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SOME PROPERTIES OF THE FEYNMAN-KAC FUNCTIONAL

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ABSTRACT

The Feynman-Kac formula and its connections with classical analysis were initiated in the now celebrated paper [6] of M. Kac. It soon became obvious that the formula provides a powerful tool for solving partial differential equations by running the Brownian motion process. K.L. Chung and K.M. Rao in [4] used it to characterize solutions of the Schrödinger equation. In this paper we study some properties of the Feynman-Kac functional using the Brownian motion process. In particular, we are going to use it in connection with the gauge function in order to obtain an energy formula similar to one obtained by G. Dal Maso and U. Mosco in [5].

Key words: Gauge Function, Brownian Motion, Subharmonic Function, Explosion Point, Kato Class, the Energy Functional, Excessive Function.

AMS subject classification: 60J45, 60J25, 35C05.

1. Introduction

In [5], G. Dal Maso and U. Mosco studied a *relaxed Dirichlet problem* in an open region Ω of R^d, $d \geq 2$, which can formally be written as:

$$-\Delta u + \mu u = 0 \text{ in } \Omega, \tag{1.1}$$

where Δ is the Laplace operator and μ is an arbitrary non-negative Borel measure not charging polar sets in R^d. The measure μ may take the value $+\infty$. Special cases of (1.1) are Dirichlet problems of the type

$$-\Delta u = 0 \text{ in } \Omega - \bar{E}, \ \ u = 0 \text{ on } E, \tag{1.2}$$

($\bar{E}$ denotes the closure of E), as well as the stationary Schrödinger equation:

$$-\Delta u + q(x)u = 0 \text{ in } \Omega,$$

where q is a non-negative potential. The main result in [5] is an approximation of equation (1.1) by a sequence of Dirichlet problems of the form (1.2). In order to carry out this procedure, one needs to study the behavior of an arbitrary local weak solution u of (1.1) such that $u \in \mathcal{H}^1_{loc}(\Omega) \cap L^2_{loc}(\Omega, \mu)$ and having finite local μ-energy given by:

$$\int_{\Omega'} |\nabla u|^2 dx + \int_{\Omega'} u^2 d\mu, \quad \Omega' \subset \Omega. \tag{1.3}$$

The methods in [5] are variational in nature. In this paper, we study the Feynman-Kac functional, directly using the Brownian motion process. Using the gauge function, we obtain an energy formula in part 5 similar to formula (1.3). Section 3 deals with the characterization of the null set of g_q. For example, in Proposition 3.3, it is shown that the set $\{g_q = 0\}$ is a polar set. Section 4 deals with continuity properties of the gauge function g_q. Specifically, Theorem 4.2 shows that if g_q is nonvanishing and continuous in Ω, then q is in local Kato class in Ω. Finally, Section 5 deals with the energy as introduced in [7]. In addition, it is shown that if s is an excessive function, then its corresponding gauge function satisfies the equation $\Delta g_s = \frac{1}{s} g_s$.

2. Notations and Preliminaries

Throughout this paper, $X = \{X_t; t \geq 0\}$ denotes the Brownian Motion process in R^d, $d \geq 2$; Ω denotes a *domain* in R^d. Let $q \geq 0$ be measurable. Let $e_q(t) = exp[-\int_0^t q(X_s)ds]$, $g_q(x) = E^x[e_q(\tau)]$, where $\tau =$ exit time from Ω. Using strong Markov property, we see that $g_q \leq E^x[g_q(X_{\tau_B})]$, where $\tau_B =$ exit time from a ball with a center x. Also, it is seen that if q is bounded, then g_q is continuous; hence, in general, g_q is *upper semi-continuous*. It follows that g_q is subharmonic in Ω.

Throughout this paper we deal with a topology on R^d that is finer than the Euclidean metric topology. Namely, the *fine topology* on R^d is the smallest topology on R^d for which all superharmonic functions are continuous in the extended sense. It is easily seen that the fine topology is larger than the Euclidean metric topology on R^d. So we speak of *fine interior*, *finely continuous*, etc.

Another concept which is used in the paper and which is related to the behavior of the Brownian motion process is that of a *regular* point of a set. Namely, given a set D denote by T the exit time for the Brownian motion from D. (Which is the same as the hitting time of the complement of D.) Then, a point $a \in \partial D$ is called *regular* for D^c if $P^a\{T = 0\} = 1$. In other words, starting at a regular point the Brownian motion hits the complement of a set in question immediately.

3. The Null Set of g_q

In this section we study the set on which function g_q vanishes.

Proposition 3.1: $q < \infty$ *a.e. on the set* $\{g_q > 0\}$.

Proof: Suppose q is bounded. Then, assuming the domain Ω is bounded,

$$1 - e_q(\tau) = \int_0^\tau q(X_s) e^{-\int_s^\tau q(X_\theta)d\theta} ds. \tag{3.1}$$

Therefore if q is *bounded*,

$$1 - g_q(x) = G[qg_q] \tag{3.2}$$

which is a consequence of (3.1) and Markov property.

For general q, let $a_n = q \wedge n$. Using (3.2), one gets,

$$1 - g_{q_n}(x) = G[q_n, g_{q_n}] \geq G[q_n g_q].$$

Let n tend to infinity in the above inequality. We get

$$1 = g_q \geq G[qg_q],$$

where qg_q is defined to be zero if $g_q = 0$. Thus, $qg_q < \infty$ *a.e.*

A more precise result is the following:

Proposition 3.2: *Suppose $g_q(\xi) = 0$ and that ξ is regular for the set $\{g_q > 0\}$. Then, ξ is an "explosion point" for q, i.e.,*

$$\forall t > 0,\ P^{\xi}\left\{ \int_0^t q(X_s)ds = \infty \right\} = 1.$$

Proof: Let $\varepsilon > 0$, $F = \{g_q \geq \varepsilon\}$ and let T be the hitting time of F. Then,

$$0 = g_q(\xi) \geq E^x[e_q(T)g_q(X_T) : T < \tau]$$

implying $e_q(T) = 0$ on the set $\{T < \tau\}$. As $\varepsilon \downarrow 0$, the hitting times decrease to the hitting time to $\{g_q > 0\}$ which is zero P^{ξ}-*a.s.* by assumption. This completes the proof.

Remark 3.1: If $g_q(\xi) = 0$ and ξ is not regular for $\{g_q > 0\}$, then ξ must be in the fine interior of $\{g_q = 0\}$. Thus,

$$\{g_q = 0\} = \{\text{finely open set}\} \cup \{\text{the set of explosion points of } q\}.$$

Remark 3.2: Suppose $g_q \not\equiv 0$. Then $\{g_q > 0\}$ is a finely open non-empty set. So, some point of $\{g_q = 0\}$ must be a regular point for $\{g_q > 0\}$. However, such a point is an explosion point for q. Thus, we can say if q has no explosion points, then g_q cannot vanish unless $g_q \equiv 0$.

Let us show that some point for which $g_q = 0$ is regular for $\{g_q > 0\}$. Let $g_q(\xi) = 0$ and $T =$ hitting time to $\{g_q > 0\}$. Since $g_q(X_t)$ is continuous in t for $t > 0$, we see that $g_q(X_T) = 0$ if $T < \infty$ and X_T is regular for $\{g_q > 0\}$, almost surely. Hence, there are points for which $g_q = 0$, and which are regular for $\{g_q > 0\}$.

Proposition 3.3: *Suppose that for some ξ, $P^{\xi}\{\int_0^{\tau} q(X_s)ds < \infty\} = 1$. This implies that $\{g_q = 0\}$ is a Polar set. Furthermore,*

$$1 - g_q = G(qg_q). \tag{3.3}$$

Proof: Indeed, g_q is finely continuous so that set $A = \{g_q = 0\}$ is finely closed. If $T =$ hitting time of A, we have

$$E^{\xi}[e^{-\int_0^{\tau}} : T < \tau] = E^{\xi}[e^{-\int_0^{T}} g_q(X_T) : T < \tau] = 0$$

because $g_q(X_T) = 0$ on the set $(T < \tau)$. Thus, $P^{\xi}[e^{-\int_0^{\tau}} 1_{T<\tau} = 0] = 1$. However, it is given that $e^{-\int_0^{\tau}} > 0$ with P^{ξ}-probability 1. Hence, we conclude that A is a Polar set.

Remark 3.3: The proof of formula (3.3) is as follows.

We have $P^{\xi}[\int_0^{\tau} q < \infty] = 1$. Therefore the following arguments are valid: Let $q_n \uparrow q$ with $G(q_n)$ bounded. Then we know that (3.3) holds with q replaced by q_n. By taking limits, we get $G(qg_q) \leq 1 - g_q$. By definition, we have always $qg_q = 0$ on the set $\{g_q = 0\}$. At ξ we have

$$G(qg_q)(\xi) = E^{\xi}\ [\int_0^{\tau} q(X_s)g_q(X_s)ds]$$

$$= E^{\xi}\ [\int_0^{\tau} q(X_s)e^{-\int_s^{\tau} q(X_u)du} ds]$$

$$= E^{\xi}[1 - e^{-\int_0^{\tau} q}] = 1 - g_q(\xi).$$

Now we can claim that $1 - g_q - G(qg_q)$ is excessive. Indeed, if $q_n \uparrow q$,

$$1 - g_q - G(qg_q) = \lim_n [1 - g_{q_n} - G(q_n g_q)]$$

$$= \lim_n [G(q_n g_{q_n}) - G(q_n g_q)]$$

$$= \lim_n G[q_n(g_{q_n} - g_q)]$$

and each is excessive. Thus, since (3.3) holds at ξ, it holds everywhere.

Remark 3.4: Under the assumption that for some ξ, $P^{\xi}[\int_0^{\tau} q < \infty] = 1$, we can show that the only zeros of g_q are the explosion points of q.

Indeed, if $g_q(x) = 0$ we have

$$0 = g_q(x) \geq E^x[e^{-\int_0^t} g_q(X_t): t < \tau].$$

Since $g_q(X_t) > 0$, $e_q(t) = 0$ for every t, which means that x is an explosion point.

Example: Let $q(x) = |x|^{-\beta}$, $2 < \beta < 3$ in R^3. Then, using scaling for Brownian motion we see that for any $t > 0$,

$$P^0\left\{\int_0^t |X_s|^{-\beta} ds < a\right\} = P^0\left\{\int_0^t |\varepsilon X_{s/\varepsilon^2}|^{-\beta} ds < a\right\}$$

$$= P^0\left\{\varepsilon^{2-\beta}\int_0^{t/\varepsilon^2} |X_s|^{-\beta} ds < a\right\}.$$

The last inequality holds for every $\varepsilon > 0$. It follows that $\int_0^t |X_s|^{-\beta} ds \equiv \infty$, $\forall t > 0$, $P^0 - a.s.$, i.e., 0 is an explosion point for q and, of course, $g_q(0) = 0$. Note that q is locally integrable. If

$\{\xi_i\}$ is a dense countable set, let $q_i = |x - \xi_i|^{-\beta}$. Then q_i is locally integrable. For suitable constants η_i, $q = \sum \eta_i q_i \in L^1_{loc}$, and $\forall t > 0$ and $\forall i$, $P^{\xi_i}[\int_0^t q(X_s)ds = \infty] = 1$. Thus, it is possible that $g_q = 0$ on a dense set *with* q *integrable.*

4. Continuity Properties of g_q

The following remark will be helpful in the proof of Proposition 4.1.

Remark: The function

$$G(x,t) = P^x[\tau < t] + E^x[g_q(X_t): t < \tau]$$

decreases as t decreases and tends to g_q as t tends to zero for each fixed x.

Indeed, if $s < t$, we may compute

$$G(x,t) - G(x,s) = P^x[s < \tau < t] + E^x[\{exp(-\int_t^\tau q(X_u)du) - exp(-\int_s^\tau q(X_u)du)\}: t < \tau]$$
$$- E^x[exp(-\int_s^\tau q(X_u)du): s < \tau < t] \geq 0.$$

Also, $G(\cdot, t)$ is continuous for each t. Thus, if g_q is continuous, $G(\cdot, t)$ tends to g_q uniformly on compacts by Dini's theorem. We conclude that g_q is continuous if and only if $E^{(\cdot)}[g_q(X_t): t < \tau]$ tends to g_q uniformly on compacts as t tends to zero.

Proposition 4.1: *Let* $g_i = g_{q_i}$ *be continuous. Then* $g = g_{q_1 + q_2}$ *is also continuous.*

Proof: If h is any of g_1, g_2 or g and r any of $q_1 + q_2$, q_1 or q_2, we have by Markov property for any t,

$$E^x[h(X_t): t < \tau] - h(x) = E^x[h(X_t)(1 - e^{-\int_0^t r(X_s)ds}): t < \tau]$$
$$- E^x[e^{-\int_0^\tau r(X_s)ds}: \tau < t].$$

The last term above clearly tends to zero uniformly on compacts because $P^x\{\tau < t\}$ does this as $t \to 0$. From the last sentence of the above remark, the continuity of h is equivalent to

$$E^{(\cdot)}[h(X_t)(1 - e_r(t)): t < \tau] \tag{4.1}$$

to tend to zero uniformly as $t \to 0$. If $h = g_i$ and $r = q_i$, $i = 1, 2$, this is the case because g_i are continuous. We have with $h = g$ and $r = q_1 + q_2$, that

$$E^{(\cdot)}[g(X_t)(1 - e_r(t)): t < \tau]$$
$$= E^{(\cdot)}[g(X_t)(1 - e_{q_1}(t) + e_{q_1}(t) - e_r(t)): t < \tau].$$

Since $g \leq g_1$,

$$E^{(\cdot)}[g(X_t)(1 - e_{q_1}(t)): t < \tau] \leq E^{(\cdot)}[g_1(X_t)(1 - e_{q_1}(t)): t < \tau],$$

using the conclusion stated in the last sentence of the previous remark, it follows that the left-hand side of this inequality tends to zero uniformly as $t \downarrow 0$, because the right-hand side does it, since g_1 is continuous. Likewise,

$$E^{(\cdot)}[g(X_t)(e_{q_1}(t) - e_r(t)): t < \tau] \leq E^{(\cdot)}[g_2(X_t)(1 - e_{q_2}(t)): t < \tau],$$

where again the left-hand side of this inequality tends to zero uniformly as $t \downarrow 0$, because the right-hand side does it, since g_2 is continuous. This concludes the proof showing that $g_{q_1+q_2}$ is also continuous.

Proposition 4.2: *If g_q is continuous, so is g_{λ_q} for all $\lambda > 0$.*

Proof: Because of Proposition 4.1, it suffices to prove Proposition 4.2 for $0 < \lambda < 1$.

As in the proof of Proposition 4.1, we need only to show that if $g_\lambda = g_{\lambda_q}$,

$$E^x[g_\lambda(X_t)(1 - e_{\lambda_q}(t)): t < \tau] \tag{4.2}$$

tends to zero uniformly on compacts. By Hölder's inequality, the expression in (4.2) is less than or equal to

$$\{E^x[g_\lambda^{\lambda^{-1}}(X_t)(1 - e_{\lambda_q}(t))^{\lambda^{-1}}: t < \tau]\}^\lambda. \tag{4.3}$$

Now we use Hölder's inequality again to obtain $g_\lambda \leq g_q^\lambda$ and clearly,

$$(1 - e_{\lambda_q}(t))^{\lambda^{-1}} \leq 1 - e_{\lambda_q} \leq 1 - e_q(t).$$

Thus, it follows from (4.3) that the expression in (4.2) is less than or equal to

$$E^x[g_q(X_t)(1 - e_q(t)): t < \tau]^\lambda$$

and the result follows.

Theorem 4.1: *Suppose Gq is locally bounded. Then g_q is continuous if and only if Gq is continuous.*

Proof: Step 1. Suppose Gq is bounded. Then g_q is continuous if and only if Gq is continuous. To see this, write

$$K_q f = E^x[\int_0^\tau e_q(t) f(X_t) dt]$$

and compute:

$$g_q = 1 - K_q q$$

$$Gq = K_q q + K_q[qGq].$$

The first equation shows that g_q is continuous if and only if $K_q q$ is continuous, and from the second equation, we see that under the condition that Gq is bounded, the continuity of Gq is equivalent to that of $K_q q$.

Step 2. Now suppose D is a relatively compact subset of Ω. Then, $G_D q \leq G_\Omega q$ and $G_\Omega q$ bounded on D implies that $G_D q$ is bounded. Suppose g_q is continuous. We have

$$g_q = E^x[e_q(\tau_D) g_q(X_{\tau_D})]. \tag{4.4}$$

Now by Jensen's inequality, $g_q(x) \geq e^{-Gq(x)}$, so that g_q is bounded from below on compacts, say, $g_q(X_{\tau_D}) \geq \varepsilon$. Then, one can write

$$g_q(x) = \varepsilon E^x[e_q(\tau_D)] + E^x[e_q(\tau_D)(g_q(X_{\tau_D}) - \varepsilon)].$$

Both terms on the right-hand side of this equation are upper-semi-continuous functions on D. However, the left-hand side is continuous on D implying that the both terms on the right-hand side are continuous. This implies that $E^x[e_q(\tau_D)]$ is continuous on D. This fact and the result from Step 1, imply that G_Dq is continuous. However, on D, $Gq = G_Dq +$ *a harmonic function*. This implies that Gq is continuous on D.

Suppose now Gq is continuous. Then, for each relatively compact open subset D, G_Dq is bounded and continuous. Hence, from Step 1, $E^{(\cdot)}[e_q(\tau_D)]$ is continuous and, from (4.4), so is g_q. This completes the proof.

Theorem 4.2: *Let g_q be nonvanishing and continuous in Ω. Then q is in local Kato class in Ω, i.e., $Gq1_F$ is continuous and bounded in Ω for each relatively compact set F.*

Proof: Step 1. First, suppose that g_q is bounded from below and continuous in Ω. Say, $g_q \geq \varepsilon$. Then, we claim that Gq is bounded and continuous in Ω. To see this, note that

$$1 - g_q \geq G[qg_q] \geq \varepsilon Gq,$$

giving $Gq \leq \frac{1-\varepsilon}{\varepsilon}$, i.e., Gq is bounded. Then, by Theorem 4.1, Gq is continuous.

Step 2. Again, we localize the problem. Let D be a relatively compact subset of Ω. We have

$$g_q = E^x[e_q(\tau_D)g_q(X_{\tau_D})].$$

Since g_q is nonvanishing and continuous, it is bounded from below on $\bar{D}$. So, as in Theorem 4.1, we see that $E^x[e_q(\tau_D)]$ is continuous D. Moreover, $E^x[e_q(\tau_D)] \geq g_q \geq \varepsilon$, say, (because g_q is bounded from below on compacts), implying from Step 1 that G_Dq is bounded and continuous on D. This completes the proof.

Finally, we have:

Theorem 4.3: *Suppose $Gq \not\equiv \infty$. If g_q is continuous, then Gq is extended continuous and $\{g_q = 0\} = \{Gq = \infty\}$. If Gq is extended continuous, then g_q is continuous at every point at which $Gq(x) < \infty$.*

Proof: Suppose g_q is continuous. Then the set $D_0 = \{g_q > 0\}$ is an open subset of Ω. Also, the set $\{Gq(x) < \infty\}$ is contained in D_0. Indeed, if $Gq(x) < \infty$, then $P^x\{\int_0^\tau q(X_s)ds < \infty\} = 1$, implying that $g_q(x) > 0$.

We will show that Gq is continuous on D_0. Gq is necessarily continuous at each x, such that $Gq(x) = \infty$, and this set contains D_0^c. Let D be a relatively compact open subset of D_0 and set $\tau_1 = \tau_D$. Then,

$$g_q(x) = E^x[e_q(\tau_1)g_q(X_{\tau_1})].$$

Since g_q is continuous and strictly positive on $\bar{D}$, $E^x[e_q(\tau_1)]$ is continuous and bounded from below on D (it dominates g_q on D). Hence, by Theorem 4.1, G_Dq is continuous and bounded in D. Since $Gq = G_Dq +$ *a harmonic function* in D, we see that Gq is continuous on D. Thus, Gq is continuous and finite on D_0. Suppose now Gq is extended continuous. Let $U = \{Gq < N\}$. U

is an open set. We need only show that g_q is continuous on U. Now, on U, $G_U q$ is bounded and continuous. So, from Theorem 4.1, $E^{(\cdot)}[e_q(\tau_U)]$ is continuous. Hence, so is g_q. This completes the proof.

Lemma 4.1: *Let f_λ, $0 \le \lambda \le 1$, be a family of lower semi-continuous functions on X. Suppose $0 \le f_\lambda \le 1$ and that $f_\lambda(x)$ is increasing in λ. Let*

$$f(x) = \int_0^1 f_\lambda(x) d\lambda.$$

Then, f continuous at x_0 implies $f_\lambda(\cdot)$ is continuous at x_0 for every λ which is a continuity point of $\lambda \to f_\lambda(x_0)$.

Proof: Suppose f is continuous at x_0 and μ is a continuity point of $\lambda \to f_\lambda(x_0)$. Since $f_\lambda(x_0)$ is increasing in λ, we see that

$$\frac{1}{h} \int_\mu^{\mu + h} f_\lambda(x) d\lambda \tag{4.5}$$

is increasing in h. Also, function $\int_\mu^{\mu + h} f_\lambda(x) d\lambda$ and hence, the function in (4.7), is continuous at x_0. As $h \downarrow 0$ these functions decrease to $f_\mu(x)$. Thus, at x_0, f_μ is upper semi-continuous function. Since f_μ is lower semi-continuous by assumption, the proof is complete.

Corollary 4.1: *g_q is continuous if and only if $x \to P^x\{A_\infty > \lambda\}$ is continuous for all $\lambda > 0$, where $A_\infty = \int_0^\tau q(X_s) ds$.*

Proof: We have

$$g_q(x) = E^x[e^{-A_\infty}] = \int_0^1 P^x\{e^{-A_\infty} > t\} dt$$

$$= \int_0^1 P^x\left\{\frac{1}{t} > e^{A_\infty}\right\} dt = 1 - \int_0^1 P^x\left\{A_\infty > \log\frac{1}{t}\right\} dt.$$

The family $f_t(x) = P^x\{A_\infty > \log\frac{1}{t}\}$ satisfies the conditions of Lemma 4.1. Thus, $x \to P^x\{A_\infty > \mu\}$ is continuous at x_0 provided $\lambda \to P^{x_0}\{A_\infty > \lambda\}$ is continuous at μ. It is easy to see that for any x, $P^x\{A_\infty = \lambda\} = 0$. In other words, $\lambda \to P^x\{A_\infty > \lambda\}$ is continuous in λ for each x. Thus, the continuity of g implies that of $x \to P^x\{A_\infty > \lambda\}$ for each $\lambda > 0$. Conversely, if $x \to P^x\{A_\infty > \lambda\}$ is continuous, then so is $x \to P^x\{e^{-A_\infty} > \lambda\}$. Finally, by bounded convergence theorem, it follows that $\int_0^1 P^x\{e^{-A_\infty} > \lambda\} d\lambda$ is also continuous.

5. Miscellaneous Results

Some simple facts: The following inequalities will be used in the sequel.

$$e^{-a} + e^{-b} \ge 1 + e^{-a-b}, \ a, b \ge 0, \tag{5.1}$$

so that if $c \le a \wedge b$,

$$e^{-(a-c)} + e^{-(b-c)} \le 1 + e^{-(a-c)-(b-c)}$$

or

$$e^{-a}+e^{-b}\le e^{-c}+e^{-a-b+c},\ a,b\ge 0,\ c\le a\wedge b. \tag{5.2}$$

Thus, if $q_1,q_2\ge 0$ and $a=\int_0^t q_1$, $b=\int_0^t q_2$, $c=\int_0^t (q_1\wedge q_2)\,(X_s)ds$ we get from (5.2) since $a+b-c=\int_0^t q_1\vee q_2$,

$$e_{q_1}(t)+e_{q_2}(t)\le e_{q_1\vee q_2}(t)+e_{q_1\wedge q_2}(t). \tag{5.3}$$

In particular,

$$g_{q_1}+g_{q_2}\le g_{q_1\vee q_2}+g_{q_1\wedge q_2}.$$

Consider $1-g_q$. We know that this function is excessive. Let

$$C(q)=\mathcal{L}[1-g_q]$$

where $\mathcal{L}$ is the mass-functional. (See [7].)

$C(q)$ has the following properties:

(A) $q_1\le q_2\Rightarrow C(q_1)\le C(q_2)$. Indeed, $g_{q_2}\le g_{q_1}\Rightarrow 1-g_{q_2}\ge 1-g_{q_1}$.

(B) $C(q)<\infty\Rightarrow 1-g_q$ is a potential. (A harmonic function has infinite mass functional.)

(C) $C(0)=0$. This property is clear.

(D) C is strongly subadditive, i.e., $C(q_1\vee q_2)+C(q_1\wedge q_2)\le C(q_1)+C(q_2)$. This follows immediately from (5.3).

(E)

$$\int|\nabla g_q|^2+\int qg_q^2\le C(q).$$

Proof: First, suppose that q is integrable and Gq bounded. Then

$$1-g_q=G(qg_q) \tag{5.4}$$

holds. Thus $C(q)=\int qg_1$. Also, the potential $G(qg_q)$ has finite energy because $\int qg_qG(qg_q)\le \int qg_q\le\int q<\infty$. The energy is $\int|\nabla G(qg_q)|^2=\int|\nabla(1-g_q)|^2=\int|\nabla g_q|^2$. After multiplying (5.4) by qg_q, integrating and using the above expression for $C(q)$, one gets

$$\int|\nabla g_q|^2+\int qg_q^2=\int qg_q=C(q).$$

Let q_n satisfy $\int q_n<\infty$ and let Gq_n be bounded and let it increase to q. Then, $1-g_{q_n}$ increases to $1-g_q$. So, $C(q_n)$ increases to $C(q)$. If $C(q)=\infty$, the statement holds. Suppose $C(q)<\infty$. Then $\mathcal{L}[1-g_{q_n}]$ is bounded. Now, $1-g_{q_n}\in\mathcal{H}_0^1(\Omega)$ and it is a bounded sequence in $\mathcal{H}_0^1(\Omega)$. This sequence tends to $1-g_q$. Hence $1-g_q\in\mathcal{H}_0^1(\Omega)$, and

$$\int|\nabla(1-g_q)|^2\le\underline{\lim}\int|\nabla(1-g_{q_n})|^2.$$

Now $q_n\uparrow q$, $g_{q_n}\downarrow g_q$, so that $\underline{\lim}(q_n,g_{q_n})\ge qg_q$, where the last is defined to be zero if $g_q=0$. So the result follows from Fatou's Lemma.

Theorem 5.1: *If s is excessive, then $G\frac{1}{s} \not\equiv \infty$.*

Proof: Let $x_0 \in D$ and $B(x_0, r) \subset D$. For $y \notin B(x_0, r)$, $G(x_0, \cdot)$ is harmonic in $D \backslash B(x_0, r)$ and continuous. So, for some M, $G(x_0, y) \leq Ms(y)$ for all $y \in \partial B(x_0, r)$. Hence, $G(x_0, y) \leq Ms(y)$ for all $y \in D \backslash B(x_0, r)$. So,

$$\int_{D \backslash B(x_0, r)} G(x_0, y)\frac{1}{s(y)}dy \leq M \cdot \lambda(D \backslash B(x_0, r)).$$

Here, λ denotes the Lebesgue measure. In $B(x_0, r)$, s is bounded from below. So,

$$\int_{B(x_0, r)} G(x_0, y)\frac{1}{s(y)}dy < \infty,$$

which completes the proof.

Thus,

$$g_s = E^x[e^{-\int_0^\tau \frac{1}{s(X_t)}dt}] > 0$$

everywhere and satisfies $\Delta g_s = \frac{1}{s} g_s$.

References

[1] Aizenman, M. and Simon, B., Brownian motion and Harnack inequality for Schrödinger operators, *Comm. Pure Appl. Math.* **35** (1982), 209-273.

[2] Chung, K.L., *On Stopped Feynman-Kac Functional*, Séminaire de Probabilités XIV, Lecture Notes in Math. **784**, Springer-Verlag, Berlin 1980.

[3] Chung, K.L. and Murali Rao, K., *Sur la Theorie du Potential avec la Fonctionelle de Feynman-Kac*, C.R. Acad. Sci., Paris **210** (1980).

[4] Chung, K.L. and Murali Rao, K, Functional and the Schrödinger equations, *Seminar on Stochastic Processes*, Birkhäuser, Boston (1981), 1-30.

[5] Dal Maso, G. and Mosco, U., Wiener criteria and energy decay for relaxed Dirichlet problems, *Arch. Rational Mech. Anal.* **95**:4 (1986), 345-387.

[6] Kac, M., On some connections between probability theory and differential and integral equations, *Proc. Second Berkeley Symposium on Math. Stat. and Probability*, Univ. of California Press, Berkeley (1951), 189-215.

[7] Pop-Stojanović, Z.R. and Murali Rao, K., Convergence in energy, *Z. Wahrsch. Geb.* **69** (1985), 593-608.