RANDOM FIXED POINTS OF WEAKLY INWARD OPERATORS IN CONICAL SHELLS

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ABSTRACT

Conditions for random fixed points of condensing random operators are obtained and subsequently used to prove random fixed point theorems for weakly inward operators in conical shells.

Key words: Banach Space, Inward Operator, Random Fixed Point.

AMS (MOS) subject classifications: 47H10, 60H25, 47H40, 47H04.

1. Introduction

The fixed point theorems play a very important role in the questions of existence, uniqueness and successive approximations for various type of equations. The random fixed point theorems are useful tools for solving various problems in the theory of random equations, which form a part of random functional analysis. In recent years, there has been exciting interaction between analysis and probability theory, furnishing a rich source of problems for analysis. Bharucha-Reid [2] proved the random version of Schauder's Fixed Pint Theorem. Random fixed point theory has further developed rapidly in recent years; see e.g. Itoh [4], Sehgal and Waters [8], Sehgal and Singh [7], Papageorgiou [6], Lin [5], Xu [10], Tan and Yuan [9] and Beg and Shahzad [1]. This paper is a continuation of these investigations. We have obtained random fixed points of weakly inward random operators in the shell, a stochastic analogue of the results of Deimling [3].

2. Preliminaries

Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ as a sigma-algebra of subsets of Ω. Let X be a real Banach space; a map $q:\Omega \to X$ is called *measurable* if for each open subset G of X, $q^{-1}(G) \in \mathcal{A}$. Let S be a nonempty subset of X; a map $f:\Omega \times S \to X$ is called a *random operator* if for each fixed $x \in S$, the map $f(\cdot, x):\Omega \to X$ is measurable. A *measurable map* $\zeta:\Omega \to S$ is a random fixed point of the random operator f if $f(\omega, \zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$.

Let $K \subset X$ be a *cone*; that is, K is closed and convex such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. Denote $\{x \in K: \|x\| \leq r\}$ by K_r and $\{x \in K: \rho \leq \|x\| \leq r\}$ by $K_{\rho, r}$ for

some $0 < \rho < r$. A map $f: K_r \rightarrow X$ is α-*condensing* if f is continuous and bounded and $\alpha(f(B)) < \alpha(B)$ for all $B \subset K_r$ with $\alpha(B) > 0$, where $\alpha(B) = \inf\{d > 0: B$ can be covered by a finite number of sets of diameter $\leq d\}$. A random operator $f: \Omega \times K_r \rightarrow X$ is α-*condensing* if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is α-condensing. An operator $f: \Omega \times C \rightarrow X$ is said to be *weakly inward* on a closed convex subset C of X if for each $\omega \in \Omega$, $f(\omega, x) \in \overline{I_c(x)}$ for all $x \in C$, where $\overline{I_c(x)}$ denotes the closure of $\{x + \lambda(y - x): \lambda \geq 0, y \in C\}$. In the case $C = K$, it simply becomes $x \in \partial K$. The latter, $x^* \in K^*$ (the dual cone), and $x^*(x) = 0$ imply that

$$x^*(f(\omega, x)) \geq 0 \tag{1}$$

for each $\omega \in \Omega$, where X^* is the dual space, $K^* = \{x^* \in X^*: x^*(x) \geq 0$ on $K\}$ and ∂K is the boundary of K. Suppose $f: \Omega \times K_r \rightarrow X$, satisfies $x \in \partial K$, $\| x \| \leq r$, $x^* \in K^*$ and $x^*(x) = 0$ imply that $x^*(f(\omega, x)) \geq 0$ for each $\omega \in \Omega$. Let $P_r: K \rightarrow K_r$ be the radial projection; that is, $P_r(x) = x$ for $\| x \| \leq r$ and $P_r(x) = \frac{rx}{\| x \|}$ for $\| x \| > r$. Then for each $\omega \in \Omega$, $f(\omega, \cdot) \circ P_r$ satisfies (1). Since $\alpha(P_r(B)) \leq \alpha(B)$ for all bounded $B \subset K$, $f(\omega, \cdot) \circ P_r$ is α-condensing if f is.

3. Main Results

Theorem 1: *Let X be a separable Banach space, $K \subseteq X$ a cone and $f: \Omega \times K_r \rightarrow X$ an α-condensing random operator, such that*

(i) $x \in \partial K$, $\| x \| \leq r, x^* \in K^*$ *and* $x^*(x) = 0$ *imply that* $x^*(f(\omega, x)) \geq 0$ *for all* $\omega \in \Omega$, *and*

(ii) $f(\omega, x) \neq \lambda x$ *on* $\| x \| = r$ *and for all* $\omega \in \Omega$ *and* $\lambda > 1$, *are satisfied.*
Then f has a random fixed point.

Proof: Define a random operator $g: \Omega \times K_r \rightarrow X$ by $g(\omega, x) = (f(\omega, \cdot) \circ P_r)x$. It is clear from Deimling [3, proof of Theorem 1] that for any $\omega \in \Omega$, $g(\omega, \cdot)$ is α-condensing and weakly inward on K_r for some $r > 0$. Further application of Xu [10, Theorem 2] or Tan and Yuan [9, Corollary 2.6] gives the desired result.   □

Theorem 2: *Let X be a separable Banach space, $K \subset X$ a cone, $f: \Omega \times K_r \rightarrow X$ an α-condensing random operators, such that (i), (ii) (from Theorem 1) and*

(iii) *there exists $\rho \in (0, r)$ and $e \in K \backslash \{0\}$ such that for any*

$$\omega \in \Omega, \quad x - f(\omega, x) \neq \lambda e \text{ for } \| x \| = \rho \text{ and } \lambda > 0$$

are satisfied.
Then f has a random fixed point in $K_{\rho, r}$.

Proof: Let P_r be as before and for all $\omega \in \Omega$,

$$C(\omega) = \sup\{\| f(\omega, x) \| : x \in K_r\}.$$

We use (ii) to get a "barrier" at $\| x \| = \rho$ as follows. Let $\phi_n: \Omega \times [0, r] \rightarrow [0, \infty)$ be a continuous random map such that $\phi_n(\omega, t) = 0$ for $t \geq \rho$ and $\phi_n(\omega, t) = \delta(\omega)$ for $t \leq \rho - \frac{1}{n}$ and large n, with a measurable map $\delta: \Omega \rightarrow (0, \infty)$ such that $\delta(\omega) \| e \| > \rho + C(\omega)$ for each ω. Let $f_n(\omega, x) = (f(\omega, \cdot) \circ P_r)x + \phi_n(\omega, \| x \|)e$. Evidently, f_n is α-condensing and weakly inward random operator on K. By Theorem 1, there exists a measurable map $\zeta_n: \Omega \rightarrow K$ such that $\zeta_n(\omega) = f_n(\omega, \zeta_n(\omega))$ for each $\omega \in \Omega$. Fix $\omega \in \Omega$ arbitrarily. We cannot have $\| \zeta_n(\omega) \| > r$. Hence, $\zeta_n(\omega) = f(\omega, \zeta_n(\omega)) + \phi_n(\omega, \| \zeta_n(\omega) \|)e$. By the choice of ϕ_n we cannot have $\| \zeta_n(\omega) \| \leq \rho - \frac{1}{n}$, and we are done if $\| \zeta_n(\omega) \| \geq \rho$. So assume that $\rho - \frac{1}{n} < \| \zeta_n(\omega) \| < \rho$ for all large n. Since $\{\phi_n(\omega, \| \zeta_n(\omega) \|)\}$ is bounded and f is α-condensing, we have without loss of generality $\zeta_n(\omega) -$

$f(\omega, \zeta_n(\omega)) \to \lambda e$ and $\zeta_n(\omega) \to \zeta_0(\omega)$ with $\| \zeta_0(\omega) \| = \rho$ where λ depends on ω. Hence, $\zeta_0(\omega) = f(\omega, \zeta_0(\omega)) + \lambda e$ and therefore $\lambda = 0$ by (iii). $\qquad\Box$

Theorem 3: *Let X be a separable Banach space, $K \subset X$ a cone such that K_1 is not compact, $f: \Omega \times K_r \to X$ a compact random operator satisfying (i) and (ii) of Theorem 1 and*

    (iv)   there exists $\rho \in (0, r)$ such that $f(\omega, x) \neq \lambda x$ for $\| x \| = \rho$ and $\lambda \in (0, 1)$ and $\inf\{ \| f(\omega, x) \| : \| x \| = \rho \} > 0$, for each $\omega \in \Omega$.

Then f has a random fixed point in $K_{\rho, r}$.

Proof: Since for any $\omega \in \Omega$ $\inf\{ \| f(\omega, x) \| : \| x \| = \rho \} > 0$, and $A = \{x \in K : \| x \| = 1\}$ is not compact, we can find $e \in A$ such that $-\lambda e \notin f(\omega, \rho A)$ for all $\lambda \geq 0$.

Consider a random operator $f_n : \Omega \times K \to X$ defined by

$$f_n(\omega, x) = \begin{cases} (f(\omega, \cdot) \circ P_r)x & \text{for } \| x \| \geq \rho \\[2mm] f(\omega, \dfrac{\rho}{\| x \|} x) & \text{for } 0 < \rho_0 \leq \| x \| < \rho \\[2mm] \dfrac{\| x \|}{\rho_0} f(\omega, \dfrac{\rho}{\| x \|} x) + \phi_n(\omega, \| x \|)e & \text{for } \| x \| < \rho_0 \end{cases}$$

where $\phi_n : \Omega \times [0, \rho_0] \to [0, \infty)$ is a continuous random map, $\phi_n(\omega, \rho_0) = 0$ and $\phi_n(\omega, t) = \delta(\omega)$ for $t \leq \rho_0(1 - \frac{1}{n})$, with a measurable map $\delta : \Omega \to (0, \infty)$ such that

$$\delta(\omega) > \rho_0 + \sup\{ \| f(\omega, x) \| : \| x \| = \rho \} \text{ for each } \omega \in \Omega.$$

Since f_n is compact and a weakly inward random operator, it has a random fixed point $\xi_n : \Omega \to K_r$ as before. We cannot have $\rho_0 \leq \| \xi_n(\omega) \| < \rho$ or $\| \xi_n(\omega) \| \leq \rho_0(1 - \frac{1}{n})$ by (iv) and the choice of ϕ_n. If, for any $\omega \in \Omega$, $\| \xi_n(\omega) \| \geq \rho$, then we are done. Now assume that there exists ω such that $\rho_0 - \frac{1}{n} < \| \zeta_n(\omega) \| < \rho_0$ for all large n. Then

$$\frac{\| \xi_n(\omega) \|}{\rho} \eta_n(\omega) - \frac{\| \xi_n(\omega) \|}{\rho_0} f(\omega, \eta_n(\omega)) + \phi_n(\omega, \| \xi_n(\omega) \|)e$$

with $\| \eta_n(\omega) \| = \| \dfrac{\rho}{\| \xi_n(\omega) \|} \xi_n(\omega) \| = \rho$. By letting $n \to \infty$ we get $\frac{\rho_0}{\rho} y_0 = f(\omega, y_0) + \lambda e$ with $y_0 \in \rho A$ and $0 \leq \lambda \leq \delta(\omega)$ depending on ω, hence $\rho_0 \to 0$ yields $-\lambda_0 e \in \overline{f(\omega, \rho A)}$ for some $\lambda_0 > 0$, contradicting the choice of e. $\qquad\Box$

Remark 4: Theorem 3 does not hold if K_1 is compact. For a counterexample, see [3].

Acknowledgement

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