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REMARKS ON THE CONTROLLABILITY OF NONLINEAR PERTURBATIONS OF VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

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ABSTRACT

Sufficient conditions for the complete controllability of nonlinear perturbations of Volterra integrodifferential systems with implicit derivative are established. The results generalize the results of Dauer and Balachandran [9] and are obtained through the notions of condensing map and measure of noncompactness of a set.

Key words: Controllability, Integrodifferential Systems, Perturbations, Fixed Point Technique.

AMS (MOS) subject classifications: 93B05.

1. Introduction

The controllability of perturbed nonlinear systems has been studied by several authors [2-4, 7-9] with the help of fixed point theorems. Dacka [6] introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivative based on the measure of noncompactness of a set and Darbo's fixed point theorem. This method has been extended to a larger class of perturbed systems by Balachandran [2, 3]. Anichini et al. [1] studied the problem through the notions of condensing map and measure of noncompactness of a set. They used the fixed point theorem due to Sadovskii [11]. In this note, we shall study the controllability of nonlinear perturbations of Volterra integrodifferential systems with implicit derivative by suitably adopting the technique of Anichini et al. [1]. The results generalize the results of Dauer and Balachandran [9].

2. Preliminaries

We first summarize some facts concerning condensing maps; for definitions and results about the measure of noncompactness and related topics, the reader can refer to the paper of Dacka [6]. Let X be a subset of a Banach space. An operator $T: X \rightarrow X$ is called condensing if, for any bounded subset E in X with $\mu(E) \neq 0$, we have $\mu(T(E)) < \mu(E)$, where $\mu(E)$ denotes the measure of noncompactness of the set E as defined in [11].

We observe that, as a consequence of the properties of μ, if an operator T is the sum of a compact and a condensing operator, then T itself is a condensing operator. Further, if the operator

$P: X \to X$ satisfies the condition $|Px - Py| \leq k|x-y|$ for $x, y \in X$, with $0 \leq k < 1$, then the operator P has a fixed point property. However, the condition $|Px - Py| < |x - y|$ for $x, y \in X$ is insufficient to ensure that P is a condensing map or that P will admit a fixed point (Browder [5]). The fixed point property holds in the condensing case (Sadovskii [11]).

Let $C_n(J)$ denote the space of continuous R^n valued functions on the interval J. For $x \in C_n(J)$ and $h > 0$, let

$$\theta(x,h) = \sup\{\,|x(t) - x(s)|\,;\ t,s \in J \text{ with } |t-s| \leq h\},$$

and write $\theta(E,h) = \sup_{x \in E} \theta(x,h)$, so that $\theta(E,\cdot)$ is the modulus of continuity of a bounded set E. Set $\theta_0(E) = \lim_{h \to 0} \theta(E,h)$. Assume that Ω is the set of functions $\omega: R^+ \to R^+$ that are right continuous and nondecreasing such that $\omega(r) < r$, for $r > 0$. Let $J = [t_0, t_1]$.

Lemma 1: [11] *Let $X \subset C_n(J)$ and let β and γ be functions defined on $[0, t_1 - t_0]$ such that $\lim_{s \to 0} \beta(s) = \lim_{s \to 0} \gamma(s) = 0$. If a transformation $T: X \to C_n(J)$ maps bounded sets into bounded sets such that*

$$\theta(T(x),h) < \omega(\theta(x,\beta(h)) + \gamma(h) \text{ for all } h \in [0, t_1 - t_0]$$

and $x \in X$ with $\omega \in \Omega$, then T is a condensing mapping.

Lemma 2: [1, 11] *Let $X \subset C_n([t_0, t_1])$, let $I = [0,1]$, and let $S \subset X$ be a bounded closed convex set. Let $H: I \times S \to X$ be a continuous operator such that, for any $\alpha \in I$, the map $H(\alpha, \cdot)$: $S \to X$ is condensing. If $x \neq H(\alpha, x)$ for any $\alpha \in I$ and any $x \in \partial S$ (the boundary of S), then $H(1, \cdot)$ has a fixed point.*

Finally, it is possible to show that for any bounded and equicontinuous set E in $C_n^1(J)$, the following relations holds:

$$\mu_{C_n^1}(E) \equiv \mu_1(E) = \mu(DE) \equiv \mu_{C_n}(DE)$$

where $DE = \{\dot{x}; x \in E\}$.

3. Main Results

Consider the nonlinear perturbations of the Volterra integrodifferential system of the form

$$\dot{x}(t) = g(t,x) + \int_{t_0}^{t} h(t,s,x(s))ds + B(t,x(t))u(t)$$
$$+ f(t, x(t), \dot{x}(t), (Sx)(t), u(t)), \ldots, \quad t \in J = [t_0, t_1] \tag{1}$$

$x(t_0) = x_0$, where the operator S is defined by

$$(Sx)(t) = \int_0^t k(t,s,x(s))ds.$$

Here, $x(t) \in R^n$, $u(t) \in R^m$ and the functions g, h, f, B and k satisfy the following hypotheses:

i) $g: J \times R^n \to R^n$ is continuous and continuously differentiable with respect to x.

ii) $h: J \times J \times R^n \to R^n$ is continuous and continuously differentiable with respect to x.

iii) $B(t, x(t))$ is a continuous family of matrices on $J \times R^n$.

iv) $f: J \times R^n \times R^n \times R^n \times R^m \to R^n$ is continuous.
v) $k: J \times J \times R^n \to R^n$ is continuous.

Let $x(t, t_0, x_0)$ be the unique solution of the equation

$$\dot{x}(t) = g(t, x) + \int_{t_0}^{t} h(t, s, x(s))ds$$

existing on some interval J.

Define

$$G(t, t_0, x_0) = g_x(t, x(t, t_0, x_0))$$

and

$$H(t, s, t_0, x_0) = h_x(t, s, x(s, t_0, x_0)).$$

Then $X(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$ exist and is the solution of

$$\dot{y}(t) = G(t, t_0, x_0)y(t) + \int_{t_0}^{t} H(t, s; t_0, x_0)y(s)ds$$

such that $X(t_0, t_0, x_0) = I$.

Then the solution of the equation (1) is given by [10]

$$x(t) = x(t, t_0, x_0) + \int_{t_0}^{t} X(t, s, x(s))[B(s, x)u(s) + f(s, x(s), \dot{x}(s), (Sx)(s), u(s))]ds$$

$$+ \int_{t_0}^{t} \int_{s}^{t} [X(t, \tau, x(\tau)) - R(t, \tau; s, x(s))]h(\tau, s, x(s))d\tau ds$$

where $R(t, s; t_0, x_0)$ is the solution of the equation

$$\frac{\partial R}{\partial s}(t, s; t_0, x_0) + R(t, s; t_0, x_0)G(s, t_0, x_0) + \int_{s}^{t} R(t, \tau; t_0, x_0)H(\tau, s; t_0, x_0)d\tau = 0$$

such that $R(t, t; t_0, x_0) = I$ on the interval $t_0 \leq s \leq t$ and

$$R(t, t_0; t_0, x_0) = X(t, t_0, x_0).$$

We say the system (1) is completely controllable on J if, for any $x_0, x_1 \in R^n$, there exists a continuous control function $u(t)$ defined on J such that the solution of (1) satisfies $x(t_1) = x_1$. Define the matrix W by

$$W(t, t_0, x) = \int_{t_0}^{t} X(t, s, x(s))B(s, x(s))B^*(s, x(s))X^*(t, s, x(s))ds,$$

where the star denotes the matrix transpose. Further define

$$q(t, t_0, x) = \int_{t_0}^{t} \int_{s}^{t} [X(t, \tau, x(\tau)) - R(t, \tau; s, x(s))]h(\tau, s, x(s))d\tau ds.$$

The main results concerning the controllability of the system (1) is given in the following theorem.

Theorem: *Let the system* (1) *satisfy all the above conditions* (i) *to* (v) *and assume the additional conditions*

(a) $\limsup_{|x| \to \infty} \frac{|f(t,x,y,Sx,u)|}{|x|} = 0,$

(b) *there exists a continuous nondecreasing function* $\omega: R^+ \to R^+$, *with* $\omega(r) < r$, *such that*

$$|f(t,x,y,Sx,u) - f(t,x,z,Sx,u)| < \omega(|y-z|) \text{ for all } (t,x,y,Sx,u) \in J \times R^{3n} \times R^m$$

(c) *there exists a positive constant* δ *such that*

$$\det W(t_0, t_1, x) \geq \delta \text{ for all } x.$$

Then the system (1) *is completely controllable on* J.

Proof: Define the nonlinear transformation

$$T: C_m(J) \times C_n^1(J) \to C_m(J) \times C_n^1(J)$$

by

$$T(u,x)(t) = (T_1(u,x)(t),\ T_2(u,x)(t))$$

where the pair of operators T_1 and T_2 are defined by

$$T_1(u,x)(t) = B^*(t,x)X^*(t_1,t,x)W^{-1}(t_1,t_0,x)[x_1 - x(t_1,t_0,x_0)$$
$$- q(t_1,t_0,x) - \int_{t_0}^{t_1} X(t_1,s,x(s))f(s,x(s),\dot{x}(s),(Sx)(s),u(s))ds]$$
$$T_2(u,x)(t) = x(t,t_0,x_0) + q(t,t_0,x) + \int_{t_0}^{t} X(t,s,x(s))B(s,x(s))T_1(u,x)(s)ds$$
$$+ \int_{t_0}^{t} X(t,s,x(s))f(s,x(s),\dot{x}(s),(Sx)(s),T_1(u,x)(s))ds.$$

Since all the functions involved in the definition of the operator T are continuous, T is continuous. Moreover, by direct differentiation with respect to t, a fixed point for the operator T gives rise to a control u and a corresponding function $x = x(t)$, solution of the system (1) satisfying $x(t_0) = x_0$, $x(t_1) = x_1$. Let

$$\eta^0 = (u^0, x^0) \in C_m(J) \times C_n^1(J)$$

$$\eta = (u, x) \neq 0 \in C_m(J) \times C_n^1(J)$$

and consider the equation

$$\eta^0 = \eta - \alpha T(\eta),$$

where $\alpha \in [0,1]$. This equation can be equivalently written as

$$u = u^0 + \alpha T_1(u,x) \tag{2}$$

$$x = x^0 + \alpha T_2(u,x). \tag{3}$$

From condition (i), for any $\epsilon > 0$ there exists $R > 0$ such that if $|x| > R$ then $|f(t,x,y,(S)(x),u)| < \epsilon|x|$. Then from (2) we get

$$\begin{aligned} |u| &\le |u^0| + |\alpha|\,|B|\,|X|\,|W^{-1}|[|x_1| + |x(t_1,t_0,x_0)| \\ &\quad + |q(t_1,t_0,x)| + |X|\epsilon|x|\delta] \\ &\le |u^0| + k_1 + |B|\,|X|^2|W^{-1}|\epsilon\,\delta|x| \end{aligned} \tag{4}$$

where $\delta = t_1 - t_0$ and

$$k_1 = |B|\,|X|\,|W^{-1}|[|x_1| + |x(t_1,t_0,x_0)| + |q(t_1,t_0,x)|].$$

From this inequality and from (3), by applying the Gronwall Lemma, we obtain

$$\begin{aligned} |x| &\le [|x^0| + |x(t,t_0,x_0)| + |T_1(u,x)|\,|X|\,|B|\delta + |q(t,t_0,x)|]\exp(|X|\epsilon\delta) \\ &\le [|x^0| + |x(t,t_0,x_0)| + (k_1 + |B|\,|X|^2|W^{-1}|\epsilon\delta|x|)|X|\,|B|\delta \\ &\quad + |q(t,t_0,x)|]\exp(|X|\epsilon\delta). \end{aligned} \tag{5}$$

Taking the derivative of (3) with respect to t, we obtain

$$\dot{x} = \frac{dx^0}{dt} + \alpha\frac{d}{dt}(T_2(u,x)(t))$$

and that results in

$$\begin{aligned} |\dot{x}| &\le |\dot{x}^0| + |g(t,x)| + \int_{t_0}^{t} |h(t,s,x(s))|\,ds + |B(t,x(t))|\,|T_1(u,x)(t)| \\ &\quad + |f(t,x(t),\dot{x}(t),(Sx)(t),u(t))| \\ &\le |\dot{x}^0| + |g(t,x)| + \int_{t_0}^{t} |h(t,s,x(s))|\,ds + |B|[k_1 + |B|\,|X|^2|W^{-1}|\epsilon\delta|x|] + \epsilon|x| \\ &= |\dot{x}^0 + k_2 + |x|[|B|^2|X|^2|W^{-1}|\epsilon\delta + \epsilon] \end{aligned} \tag{6}$$

where $k_2 = |g(t,x)| + \delta|h(t,s,x(s))| + |B|\,|k_1|$.

From (4)

$$|u| - |B|\,|X|^2|W^{-1}|\epsilon\delta|x| \le |u^0| + k_1 \tag{7}$$

and from (5)

$$|x|[\exp(-|X|\epsilon) - |B|^2|X|^3|W^{-1}|\epsilon\delta^2] \le |x^0| + k_3 \tag{8}$$

where $k_3 = |x(t,t_0,x_0)| + k_1|X|\,|B|\delta + |q(t,t_0,x)|$ and from (6)

$$|\dot{x}| - |x|[|B|^2|X|^2|W^{-1}|\epsilon\delta + \epsilon] \le k_2 + |\dot{x}^0|. \tag{9}$$

Taking the sum of all the inequalities (7), (8) and (9), we obtain

$$|u| - |x|\{|B|\,|X|^2|W^{-1}|\epsilon\delta - \exp(-|X|\epsilon\delta) + |B|^2|X|^3|W^{-1}|\epsilon\delta^2$$
$$+ |B|^2|X|^2|W^{-1}|\epsilon\delta + \epsilon\} + |\dot{x}| \le |u^0| + |x^0| + |\dot{x}^0| + k$$

where $k = k_1 + k_2 + k_3$.

That is,

$$|u| - \lambda|x| + |\dot{x}| \le |u^0| + |x^0| + |\dot{x}^0| + k$$

where $\lambda = |B|\,|X|^2|W^{-1}|\epsilon\delta\{1 + |B|\,|X|\delta + |B|\} + \epsilon - \exp(-|X|\epsilon\delta)$.

Then, for suitable positive constants a, b, c we can write

$$|u| - [\epsilon a - \exp(-\epsilon b)]|x| + |\dot{x}| \le |u^0| + |x^0| + |\dot{x}^0| + c,$$

so we divide by $|u| + |x| + |\dot{x}|$ and from the arbitrariness of ϵ, we get the existence of a ball S in $C_m(J) \times C_n^1(J)$ sufficiently large such that

$$|\eta - \alpha T(\eta)| > 0 \text{ for } \eta = (u,x) \in \partial S.$$

We want to show that T is a condensing map. To this aim, we note that $T_1: C_m(J) \to C_m(J)$ is a compact operator and then, if E is a bounded set, $\mu(T_1(E)) = 0$. Then it will be enough to show that T_2 is a condensing operator. For that, let us consider the modulus of continuity of $DT_2(u,x)(\;)$. Now, for $t, s \in J$, we have

$$|DT_2(u,x)(t) - DT_2(u,x)(s)| \le |g(t,x(t)) - g(s,x(s))| + \Big|\int_{t_0}^{t} h(t,\tau,x(\tau))d\tau$$
$$- \int_{t_0}^{s} h(s,\tau,x(\tau))d\tau\Big| + |B(t,x(t))T_1(u,x)(t) - B(s,x(s))T_1(u,x)(s)|$$
$$+ |f(t,x(t),\dot{x}(t),(Sx)(t),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),(Sx)(s),T_1(u,x)(s))|.$$

For the first three terms of the right hand side of the inequality, we may given the upper estimate as $\beta_0(|t-s|)$ with $\lim_{h\to 0}\beta_0(h) = 0$ and it may be chosen independent of the choice of (u,x). For the fourth term, we can given the following estimate:

$$|f(t,x(t),\dot{x}(t),(Sx)(t),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),(Sx)(s),T_1(u,x)(s))|$$
$$\le |f(t,x(t),\dot{x}(t),(Sx)(t),T_1(u,x)(t)) - f(t,x(t),\dot{x}(s),(Sx)(t),T_1(u,x)(t))|$$
$$+ |f(t,x(t),\dot{x}(s),(Sx)(t),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),(Sx)(s),T_1(u,x)(s))|.$$

For the first term we have the upper estimate $\omega(|\dot{x}(t) - \dot{x}(s)|)$ whereas for the second term, we may find an estimate

$$\beta_1(|t-s|) \text{ with } \lim_{h\to 0}\beta_1(h) = 0.$$

Hence

$$\theta(DT_2(u,x),h) \leq \omega(\theta(DE,h) + \beta(h)$$

where $\beta = \beta_0 + \beta_1$. Therefore, by Lemma 1, we get

$$\theta_0(DT_2(E)) < \theta_0(DE).$$

Hence, from

$$2\mu_1(T_2(E)) = 2\mu(DT_2(E)) = \theta_0(DT_2(E)) < \theta_0(DE))$$
$$= 2\mu(DE) = 2\mu_1(E),$$

it follows that $\mu_1(T_2(E)) < \mu_1(E)$. Then the existence of a fixed point of the operator T follows from Lemma 2. In other words, there exists functions $u \in C_m(J)$ and $x \in C_n^1(J)$ such that

$$T(u,x) = (u,x)$$

and

$$u(t) = T_1(u,x)(t), \quad x(t) = T_2(u,x)(t).$$

These functions are the required solutions. Further, it is easy to verify that the function $x(\cdot)$ given by the systems (1) satisfies the boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$. Hence, the system (1) is completely controllable.

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