi^{-3}$, $\varphi'(\pi/2) = 16\pi^{-3}$, $f(\pi/2) = 1 - 2\pi^{-1}$ and $\varphi(\pi/2) = 1 - 4\pi^{-2}$, we conclude that the fourth inequality in (5.7) is true when $z \geq 6\pi$.

It follows from (2.19) that

$$|f_4| \leq |g_0k_4| + |g_1k_3| + |g_2k_2| + 2|h_1h_3| + h_2^2. \tag{5.27}$$

We infer from (2.16) and (2.8) that

$$|g_0| = g_0 \leq \left(\tfrac{1}{2}\right) + z^{-1} < 0.553052 \tag{5.28}$$

when $z \geq 6\pi$, and from (2.14) that

$$k_4 = f'''(x)(1 - \cos z) - \psi(x), \tag{5.29}$$

in which

$$\psi(x) = \varphi''(x) + f'''(x) - \left(\tfrac{1}{4}\right) \tag{5.30}$$

is an increasing function for which $\psi(0) = 0$, $\psi(\pi/2) = 7/4$. Hence

$$k_4 \leq f'''(x)(1 - \cos z) \leq 2f'''(\pi/2) = 192\pi^{-4} < 1.97107,$$

$$k_4 \geq -\psi(x) \geq -7/4 = -1.75,$$

and we deduce from (5.28) that

$$|g_0 k_4| < 1.09011. \tag{5.31}$$

We infer from (2.13) that $|k_3| \leq 3f''(\pi/2) = 3(1 - 16\pi^{-3}) < 1.45193$, so that an appeal to Lemma 12 shows that

$$|g_1 k_3| \leq 0.527604. \tag{5.32}$$

Similarly, it follows from (2.13) and Lemma 12 that

$$|g_2 k_2| < \left(\tfrac{3}{2}\right)\left\{\left(\tfrac{1}{2}\right) + 3f'\left(\tfrac{\pi}{2}\right)\right\} = \left(\tfrac{3}{4}\right) + \left(\tfrac{18}{\pi^2}\right) < 2.57379, \tag{5.33}$$

and from (2.11) and (5.5) that

$$2|h_1 h_3| \leq 2f(x)\{\varphi'(x) + f''(x)\} \leq 2f\left(\tfrac{\pi}{2}\right) < 0.726762, \tag{5.34}$$

$$h_2^2 \leq 4f'^2(x) \leq 4f'^2\left(\tfrac{\pi}{2}\right) = 64\pi^{-4} < 0.657023. \tag{5.35}$$

If we use (5.24) and (5.31) through (5.35), we conclude that the first inequality in (5.8) is true when $z \geq 6\pi$.

It follow from (2.19) and some results from the previous paragraph that

$$\begin{aligned} |f_5| &\leq |g_1 k_4| + |g_2 k_3| + 2|h_2 h_3| \\ &\leq 192\pi^{-4}(1 - 2\pi^{-1}) + \left(\tfrac{9}{2}\right)(1 - 16\pi^{-3}) + 16\pi^{-2}. \end{aligned} \tag{5.36}$$

Therefore, the second inequality in (5.8) is true. Similarly,

$$|f_6| \leq |g_2 k_4| + h_3^2 \leq 288\pi^{-4} + 1, \tag{5.37}$$

so that the third inequality of (5.8) is true. The proof of Lemma 14 is now complete.

Lemma 15: *The left hand side of* (2.22) *does not exceed* 0.919051 *when* $0 \leq x \leq \pi/2$, $n \geq 2$ *and* $z = (2n+1)x \leq 6\pi$.

We define ξ so that

$$\xi = \min\left(\tfrac{z}{5}, \tfrac{\pi}{2}\right). \tag{5.38}$$

It is clear that $0 \leq x \leq \xi$ when $n \geq 2$. It then follows from (3.14) that

$$|f_1| \leq P_0(z)f(\xi), \tag{5.39}$$

in which

$$P_0(z) = |4^{-1}\sin z + 2z^{-1} - z^{-2}\sin z - 2z^{-3}(1-\cos z)(3+\cos z) + 6z^{-4}(1-\cos z)\sin z|. \tag{5.40}$$

Similarly, it follows from (5.13) and (5.22) that

$$|f_2| \le 24^{-1}[\{6f'(\xi)-1\}|P_1(z)| + |P_2(z)|] + f^2(\xi) + \varphi(\xi)k_0(z), \tag{5.41}$$

$$|f_3| \le f''(\xi)|P_3(z)| + 2\varphi'(\xi)|h_0(z)| + f(\xi)\varphi(\xi)|\sin z|. \tag{5.42}$$

(We have used the implication of (2.21) that $k_0(z) \ge 0$.)

The analysis for f_4 is more elaborate. We first deduce from (5.29) that

$$\{f'''(0)(1-\cos z) - \psi(\xi)\}g_0(z) \le g_0 k_4 \le f'''(\xi)(1-\cos z)g_0(z), \tag{5.43}$$

and from (2.8) and (2.31) that

$$0 \le g_1 k_3 = 3f(x)f''(x)\sin^2 z \le 3f(\xi)f''(\xi)\sin^2 z. \tag{5.44}$$

We next use (2.9) and (2.13) to see that

$$g_2 k_2 = \{\left(\tfrac{1}{2}\right) + \varphi(x) + f'(x)\}\{\left(\tfrac{1}{2}\right) - 3f'(x)\} + \{1 + 3\varphi(x) + 6f'(x)\}f'(x)(1-\cos z) - 3f'^2(x)(1-\cos z)^2,$$

so that

$$\{\left(\tfrac{1}{2}\right) + \varphi(\xi) + f'(\xi)\}\{\left(\tfrac{1}{2}\right) - 3f'(\xi)\} + \{1 + 3\varphi(0) + 6f'(0)\}f'(0)(1-\cos z) - 3f'^2(\xi)(1-\cos z)^2 \le g_2 k_2 \le \{1 + 3\varphi(\xi) + 6f'(\xi)\}f'(\xi)(1-\cos z) - 3f'^2(0)(1-\cos z)^2. \tag{5.45}$$

We now use (2.11) to see that

$$-2h_1h_3 = 2\{\varphi'(x) + f''(x)\cos z\}f(x)\cos z \le 2\{\varphi'(\xi) + f''(\xi)\}f(\xi).$$

We observe that $-2h_1h_3$ is a quadratic function of $\cos z$ whose absolute minimum is $-\varphi'^2(x)f(x)/2f''(x)$. We have tabulated the function $\varphi'^2(x)f(x)/f''(x)$ when $x = 0\ (\pi/36)\pi/2$, and have observed that it is an increasing function of x. Therefore,

$$-\varphi'^2(\xi)f(\xi)/2f''(\xi) \le -2h_1h_3 \le 2\{\varphi'(\xi) + f''(\xi)\}f(\xi). \tag{5.46}$$

Finally, we use (2.11) to see that

$$-4f'^2(\xi)\sin^2 z \le -h_2^2 = -4f'^2(x)\sin^2 z \le -4f'^2(0)\sin^2 z. \tag{5.47}$$

It now follows from the definition of (2.19) of f_4, the inequalities (5.43) through (5.47), and the numerical values, $f'(0) = 1/6$, $f'''(0) = 7/60$, $\varphi(0) = 1/3$, obtained from (3.22), that

$$|f_4| \le \max\{|P_4(z)|, |P_5(z)|\}, \tag{5.48}$$

in which

$$P_4(z) = [f'''(\xi)g_0(z) + \{1 + 3\varphi(\xi) + 6f'(\xi)\}f'(\xi) \tag{5.49}$$
$$- 12^{-1}(1 - \cos z)](1 - \cos z) + \{3f(\xi)f''(\xi) - \left(\tfrac{1}{9}\right)\}\sin^2 z$$
$$+ 2\{\varphi'(\xi) + f''(\xi)\}f(\xi),$$

$$P_5(z) = \{\left(\tfrac{7}{60}\right)(1 - \cos z) - \psi(\xi)\}g_0(z) + \{\left(\tfrac{1}{2}\right) + \varphi(\xi) \tag{5.50}$$
$$+ f'(\xi)\}\{\left(\tfrac{1}{2}\right) - 3f'(\xi)\} + 2^{-1}(1 - \cos z) - 3f'^2(\xi)(1 - \cos z)^2$$
$$- \varphi'^2(\xi)f(\xi)/2f''(\xi) - 4f'^2(\xi)\sin^2 z.$$

It follows from (2.19), (2.9), (2.11), (2.13) and (2.14) that

$$f_5 = \{P_6(x)\cos z + P_7(x)\}\sin z, \tag{5.51}$$

in which

$$P_6(x) = f(x)f'''(x) - f'(x)f''(x), \tag{5.52}$$

$$P_7(x) = f(x)\{\varphi''(x) - \left(\tfrac{1}{4}\right)\} - 4f'(x)\varphi'(x) + 3f''(x)\{\varphi(x) + \left(\tfrac{1}{2}\right)\}. \tag{5.53}$$

We have tabulated the functions $P_6(x)$ and $P_7(x) - P_6(x)$ when $x = 0(\pi/36)\pi/2$, and have observed that each of them is a positive increasing function of x when $0 < x \leq \pi/2$. We conclude that

$$|f_5| \leq \{P_7(\xi) + P_6(\xi)\cos z\}\,|\sin z|. \tag{5.54}$$

It follows from (2.19), (2.9), (2.11) and (2.14) that

$$f_6 = P_8(x) + P_9(x)\cos z + P_{10}(x)\cos^2 z, \tag{5.55}$$

in which

$$P_8(x) = -\{\left(\tfrac{1}{2}\right) + \varphi(x)\}\{\left(\tfrac{1}{4}\right) - \varphi''(x)\} - \varphi'^2(x), \tag{5.56}$$

$$P_9(x) = -\{\left(\tfrac{1}{4}\right) - \varphi''(x)\}f'(x) + \{\left(\tfrac{1}{2}\right) + \varphi(x)\}f'''(x) - 2\varphi'(x)f''(x), \tag{5.57}$$

$$P_{10}(x) = f'(x)f'''(x) - f''^2(x). \tag{5.58}$$

We rewrite (5.55) in the form

$$f_6 = (P_8 + P_9 + P_{10}) - (P_9 + 2P_{10})(1 - \cos z) + P_{10}(1 - \cos z)^2, \tag{5.59}$$

and observe, after tabulating $P_9(x)$ and $P_{10}(x)$ when $x = 0(\pi/36)\pi/2$, that each of them is a positive increasing function of x when $0 \leq x \leq \pi/2$. Moreover,

$$8\{P_8(x) + P_9(x) + P_{10}(x)\} = (13 - 3\cos x - 9\cos^2 x - \cos^3 x)/(1 + \cos x)^3, \tag{5.60}$$

so that $P_8(x) + P_9(x) + P_{10}(x)$ is also a positive increasing function of x when $0 < x \leq \pi/2$. We conclude that

$$-\{P_9(x)+2P_{10}(x)\}(1-\cos z) \leq f_6 \leq P_8(x)+P_9(x)+P_{10}(x), \tag{5.61}$$

so that

$$|f_6| \leq \max[P_8(\xi)+P_9(\xi)+P_{10}(\xi), \{P_9(\xi)+2P_{10}(\xi)\}(1-\cos z)]. \tag{5.62}$$

If we let $f_r^*(z)$ $(r = 1, 2, \ldots, 6)$ be the right hand sides of the inequalities (5.39), (5.41), (5.42), (5.48), (5.54) and (5.62), respectively, then the left hand side of (2.22) does not exceed

$$\theta(z) = f_0^{-1}(z)\sum_{r=1}^{6} 5^{-r} f_r^*(z) \tag{5.63}$$

when $n \geq 2$. We have tabulated $\theta(z)$ when $z = 0\ (\pi/18)6\pi$, and used these results to search, by appropriate subtabulation, for $\max_{0 \leq z \leq 6\pi}\theta(z)$. In this way, we find that $\theta(z) < 0.919051$. This completes the proof of Lemma 15.

Lemma 16: *The inequality* (2.22) *is true when* $n \geq 2$ *and* $0 \leq x \leq \pi/2$. *Therefore, Lemmas* 4, 5 *and* 6 *are also true when* $n_0 = 2$.

The first sentence in Lemma 16 is an immediate consequence of Lemmas 14 and 15. Therefore, Lemma 4 is true when $n_0 = 2$. This remark and Lemma 13 then show that Lemmas 5 and 6 are true when $n_0 = 2$.

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