Journal of Applied Mathematics and Stochastic Analysis
7, Number 4, 1994, 467-486

NON-COMPACT RANDOM GENERALIZED GAMES AND RANDOM QUASI-VARIATIONAL INEQUALITIES

XIAN-ZHI YUAN
Department of Mathematics, Statistics, and Computing Science
Dalhousie University, Halifax, N.S., Canada B3H 3J5
and
Department of Mathematics
The University of Queensland, Brisbane, Australia 4072

(Received October, 1993; revised June, 1994)

ABSTRACT

In this paper, existence theorems of random maximal elements, random equilibria for the random one-person game and random generalized game with a countable number of players are given as applications of random fixed point theorems. By employing existence theorems of random generalized games, we deduce the existence of solutions for non-compact random quasi-variational inequalities. These in turn are used to establish several existence theorems of non-compact generalized random quasi-variational inequalities which are either stochastic versions of known deterministic inequalities or refinements of corresponding results known in the literature.

Key words: Polish Space, Suslin Space, Measurable Space, Suslin Family, (Random) Fixed Point, (Random) Maximal Element, (Random) Equilibria, (Random) Qualitative Game, (Random) Generalized Game, (Random) Variational Inequality, (Random) Quasi-Variational Inequality, Class L, L-Majorized, Measurable Selection Theorem, Property (K), Random Operator.

AMS (MOS) subject classifications: Primary 47B80, 47H04, 47H10, 47H40, 47N10, 49J40, 49J45, 49J55, 54C60, 60H25, 90A14, 90D13, 99D15, 93E05; Secondary 28A05, 28A20.

1. Introduction

Since Spacek [34] and Hans [14] established some existence results of random fixed point theorems in the fifties, random fixed point theory has received much more attention in recent years, e.g., see Bharucha-Reid [5], Bocsan [8], Engl [13], Itoh [16], Kucia and Nowak [21], Lin [23], Liu and Chen [24], Nowak [26], Papageorgiou [27], Rybinski [28], Sarbadhikari and Srivastava [31], Sehgal and Singh [32], Tan and Yuan [37-38] and Xu [45], etc. Recently, we proved a very general random fixed point theorem in [37] (e.g., see Theorem A below). In this paper, as applications of random fixed point theorem in [37], existence theorems of random maximal elements, random equilibria for a random one-person game and random generalized games with a countable number of players are given. By employing existence theorems of random generalized games, we deduce the existence of solutions for non-compact random quasi-variational inequalities which in turn are used to establish several existence theorems of non-compact generalized random quasi-variational inequalities which are either stochastic versions or improvements of corresponding results in the literature, e.g., Aliprantis et al. [1], Arrow and Debreu [2], Aubin [3], Aubin and Ekeland [4], Bharucha-Reid [5], Borglin and Keiding [6], Border [7], Bocsan [8], Hans [14], Kucia and Nowak [21], Liu and Chen [24], Mas-Colell and Zame [25], Nowak [26], Papageorgiou [27], Rybinski [28], Shih and Tan [33], Spacek [34], Tan [35-36], Tan and Yuan [39], Tarafdar and Mehta [41], Toussaint [42], Tulcea [43], Yannelis and Prabhakar [46], Zhang (Chang) [47], and Zhou and Chen [48].

2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$ and the set of natural numbers is denoted by $\mathbb{N}$. If X is a set, we shall denote by 2^X the family of all subsets of X. Let A be a subset of a topological space X. The set A is said to be compactly open if A is relatively open in each non-empty compact subset of X. We shall denote by $int_X(A)$ the interior of A in X and by $cl_X(A)$ the closure of A in X. If A is a subset of a vector space, we shall denote by coA the convex hull of A. If A is a non-empty subset of a topological vector space E and S, $T: A \to 2^E$ are correspondences, then coT, $T \cap S$: $A \to 2^E$ are correspondences defined by $(coT)(x) = coT(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$. If X and Y are topological spaces and (Ω, Σ) is a measurable space (see definition below), and $T: \Omega \times X \to 2^Y$ is a correspondence, the Graph of T, denoted by $GraphT$, is the set $\{(\omega, x, y) \in \Omega \times X \times Y: \ y \in T(\omega, x)\}$ and the correspondence $\bar{T}: \Omega \times X \to 2^Y$ is defined by $\bar{T}(\omega, x) = \{y \in Y: (x, y) \in cl_{X \times Y} GraphT(\omega, \cdot)\}$, and $clT: \Omega \times X \to 2^Y$ is defined by $clT(\omega, x) = cl_Y(T(\omega, x))$ for each $(\omega, x) \in \Omega \times X$. It is easy to see that $clT(\omega, x) \subset \bar{T}(\omega, x)$ for each $(\omega, x) \in \Omega \times X$.

If X and Y are topological spaces, $A \subset X \times Y$, and $F: X \to 2^Y$, then

(1) the domain of F, denoted by $DomF$, is the set $\{x \in X: F(x) \neq \emptyset\}$;

(2) the projection of A into X, denoted by $Proj_X A$, is the set $\{x \in X$: there exists some $y \in Y$ such that $(x, y) \in A\}$;

(3) F is said to be lower (respectively, upper) semicontinuous if for each closed (respectively, open) subset C of Y, the set $\{x \in X: F(x) \subset C\}$ is closed (respectively, open) in X;

(4) F is said to be compact if for each $x \in X$, there exists a open neighborhood V_x of x in X such that $F(V_x) = \cup_{z \in V_x} F(z)$ is relatively compact in Y; and

(5) $x \in X$ is a maximal element of F if $F(x) = \emptyset$.

Note that $DomF = Proj_X GraphF$.

Let X be a subset of a topological vector space E. The set X is said to have the property (K) (see [43]) if for each compact subset S of X, the convex hull coB of B is relatively compact in X.

Let X be a topological space, Y a non-empty subset of a vector space E, $\theta: X \to E$ a (single-valued) mapping and $\phi: X \to 2^Y$ a mapping. Then

(1) ϕ is said to be of class L_θ if for every $x \in X$, $co\phi(x) \subset Y$ and $Q(x) \notin co\phi(x)$ and for each $y \in Y$, $\phi^{-1}(y) := \{x \in X: \ y \in \phi(x)\}$ is compactly open in X;

(2) a correspondence $\phi_x: X \to 2^Y$ is said to be an L_θ-majorant of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that
   - (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin co\phi_x(z)$,
   - (b) for each $z \in X$, $co\phi_x(z) \subset Y$ and
   - (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X;

(3) ϕ is L_θ-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an L_θ-majorant of ϕ at $x \in X$.

We shall only deal with either the case (I) $X = Y$ and which is a non-empty convex subset of a topological vector space and $\theta = I_X$, the identity mapping on X (in this case, the above notions coincide with the corresponding notions introduced in [46]), or the case (II) $X = \Pi_{i \in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $X_j = Y$ is a non-empty convex subset of atopological vector space. In both cases (I) and (II), we shall write L in place of L_θ.

A measurable space (Ω, Σ) is a pair where Ω is a set and Σ is a σ-algebra of subsets of Ω. If X is a set, $A \subset X$, and $\mathfrak{D}$ is a non-empty family of subsets of X, we shall denote by $\mathfrak{D} \cap A$ the family $\{D \cap A : D \in \mathfrak{D}\}$ and by $\sigma_X(\mathfrak{D})$ the smallest σ-algebra on X generated by $\mathfrak{D}$. If X is a topological space with topology τ_X, we shall use $\mathfrak{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ-algebra on X. If (Ω, Σ) and (Φ, Γ) are two measurable spaces, then $\Sigma \otimes \Gamma$ denotes the smallest σ-algebra on $\Omega \times \Phi$ which contains all the sets $A \times B$, where $A \in \Sigma$, $B \in \Gamma$, i.e., $\Sigma \otimes \Gamma = \sigma_{\Omega \times \Phi}(\Sigma \times \Gamma)$. We note that the Borel σ-algebra $\mathfrak{B}(X_1 \times X_2)$ contains $\mathfrak{B}(X_1) \otimes \mathfrak{B}(X_2)$ in general. A mapping $f: \Omega \to \Phi$ is said to be (Σ, Γ) measurable (or simply, measurable) if for each $B \in \Gamma$, $f^{-1}(B) = \{x \in \Omega : f(x) \in B\} \in \Sigma$. Let X be a topological space and $F: (\Omega, \Sigma) \to 2^X$ be a mapping. Then F is said to be measurable (respectively, weakly measurable)) if $F^{-1}(B) = \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \Sigma$ for each closed (respectively, open) subsets B of X. The map F is said to have a measurable graph if $\mathit{Graph}F := \{\omega, y) \in \Omega \times X : y \in F(\omega)\} \in \Sigma \otimes \mathfrak{B}(X)$. A function $f: \Omega \to X$ is a measurable selection of F if f is a measurable function such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

If (Ω, Σ) and (Φ, Γ) are measurable spaces, Y is a topological space, then a mapping $F: \Omega \times \Phi \to 2^Y$ is called (jointly) measurable (respectively, weakly measurable) if for every closed (respectively, open) subset B of Y, $F^{-1}(B) = \{(\omega, x) \in \Omega \times \Phi : F(\omega, x) \cap B \neq \emptyset\} \in \Sigma \otimes \Gamma$. In the case $\Phi = X$, a topology space, then it is understood that Γ is the Borel σ-algebra $\mathfrak{B}(X)$.

A topological space X is

- (i) a Polish space if X is separable and metrizable by a complete metric;
- (ii) a Suslin space if X is a Hausdorff topological space and the continuous image of a Polish space.

A Suslin subset in a topological space is a subset which is a Suslin space. "Suslin" sets play very important roles in measurable selection theory. We also note that if X_1 and X_2 are Suslin spaces, then $\mathfrak{B}(X_1 \times X_2) = \mathfrak{B}(X_1) \times \mathfrak{B}(X_2)$ (e.g., see [29, p. 113]).

Denote by $\mathfrak{J}$ and $\mathfrak{F}$ the sets of infinite and finite sequences of positive integers respectively. Let $\mathfrak{G}$ be a family of sets and $F: \mathfrak{Gl} \to \mathfrak{F}$ be a map. For each $\sigma = (\sigma_i)_{i=1}^{\infty} \in \mathfrak{J}$ and $n \in \mathbb{N}$, we shall denote $(\sigma_1, \ldots, \sigma_n)$ by $\sigma \mid n$; then $\bigcup_{\sigma \in \mathfrak{J}} \bigcap_{n=1}^{\infty} F(\sigma \mid n)$ is said to be obtained from $\mathfrak{G}$ by the Suslin operation. Now, if every set obtained from $\mathfrak{G}$ in this way is also in $\mathfrak{G}$, then $\mathfrak{G}$ is called a Suslin family (e.g., see [22], [30], [44], etc.).

Let X and Y be topological spaces, (Ω, Σ) a measurable space and $F: \Omega \times X \to 2^Y$ be a mapping. Then

- (a) F is a random operator if for each fixed $x \in X$, the mapping $F(\cdot, x): \Omega \to 2^Y$ is a measurable map;
- (b) F is random lower semicontinuous (respectively, random upper semicontinuous, random continuous) if F is a random operator and for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$: $X \to 2^Y$ is lower semicontinuous (respectively, upper semicontinuous, continuous); and
- (c) a measurable (single-valued) mapping $\psi: \Omega \to X$ is said to be a random maximal element of the correspondence F if $F(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.

Let (Ω, Σ) be a measurable space, X a topological space and $F: \Omega \times X \to 2^X$ a mapping. The (single-valued) mapping $\varphi: \Omega \to X$ is said to be

- (i) a deterministic fixed point of F if $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$; and
- (ii) a random fixed point of F if φ is a measurable mapping and $\varphi(\omega) \in F(\omega, \varphi(x))$ for all $\omega \in \Omega$.

It should be noted here that some authors define a random fixed point of F to be a measurable mapping φ such that $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for almost every $\omega \in \Omega$, e.g., see [27], [28] and the references therein.

Let I be any set of players and (Ω, Σ) be a measurable space. For each $i \in I$, let its strategy set X_i be a non-empty subset of a topological vector space. Let $X = \Pi_{i \in I} X_i$. For each $i \in I$, let $P_i: \Omega \times X \to 2^{X_i}$ be a correspondence. The collection $\Gamma = (\Omega, X_i, P_i)_{i \in I}$ will be called a random qualitative game. A measurable map $\psi: \Omega \to X$ is said to be a random equilibrium of the random qualitative game Γ if $P_i(\omega, \psi(\omega)) = \emptyset$ for all $i \in I$ and all $\omega \in \Omega$.

A random generalized game (abstract economy) is a collection $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ where I is a (finite or infinite) set of players (agents) such that for each $i \in I$, X_i is a non-empty subset of a topological vector space and $A_i, B_i: \Omega \times X \to 2^{X_i}$ are random constraint correspondences where $X = \Pi_{i \in I} X_i$, and $P_i: \Omega \times X \to 2^{X_i}$ is a preference correspondence (which are interpreted as for each player (or agent) $i \in I$, the associated constraint and preferences A_i, B_i and P_i have stochastic actions). A random equilibrium of Γ is a (single-valued) measurable mapping $\Psi: \Omega \to X$ such that for each $i \in I$, $\pi_i(\psi(\omega)) \in \overline{B_i}(\omega, \psi(\omega))$ and $A_i(\omega, \psi(\omega)) \cap P_i(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$. Here, π_i is the projection from X onto X_i. If $x \in X$, we shall also write x_i in place of $\pi_i(x)$ if there is no ambiguity. We remark that if A_i, B_i and P_i of the random generalized game $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ are independent of the variable $\omega \in \Omega$, i.e., $A_i(\omega, \cdot) = A_i(\cdot)$, $B_i(\omega, \cdot) = B_i(\cdot)$ and $P_i(\omega, \cdot) = P_i(\cdot)$ for all $\omega \in \Omega$, when $\overline{B_i}(\widehat{x}) = cl_{X_i} B_i(\widehat{x})$ for each $\widehat{x} \in X$ (which is the case when B_i has a closed graph in $X \times X_i$; in particular, when clB_i is upper semicontinuous with closed values), our definition of an equilibrium point coincides with that of Ding et al. [12] in the deterministic case; and if in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition of the deterministic case, e.g., in Borglin and Keiding [7], Tulcea [43], and Yannelis and Prabhakar [46].

We shall now list some results which will be needed in this paper. The following very general random fixed point theorem is Theorem 2.2 of Tan and Yuan in [37].

Theorem A. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a Suslin space. Suppose $F: \Omega \times X \to 2^X \setminus \{\emptyset\}$ is such that $GraphF \in \Sigma \otimes \mathfrak{B}(X \times X)$. Then F has a random fixed point if and only if F has a deterministic fixed point in X, i.e., for each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X.*

For a non-self mapping generalization of the above result, we refer the reader to [38, Theorem 2.3]. The following measurable selection theorem is due to Leese [22, Corollary, p. 408-409].

Theorem B. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a Suslin space. Suppose $F: \Omega \to 2^X$ has non-empty values such that $GraphF \in \Sigma \otimes \mathfrak{B}(X)$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of measurable selections of F such that for each $\omega \in \Omega$, the set $\{g_n(\omega): n \in \mathbb{N}\}$ is dense in $F(\omega)$.*

The following lemma is Theorem 3.3 of Tan and Yuan in [39].

Lemma 1. *Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (a) *for each $i \in I$, X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i;*
- (b) *for each $i \in I$, $A_i: X \to 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subseteq B_i(x)$;*
- (c) *for each $i \in I$, $A_i \cap P_i$ is L_C-majorized;*
- (d) *for each $i \in I$, the set $E^i = \{x \in X: (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;*
- (e) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact*

subset K of X such that for each $y \in X \backslash K$ there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then $\mathcal{G}$ has an equilibrium point in K, i.e., there exists a point $\widehat{x} = (\widehat{x}_i)_{i \in I} \in K$ such that for each $i \in I$, $\widehat{x}_i \in \overline{B_i}(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$.

The following result is Theorem 5.3 of Tan and Yuan in [40].

Lemma 2. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (*a*) *for each $i \in I$, X_i is a non-empty closed convex subset of a locally convex Hausdorff topological vector space E_i and X_i has the property (K);*
- (*b*) *for each $i \in I$, B_i is compact and upper semicontinuous with non-empty compact convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;*
- (*c*) *for each $i \in I$, P_i is lower semicontinuous and L_C-majorized;*
- (*d*) *for each $i \in I$, $E^i = \{x \in X: (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;*
- (*e*) *there exist a nonempty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \backslash K$ there is an $x \in co(X_0 \cup \{y\})$ with $x_i \in co(A_i(y) \cap P_i(y))$ for all $i \in I$.*

Then there exists $\overline{x}\,(\widehat{x}_i)_{i \in I} \in K$ such that for each $i \in I$, $\overline{x}_i \in \overline{B_i}(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

We also need the following result (e.g., see Theorem 1 of Ding and Tan [11]).

Lemma 3. *Let X be a non-empty paracompact convex subset of a Hausdorff topological vector space and $P: X \rightarrow 2^X$ be L-majorized (i.e., L_{I_X}-marjorized). Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in z \backslash K$, there exists $x \in co(X_0 \cup \{y\})$ with $x \in coP(y)$. Then there exists an $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$.*

3. Random Equilibria of Random Games

As an application of our random fixed point theorem, namely, Theorem A above, we shall first prove the following existence theorem of random maximal elements:

Theorem 1. *Let (Ω, Σ) be a measurable space, Σ Suslin family, X a non-empty paracompact convex and Suslin subset of a Hausdorff topological vector space E and $Q: \Omega \times X \rightarrow 2^X$ such that for each given $\omega \in \Omega$, $Q(\omega, \cdot)$ is L_{I_X}-majorized and $DomQ \in \Sigma \otimes \mathfrak{B}(X)$. Suppose that for each fixed $\omega \in \Omega$, there exists a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \backslash K(\omega)$ there is an $x \in co(X_0(\omega) \cup \{y\})$ with $x \in coQ(\omega, y)$. Then Q has a random maximal element, i.e., there exists a measurable mapping $\psi: \Omega \rightarrow X$ such that $Q(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.*

Proof. By Lemma 3, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $Q(\omega, x_\omega) = \emptyset$. Define $F: \Omega \times X \rightarrow 2^X$ by $F(\omega, x) = \{y \in X: Q(\omega, y) = \emptyset\}$ for each $(\omega, x) \in \Omega \times X$. Then for each fixed $\omega \in \Omega$, x_ω is a fixed point of $F(\omega, \cdot)$. In order to prove that $GraphF \in \Sigma \otimes \mathfrak{B}(X \times X)$, we define a mapping $C: \Omega \times X \times X \rightarrow \Omega \times X \times X$ by

$$C(\omega, x, y) = (\omega, y, x)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. By hypothesis, $DomQ \in \Sigma \otimes \mathfrak{B}(X)$. Since

$$GraphF = \{(\omega, x, y) \in \Omega \times X \times X: Q(\omega, y) = \emptyset\}$$

$$= C^{-1}[(\Omega \times X \backslash DomQ) \otimes X] \in \Sigma \otimes \mathfrak{B}(X \times X),$$

then, by Theorem A, F has a random fixed point ψ, i.e., there exists $\psi: \Omega \to X$ is measurable such that $\psi(\omega) \in F(\omega, \psi(\omega))$ for all $\omega \in \Omega$ which imp! s that $Q(\omega, \psi(\omega)) = \emptyset$ for each $\omega \in \Omega$. □

As an application of Theorem A again, we have the following existence theorem of random equilibria for random one-person games:

Theorem 2. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a non-empty precompact convex and Suslin subset of a Hausdorff topological vector space. Let $A, B, P: \Omega \times X \to 2^X$ be such that*

(i) *for each $\omega \in \Omega$, $A(\omega, \cdot) \cap P(\omega, \cdot)$ is L-majorized;*
(ii) *$A(\omega, x)$ is non-empty and $coA(\omega, x) \subset B(\omega, x)$ for each $(\omega, x) \in \Omega \times X$;*
(iii) *$(A(\omega, \cdot))^{-1}(y) = \{x \in X: y \in A(\omega, x)\}$ is open in X for each $(\omega, y) \in \Omega \times X$;*
(iv) *$Dom(A \cap P)$ and $Proj_{\Omega \times X}[(Graph\bar{B}) \cap (\Omega \times \Delta)] \in \Sigma \otimes \mathfrak{B}(X)$ where $\Delta = \{(x, x): x \in X\}$;*
(v) *for each fixed $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \backslash K(\omega)$ there is an $x \in co(X_0(\omega) \cap \{y\})$ with $x \in co(P(\omega, y) \cap A(\omega, y))$.*

Then the random one-person game $(\Omega; X; A, B; P)$ has a random equilibrium, i.e, there exists a measurable mapping $\psi: \Omega \to X$ such that $\psi(\omega) \in \bar{B}(\omega, \psi(\omega))$ and $A(\omega, \psi(\omega)) \cap P(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.

Proof. Define $\Psi: \Omega \times X \to 2^X$ by

$$\Psi(\omega, x) = \{y \in X: A(\omega, y) \cap P(\omega, y) = \emptyset \text{ and } y \in \bar{B}(\omega, y)\}$$

for each $(\omega, x) \in \Omega \times X$. Then by Theorem 2 of Ding and Tan [11], for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $x_\omega \in \Psi(\omega, x)$ for all $x \in X$. It follows that $\Psi: \Omega \times X \to 2^X \backslash \{\emptyset\}$ and $x_\omega \in \Psi(\omega, x_\omega)$ for all $\omega \in \Omega$ so that Ψ has a deterministic fixed point in X. Now define a mapping $C: \Omega \times X \times X \to \Omega \times X \times X$, by $C(\omega, x, y) = (\omega, y, x)$ for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. Note that

$$\begin{aligned} Graph\Psi &= C^{-1}([\Omega \times X) \backslash Dom(A \cap P)] \times X) \\ &\quad \cap C^{-1}(Proj_{\Omega \times X}[Graph\bar{B} \cap (\Omega \times \Delta)] \times X) \\ &\quad \in \Sigma \otimes \mathfrak{B}(X \times X), \end{aligned}$$

so that $Graph\Psi \in \Sigma \otimes \mathfrak{B}(X \times X)$. By Theorem A, Ψ has a random fixed point ψ, i.e., $\psi: \Omega \times X$ is measurable such that $A(\omega, \psi(\omega)) \cap P(\omega, \psi(\omega)) = \emptyset$ and $\psi(\omega) \in \bar{B}(\omega, \psi(\omega))$ for all $\omega \in \Omega$. □

As another application of Theorem A, we have the following:

Theorem 3. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ a random generalized game such that I is countable and $X = \Pi_{i \in I} X_i$ is paracompact. For each $i \in I$, suppose that the following conditions are satisfied:*

(I) *X_i is a non-empty convex and Suslin subset of a locally convex Hausdorff topological vector space;*
(II) *$Dom(A_i \cap P_i)$, $Proj_{\Omega \times X}[Graph\overline{B_i} \cap (\Omega \times \Delta_i)] \in \Sigma \otimes \mathfrak{B}(X)$ where $\Delta_i = \{(x, \pi_i(x)): x \in X\}$.*
(III) *for each $\omega \in \Omega$, $E_i(\omega) = \{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X;*
(IV) *for each fixed $\omega \in \Omega$, either*

(i) (a) *$A_i(\omega, \cdot): X \to 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(\omega, x)$ is non-empty and $coA_i(\omega, x) \subset B_i(\omega, x)$, and*

(b) $A_i(\omega, \cdot) \cap P_i(\omega, \cdot)$ *is L-majorized;*

or

(ii) (a) $B_i(\omega, \cdot)$ *is upper semicontinuous with non-empty compact and convex values such that for each* $x \in X$, $A_i(\omega, x) \subset B_i(\omega, x)$, *and*

(b) $P_i(\omega, \cdot)$ *is lower semicontinuous and L-majorized, and* X_i *is closed and has the property* (K);

(V) *for each fixed* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $y \in X \backslash K(\omega)$ *there is an* $x \in co(X_0(\omega) \cup \{y\})$ *with* $x_i \in co(A_i(\omega, y) \cap P_i(\omega, y))$ *for all* $i \in I$.

Then Γ *as a random equilibrium.*

Proof. First we note that as each X_i is a Suslin space and I is countable, X is also a Suslin space. For each $i \in I$, define $\Psi_i : \Omega \times X \to 2^X$ by

$$\Psi_i(\omega, x) = \{y \in X : A_i(\omega, y) \cap P_i(\omega, y) = \emptyset \text{ and } \pi_i(y) \in \overline{B_i}(\omega, y)\}$$

for each $(\omega, x) \in \Omega \times X$. Define $\Psi : \Omega \times X \to 2^X$ by $\Psi(\omega, x) = \bigcap_{i \in I} \Psi_i(\omega, x)$ for each $(\omega, x) \in \Omega \times X$. Then by Lemma 1 or Lemma 2, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $x_\omega \in \Psi_i(\omega, x)$ for all $x \in X$ and for all $i \in I$ so that $x_\omega \in \Psi(\omega, x)$ for all $x \in X$. It follows that $\Psi : \Omega \times X \to 2^X \backslash \{\emptyset\}$ and $x_\omega \in \Psi(\omega, x_\omega)$ for all $\omega \in \Omega$ so that Ψ has a deterministic fixed point in X. Now define a mapping $C : \Omega \times X \times X \to \Omega \times X \times X$ by

$$C(\omega, x, y) = (\omega, y, x)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. Note that

$$\begin{aligned} Graph \Psi_i &= C^{-1}([\Omega \times X \backslash Dom(A_i \cap P_i)] \times X) \\ &\quad \cap C^{-1}(Proj_{\Omega \times X}[Graph \overline{B_i} \cap (\Omega \times \Delta_i)] \times X) \\ &\quad \in \Sigma \otimes \mathfrak{B}(X \times X), \end{aligned}$$

and I is countable, we have $Graph \Psi = \bigcap_{i \in I} Graph \Psi_i \in \Sigma \otimes \mathfrak{B}(X \times X)$. By Theorem A, there exists a measurable mapping $\psi : \Omega \to X$ such that $\psi(\omega) \in \Psi(\omega, \psi(\omega))$ for all $\omega \in \Omega$; i.e., $A_i(\omega, \psi(\omega) \cap P_i(\omega, \psi(\omega)) = \emptyset$ and $\pi_i(\psi_i(\omega)) \in \overline{B_i}(\omega, \psi(\omega))$ for all $\omega \in \Omega$ and for all $i \in I$. □

As a consequence of Theorem 3, we have the following existence theorem of random qualitative games:

Theorem 4. *Let* (Ω, Σ) *be a measurable space with* Σ *a Suslin family and* $\Gamma = (\Omega; X_i; P_i)_{i \in I}$ *a random qualitative game such that* I *is countable and* $X = \Pi_{i \in I} X_i$ *is paracompact. For each* $i \in I$, *suppose that the following conditions are satisfied:*

(i) X_i *is a non-empty convex and Suslin subset of a locally convex Hausdorff topological vector space;*

(ii) $Dom P_i \in \Sigma \otimes \mathfrak{B}(X)$;

(iii) *for each* $\omega \in \Omega$, $Dom P_i(\omega, \cdot)$ *is open in* X;

(iv) *for each fixed* $\omega \in \Omega$, $P_i(\omega, \cdot)$ *is L-majorized;*

(v) *for each fixed* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $y \in X \backslash K(\omega)$ *there is an* $x \in co(X_0(\omega) \cup \{y\})$ *with* $x_i \in co(P_i(\omega, y))$ *for all* $i \in I$.

Then Γ *has a random equilibrium.*

Proof. For each $i \in I$, define $A_i, B_i : \Omega \times X \to 2^{X_i}$ by $A_i(\omega, x) = B_i(\omega, x) = X$ for each

$(\omega, x) \in \Omega \times X$. Then it is easily seen that all hypotheses of Theorem 3 are satisfied. By Theorem 3, the conclusion follows. □

4. Random Quasi-Variational Inequalities

In this section, by our existence theorems of random equilibria for random generalized games, namely, Theorem 3, some existence theorems of random quasi-variational inequalities and generalized random quasi-variational inequalities are given. Our results not only generalize the results of Tan [36] and Zhang [47], but also they are the stochastic versions of corresponding results in the literatures, e.g., see Aubin [3], Aubin and Ekeland [4], Hildenbrand and Sonnenschein [15], Shih and Tan [33], Tan [35, 36], Zhang [47], Zhou and Chen [48] and the references therein.

Here we emphasize that our arguments for the existence of solutions for non-compact random quasi-variational inequalities are different from the approaches used in the literatures by Tan [36] and Zhang [47].

Theorem 5. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and I be countable. For each $i \in I$, suppose that the following conditions are satisfied:*

- (a) *X_i is a non-empty convex and closed Suslin subset of a locally convex Hausdorff topological vector space such that X_i has the property (K) and $X = \Pi_{i \in I} X_i$ is paracompact;*
- (b) *for each fixed $\omega \in \Omega$, $A_i(\omega, \cdot): X = \Pi_{i \in I} X_i \to 2^{X_i}$ is upper semicontinuous with non-empty compact and convex values;*
- (c) *$\psi_i: \Omega \times X \times X_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:*
- $(c)_1$: *$x \mapsto \psi_i(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X_i$;*
- $(c)_2$: *$x_i \notin co\{y \in X_i: \psi_i(\omega, x, y) > 0\}$ for each fixed $(\omega, x) \in \Omega \times X$;*
- $(c)_3$: *for each fixed $\omega \in \Omega$, the set $\{x \in X: \alpha_i(\omega, x) > 0\}$ is open in X, where $\alpha_i: \Omega \times X \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $\alpha_i(\omega, x) = sup_{y_i \in A_i(\omega, x)} \psi_i(\omega, x, y_i)$ for each $(\omega, x) \in \Omega \times X$;*
- (d) *$\{(\omega, x) \times X: \alpha_i(\omega, x) > 0\}$, and $\{(\omega, x) \in \Omega \times X: \pi_i(x) \in A_i(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;*
- (e) *for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \backslash K(\omega)$ there exists $x \in co(X_0(\omega) \cup \{y\})$ with $x_i \in co(A_i(\omega, y) \cap \{z \in X_i: \psi_i(\omega, y, z) > 0\})$.*

Then there exists a measurable mapping $\phi: \Omega \to X$ such that for $i \in I$, $\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega))$ and

$$\sup_{y \in A_i(\omega, \phi(\omega))} \psi_i(\omega, \phi(\omega), y) \leq 0$$

for all $\omega \in \Omega$.

Proof. For each $i \in I$, define $P_i: \Omega \times X \to 2^{X_i}$ by $P_i(\omega, x) = \{y \in X_i: \psi_i(\omega, x, y) > 0\}$ for each $(\omega, x) \in \Omega \times X$. We shall show that $G = (\Omega; X_i; A_i; P_i)_{i \in I}$ satisfies all hypotheses of Theorem 3 with $A_i = B_i$ for all $i \in I$.

Suppose $i \in I$ and $\omega \in \Omega$. By $(c)_1$, for each fixed $y \in X_i$, $(P_i(\omega, \cdot))^{-1}(y) = \{x \in X: \psi_i(\omega, x, y) > 0\}$ is open in X and by $(c)_2$, $x_i \notin coP_i(\omega, x)$ for each $x \in X$. This shows that $P_i(\omega, \cdot)$ is lower semicontinuous and is of class L and hence is L-majorized. By the definition of α_i, we note that $\{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\} = \{x \in X: \alpha_i(\omega, x) > 0\}$, so that $\{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X by $(c)_3$. By (d), we know that $Dom(A_i \cap P_i) \in \Sigma \otimes \mathfrak{B}(X)$ and

$$Proj_{\Omega \times X}[Graph A_i \cap (\Omega \times \Delta_i)] \in \Sigma \otimes \mathfrak{B}(X).$$

Thus $G = (\Omega, X_i, A_i, P_i)_{i \in I}$ satisfies all hypothesis of Theorem 3 with $A_i = B_i$ for each $i \in I$. By

Theorem 3, there exists a measurable mapping $\phi:\Omega\to X$ such that for each $i\in I$,

$$\pi_i(\phi(\omega))\in A_i(\omega,\phi(\omega)) \text{ and } A_i(\omega,\phi(\omega))\cap P_i(\omega,\phi(\omega))=\emptyset$$

for all $\omega\in\Omega$, i.e.,

$$\pi_i(\phi(\omega))\in A_i(\omega,\phi(\omega)) \text{ and } \sup_{y\in A_i(\omega,\phi(\omega))}\phi_i(\omega,\phi(\omega),y)\le 0$$

for all $\omega\in\Omega$. □

Letting $I=\{1\}$ in Theorem 5, we have the following existence results on random quasi-variational inequalities:

Theorem 6. *Let (Ω,Σ) be a measurable space with Σ a Suslin family. Suppose that the following conditions are satisfied:*

- (a) *X is a non-empty closed paracompact convex and Suslin subset of a locally convex Hausdorff topological vector space, and X has the property (K);*
- (b) *for each fixed $\omega\in\Omega$, $A(\omega,\cdot):X\to 2^X$ is upper semicontinuous with non-empty compact and convex values;*
- (c) *$\psi:\Omega\times X\times X\to\mathbb{R}\cup\{-\infty,+\infty\}$ is such that:*
- $(c)_1$ *$x\mapsto\psi(\omega,x,y)$ is lower semicontinuous on X for each fixed $(\omega,y)\in\Omega\times X$;*
- $(c)_2$ *$x\notin co\{y\in X:\psi(\omega,x,y)>0\}$ for each fixed $(\omega,x)\in\Omega\times X$;*
- $(c)_3$ *for each fixed $\omega\in\Omega$, the set $\{x\in X:\alpha(\omega,x)>0\}$ is open in X, where $\alpha:\Omega\times X\mapsto\mathbb{R}\cup\{-\infty,+\infty\}$ is defined by $\alpha(\omega,x)=\sup_{y\in A(\omega,x)}\psi(\omega,x,y)$ for each $(\omega,x)\in\Omega\times X$;*
- (d) *$\{(\omega,x)\in\Omega\times X:\alpha(\omega,x)>0\}$, and $\{(\omega,x)\in\Omega\times X:x\in A(\omega,x)\}\in\Sigma\otimes\mathfrak{B}(X)$;*
- (e) *for each given $\omega\in\Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y\in X\backslash K(\omega)$ there exist $x\in co(X_0(\omega)\cup\{y\})$ with $x\in co(A(\omega,y)\cap\{z\in X:\psi(\omega,y,z)>0\})$.*

Then there exists a measurable mapping $\phi:\Omega\to X$ such that $\phi(\omega)\in A(\omega,\phi(\omega))$ and

$$\sup_{y\in A(\omega,\phi(\omega))}\psi(\omega,\phi(\omega),y)\le 0$$

for all $\omega\in\Omega$.

4. Generalized Random Quasi-Variational Inequalities

Let (Ω,Σ) be a measurable space, X a non-empty compact convex subset of a locally convex Hausdorff topological vector E and E^* the dual space of E. Suppose the correspondences $F:\Omega\times X\to 2^X$, $T:\Omega\times X\to 2^{E^*}$ and the function $f:\Omega\times X\times X\to\mathbb{R}\cup\{-\infty,+\infty\}$ are given. We want to find a measurable mapping $\psi:\Omega\to X$ which satisfies the following generalized random quasi-variational inequalities:

$$\begin{cases}\psi(\omega)\in F(\omega,\psi(\omega))\\ \quad \sup_{y\in F(\omega,\psi(\omega))}[\sup_{u\in T(\omega,\psi(\omega))}Re\langle u,\psi(\omega)-y\rangle+f(\omega,\psi(\omega,y)]\le 0\end{cases}\tag{$*$}$$

for each $\omega\in\Omega$. We also want to find two measurable maps $\psi:\Omega\to X$ and $\phi:\Omega\to E^*$ such that

$$\begin{cases}\psi(\omega)\in F(\omega,\psi(\omega)) \text{ and } \phi(\omega)\in T(\omega,\psi(\omega))\\ \quad Re\langle\phi(\omega),\psi(\omega)-y\rangle+f(\omega,\psi(\omega),y)\le 0\end{cases}\tag{$**$}$$

for all $y \in F(\omega, \psi(\omega))$ and for all $\omega \in \Omega$.

In this section, by applying results in Section 3, we shall consider the generalized random variational inequality problems (*) and (**) above.

Now we recall some definitions (e.g., see [48]). Let X be a non-empty convex subset of topological vector space E. A function $\psi(\omega, y): X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be

(1) γ-diagonally quasi-convex (respectively, γ-diagonally quasi-concave) in y, in short $\gamma-$DQCX (respectively, γ-DQCV) in y, if for each $A \in \mathfrak{F}(X)$ and each $y \in co(A)$, $\gamma \leq max_{x \in A}\psi(y,x)$ (respectively, $\gamma \geq inf_{x \in A}\psi(y,x)$);

(2) γ-diagonally convex (respectively, γ-diagonally concave) in y, in short γ-DCX (respectively, γ-DCV) in y, if for each $A \in \mathfrak{F}(X)$ and each $y \in co(A)$ with $y = \Sigma_{i=1}^m \lambda_i y_i$ ($\lambda_i \geq 0$, and $\Sigma_{i=1}^m \lambda_i = 1$), we have $\gamma \leq \Sigma_{i=1}^m \lambda_i \psi(y, y_i)$ (respectively, $\gamma \geq \Sigma_{i=1}^m \lambda_i \psi(y, y_i)$).

Let X and Y be two non-empty convex subsets of E, we also recall that a function $\psi: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ is quasi-convex (respectively, quasi-concave) in y, if for each fixed $x \in X$, for each $A \in \mathfrak{F}(Y)$ and each $y \in co(A)$, $\psi(x,y) \leq max_{z \in A}\psi(x,z)$ (respectively, $\psi(x,y) \geq min_{z \in A}\psi(x,z)$). Moreover, it is easy to verify that

(*i*) if $\psi(x,y)$ is γ-DCX (respectively, γ-DCV) in y, then $\psi(x,y)$ is γ-DQCX (respectively, γ-DQCV) in y,

(*ii*) if $\psi_i: X \times Y \to \mathbb{R}$ is γ-DCX (respectively, γ-DCV) in y for each $i = 1, 2, \ldots, m$, then $\psi(x,y) = \Sigma_{i=1}^m a_i(x)\psi_i(x,y)$ is also γ-DCX (respectively, γ-DCV) in y, where $a_i: X \to \mathbb{R}$ with $a_i(x) \geq 0$ and $\Sigma_{i=1}^m a_i(x) = 1$ for each $x \in X$, and

(*iii*) the function $\psi(x,y): X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is 0-DQCV in y if and only if $x \notin co(\{y \in X: \psi(x,y) > 0\})$ for each $x \in X$.

In what follows, we first consider the existence of solutions of problem (*) for which monotonicity is needed.

Theorem 7. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty closed paracompact convex and Suslin subset of a locally convex Hausdorff topological vector space E such that X has the property (K). Suppose that the following conditions are satisfied:*

(*i*) *$F: \Omega \times X \to 2^X$ is such that for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact and convex values;*

(*ii*) *$T: \Omega \times X \to 2^{E^*}$ is such that for each fixed $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone (i.e., $Re\langle u - v, y - x\rangle \geq 0$ for all $u \in T(\omega, y)$ and $v \in T(\omega, x)$ for all $x, y \in X$) with non-empty values and for each one-dimensional flat $L \subset E$, $T(\omega, \cdot)|_{L \cap X}$ is lower semicontinuous from the relative topology of X into the weak*-topology $\sigma(E^*, E)$ of E^*;*

(*iii*) *$f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X$ and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;*

(*iv*) *for each fixed $\omega \in \Omega$, the set $\{x \in X: sup_{y \in F(\omega,x)}[sup_{u \in T(\omega,y)} Re\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$ is open in X;*

(*v*) *$\{(\omega, x) \in \Omega \times X: sup_{y \in F(\omega x)}[sup_{u \in T(\omega,y)} Re\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;*

(*vi*) *$\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;*

(*vii*) *for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \backslash K(\omega)$ there exists $y \in co(X_0(\omega) \cup \{x\})$ with $y \in co(F(\omega, x) \cap \{z \in X: sup_{u \in T(\omega,z)} Re\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.*

Then there exists a measurable mapping $\phi: \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $y \in F(\omega, \phi(\omega))$ *and* $\omega \in \Omega$.

Proof. Define a function $\psi: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega, x, y) = \sup_{u \in T(\omega, y)} Re\langle u, x - y\rangle + f(\omega, x, y)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. By (iii), $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X for each $(\omega, y) \in \Omega \times X$. For each $\omega \in \Omega$, since $T(\omega, \cdot)$ is monotone, by (iii), it is easy to verify that $\psi(\omega, x, y)$ is 0-DCV in y by Proposition 3.2 of Zhou and Chen [48]. The conditions (i)-(vi) imply that all hypotheses of Theorem 6 are satisfied. By Theorem 6, there exists a measurable mapping $\Phi: \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \sup_{u \in T(\omega, y)} [Re\langle u, \phi(\omega) - y\rangle + (\omega, \phi(\omega), y)] \leq 0 \tag{1}$$

for all $\omega \in \Omega$. We shall now prove that

$$\sup_{y \in F(\omega, \phi(\omega))} \sup_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for each $\omega \in \Omega$.

Fix an $\omega \in \Omega$. Let $x \in F(\omega, \phi(\omega))$ be arbitrarily given and let $z_t(\omega) = tx + (1-t)\phi(\omega) = \phi(\omega) - t(\phi(\omega) - x)$ for $t \in [0,1]$. As $F(\omega, \phi(\omega))$ is convex, we have $z_t(\omega) \in F(\omega, \phi(\omega))$ for $t \in [0,1]$. Therefore, by (1) we have

$$\sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - z_t(\omega)\rangle + f(\omega, \phi(\omega), z_t(\omega))] \leq 0$$

for all $t \in [0,1]$.

Since for each $x \in X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$, it follows that for $t \in (0,1]$,

$$t \cdot \{ \sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - x\rangle] + f(\omega, \phi(\omega), x)\}$$

$$\leq \sup_{u \in T(\omega, z_t(\omega))} t \cdot [Re\langle u, \phi(\omega) - x\rangle] + f(\omega, \phi(\omega), tx + (1-t)\phi(\omega))$$

$$= \sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - z_t(\omega)\rangle] + f(\omega, \phi(\omega), z_t(\omega)) \leq 0$$

which implies that for $t \in (0,1]$,

$$\sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - x\rangle] + f(\omega, \phi(\omega), x)] \leq 0. \tag{2}$$

Let $z_0 \in T(\omega, \phi(\omega))$ be arbitrarily fixed. For each $\epsilon > 0$, let

$$U_{z_0} = \{z \in E^*: |\, Re\langle z_0 - z, \phi(\omega) - x\rangle \,| < \epsilon\}.$$

Then U_{z_0} is a $\sigma(E^*, E)$-neighborhood of z_0. Since $T(\omega, \cdot)|_{L \cap X}$ is lower semicontinuous where $L: = \{z_t(\omega): t \in [0,1]\}$, and $U_{z_0} \cap T(\omega, \phi(\omega)) \neq \emptyset$, there exists a neighborhood $N(\phi(\omega))$ of $\phi(\omega)$ in L such that if $z \in N(\phi(\omega))$, then $T(\omega, \phi(\omega)) \cap U_{z_0} \neq \emptyset$. But then there exists $\delta \in (0,1]$ such that $z_t(\omega) \in N(\phi(\omega))$ for all $t \in (0, \delta)$. Fixing any $t \in (0, \delta)$ and $u \in T(\omega, z_t(\omega)) \cap U_{z_0}$, we have

$|Re\langle z_0 - u, \phi(\omega) - z\rangle| < \epsilon$. This implies that

$$Re\langle z_0, \phi(\omega) - x\rangle < Re\langle u, \phi(\omega) - z\rangle + \epsilon.$$

Thus

$$Re\langle z_0, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) < Re\langle u, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) + \epsilon < \epsilon$$

by (2). Since $\epsilon > 0$ is arbitrary, $Re\langle z_0, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) \le 0$. As $z_0 \in T(\omega, \phi(\omega))$, is arbitrary, we have the following

$$\sup_{z \in T(\omega, \phi(\omega))} [Re\langle z, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x)] \le 0$$

for all $x \in F(\omega, \phi(\omega))$. □

Corollary 8. *Let (Ω, Σ) be a measurable space with Σ a Suslin family, X a non-empty compact convex Suslin subset of a locally convex Hausdorff topological vector space E and $F: \Omega \times X \to 2^X$ be such that $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$. If for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact convex values, then F has a random fixed point.*

We shall now observe that in Theorem 7, the interaction between the correspondences T and F (namely, the condition (*iv*)) can be achieved by imposing additional continuity conditions on T and F.

Theorem 9. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty closed paracompact convex and Suslin bounded subset of a locally convex Hausdorff topological vector space E such that X has the property (K). If $F: \Omega \times X \to 2^X$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values, and $T: \Omega \times X \to 2^{E^*}$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values, and $T: \Omega \times X \to 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone with non-empty values and is lower semicontinuous from the relative topology of X to the strong topology of E^*. Suppose that*

- (*i*) *$f: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that for each given $\omega \in \Omega$, $(x, y) \mapsto f(\omega, x, y)$ is lower semicontinuous and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;*
- (*ii*) *the set $\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \sup_{u \in T(\omega, y)} [Re\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;*
- (*iii*) *$\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;*
- (*iv*) *for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \backslash K(\omega)$ there exists $y \in co(X_0(\omega) \cup \{x\})$ with $y \in co(F(\omega, x) \cap \{z \in X: \sup_{u \in T(\omega, z)} Re\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.*

Then there exists a measurable mapping $\phi: \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [\sup_{u \in T(\omega, \phi(\omega))} Re\langle u\phi(\omega) - y) + f(\omega, \phi(\omega), y\rangle] \le 0$$

for all $\omega \in \Omega$.

Proof. By Theorem 7, we need only show that for each given $\omega \in \Omega$, the set

$$\Sigma(\omega) := \{x \in X: \sup_{y \in F(\omega, x)} [\sup_{u \in T(\omega, y)} Re\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$$

is open in X.

Since X is bounded and $f(\omega, \cdot, \cdot)$ is lower semicontinuous, the function $(u,x,y) \mapsto Re\langle u, x-y\rangle + f(\omega,x,y)$ is lower semicontinuous from $E^* \times X \times X$ to $\mathbb{R}$ for each fixed $\omega \in \Omega$. Therefore $(x,y) \mapsto sup_{u \in T(\omega,y)}[Re\langle u, x-y\rangle + f(\omega,x,y)]$ is also lower semicontinuous by lower semicontinuity of $T(\omega, \cdot)$ and Proposition III-19 of Aubin and Ekeland [4, p.118]. Since $F(\omega, \cdot)$ is lower semicontinuous, $x \mapsto sup_{y \in F(\omega,x)} sup_{u \in T(\omega,y)}[Re\langle u, x-y\rangle + f(\omega,x,y)]$ is lower semicontinuous by Proposition III-19 of [4, p. 118] again for each fixed $\omega \in \Omega$. Thus the set $\Sigma(\omega) := \{x \in X: sup_{y \in F(\omega,x)} sup_{u \in T(\omega,x)}[Re\langle u, x-y\rangle + f(\omega,x,y)] > 0\}$ is open in X. □

Now we will consider the existence of solutions for the problems (*) and (**) without assuming the monotonicity as in Theorem 9.

Theorem 10. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex and Polish subset of a locally convex Hausdorff topological vector space E. Suppose that:*

(i) *$F: \Omega \times X \to 2^X$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact and convex values;*

(ii) *$T: \omega \times X \to 2^{E^*}$ is such that $x \mapsto inf_{u \in T(\omega,x)} Re\langle u, x-y\rangle$ is lower semicontinuous for each $(\omega, y) \in \Omega \times X$;*

(iii) *$f: \Omega \times X \times X \to \mathbb{R}$ is such that $x \mapsto f(\omega,x,y)$ is lower semicontinuous on X for each fixed $(\omega,y) \in \Omega \times X$; and for each fixed $(\omega,x) \in \Omega \times X$, $y \mapsto f(\omega,x,y)$ is 0-diagonal concave;*

(iv) *for each given $\omega \in \Omega$, the set*

$$\{x \in X: \sup_{y \in F(\omega,x)} [\inf_{u \in T(\omega,x)} Re\langle u, x-y\rangle + f(\omega,x,y)] > 0\}$$

is open in X;

(v) *$\{(\omega,x) \in \Omega \times X: sup_{y \in F(\omega,x)} inf_{u \in T(\omega,x)}[Re\langle u, x-y\rangle + f(\omega,x,y)] > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;*

(vi) *$\{(\omega,x) \in \Omega \times X: x \in F(\omega,x)\} \in \Sigma \otimes \mathfrak{B}(X)$;*

(vii) *for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \backslash K(\omega)$ there exists $y \in co(X_0(\omega) \cap \{x\})$ with $y \in co(F(x) \cap \{z \in X: sup_{u \in T(\omega,z)} Re\langle u, x-z\rangle + f(\omega,x,z) > 0\})$.*

Then there exists a measurable mapping $\phi: \Omega \times X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\inf_{u \in T(\omega,\phi(\omega))} [Re\langle u, \phi(\omega) - y\rangle] + f(\omega, \phi(\omega), y) \leq 0$$

for all $y \in F(\omega, \phi(\omega))$ and $\omega \in \Omega$.

Suppose that in addition,

(1) *for each fixed $(\omega,x) \in \Omega \times X$, $y \mapsto f(\omega,x,y)$ is lower semicontinuous and concave and f is measurable;*

(2) *there exists a non-empty Polish subset E_0^* of E^* such that $T(\Omega, X) \subset E_0^*$, T is measurable with non-empty strongly compact convex values; and*

(3) *F is measurable.*

Then there exist a measurable function $\rho: \Omega \to E^*$ such that $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega,\phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

Proof. Define $\psi: \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega,x,y) = \inf_{u \in T(\omega,x)} [Re\langle u, x-y\rangle + f(\omega,x,y)],$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then by (ii), (iii) and (iv) we have:

(a) for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X and $x \notin co(\{y \in X: \psi(\omega, x, y) > 0\})$ for each $(\omega, x) \in \Omega \times X$;

(b) for each $\omega \in \Omega$, the set $\{x \in X: sup_{y \in F(\omega, x)} \psi(\omega, x, y) > 0\}$ is open in X.

Therefore, F and ψ satisfy all conditions of Theorem 6. By Theorem 6 there exists a measurable mapping $\phi: \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \inf_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

If, in addition, the conditions (1), (2) and (3) hold, we shall find another measurable (single-valued) mapping $\rho: \Omega \to E^*$ such that $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for each $\omega \in \Omega$.

Fix any $\omega \in \Omega$. Define $f_1: F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega)) \to \mathbb{R}$ by

$$f_1(y, u) = Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega, y)$$

for each $(y, u) \in F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega))$. Then for each $y \in F(\omega, \phi(\omega))$, $u \mapsto f_1(y, u)$ is lower semicontinuous and convex and for each fixed $u \in T(\omega, \phi(\omega))$, $y \mapsto f_1(y, u)$ is concave. By Kneser's Minimax Theorem [20],

$$\inf_{u \in T(\omega, \phi(\omega))} \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] =$$

$$\sup_{y \in F(\omega, \phi(\omega))} \inf_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0.$$

Since $T(\omega, \phi(\omega))$ is compact, there exists $u_0 \in T(\omega, \phi(\omega))$ such that

$$\sup_{y \in F(\omega, \phi(\omega))} [Re\langle u_0, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0.$$

Now we define $\Phi, T_1: \Omega \to 2^X$ by

$$\Phi(\omega) = \{u \in T(\omega, \phi(\omega)): \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0\}$$

and

$$T_1(\omega) = T(\omega, \phi(\omega))$$

for each $\omega \in \Omega$. Note that $\Phi(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Since T and ϕ are measurable, T_1 is also measurable by Lemma 3 in [28, p. 55]. Define $g_1: \Omega \times X \times X \times E_0^* \to \mathbb{R}$ by

$$g_1(\omega, x, y, u) = Re\langle u, x - y \rangle + f(\omega, x, y)$$

for each $(\omega, x, y, u) \in \Omega \times X \times X \times E_0^*$. Then g_1 is measurable. Also we define $g_2: \Omega \times X \times E_0^* \to \mathbb{R}$ by

$$g_2(\omega, y, u) = Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)$$

for each $(\omega, y, u) \in \Omega \times X \times E_0^*$. Now define $F_1: \Omega \times 2^X$ by

$$F_1(\omega) = F(\omega, \phi(\omega))$$

for each $\omega \in \Omega$. Since ϕ is measurable and F is also measurable, g_2 and F_1 are measurable by Lemma 3 in [28, p. 55] again. Let $g_3: \Omega \times E_0^* \to \mathbb{R}$ by

$$g_3(\omega, u) = \sup_{y \in F(\omega, \phi(\omega))} g_2(\omega, y, u) = \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)]$$

for each $(\omega, u) \in \Omega \times E_0^*$. We shall show that g_3 is measurable. Since F_1 is measurable by Theorem B, there exists a countable family of measurable mappings $p_n: \Omega \to X$ such that $F_1(\omega) = cl\{p_n(\omega): n = 1, 2, \ldots\}$ for each $\omega \in \Omega$. Since ϕ is measurable, for each fixed $(u, y) \in E^* \times X$, the mapping $\omega \mapsto Re\langle u, \phi(\omega) - y\rangle$ is measurable. Note that the mapping $(u, y) \mapsto Re\langle u, \phi(\omega) - y\rangle$ is continuous, so that the mapping $(\omega, u, y) \mapsto Re\langle u, \phi(\omega) - y\rangle$ is measurable by Theorem III.14 of Castaing and Valadier [9, p. 70]. For each $n \in \mathbb{N}$, the function $g_n': \Omega \times E^* \to \mathbb{R}$, defined by

$$g_n'(\omega, u) = Re\langle u, \phi(\omega) - p_n(\omega)\rangle + f(\omega, \phi(\omega), p_n(\omega))$$

for each $(\omega, u) \in \Omega \times E^*$, is measurable. Therefore, for each $n \in \mathbb{N}$, the mapping $(\omega, u) \mapsto Re\langle u, \phi(\omega) - p_n(\omega)\rangle + f(\omega, \phi(\omega), p_n(\omega))$ is also measurable. Since for each $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is lower semicontinuous, it follows that for each $r \in \mathbb{R}$,

$$\{(\omega, u) \in \Omega \times E^*: g_3(\omega, u) \leq r\} = \bigcap_{n=1}^{\infty} \{(\omega, u) \in \Omega \times E^*: g_n'(\omega, u) \leq r\} \in \Sigma \otimes \mathcal{B}(E^*).$$

Therefore the function g_3 is measurable so that the set $M_0 = \{(\omega, u) \in \Omega \times E_0^*: g_3(\omega, u) \leq 0\} \in \Sigma \otimes \mathcal{B}(E^*)$. Hence $Graph\Phi = (Graph T_1) \cap M_0 \in \Sigma \otimes \mathcal{B}(E_0^*)$. By Theorem B, there exists a measurable mapping $\rho: \Omega \to E_0^*$ such that $\rho(\omega) \in \Phi(\omega)$ for each $\omega \in \Omega$. By the definition of Φ, the measurable mapping ρ satisfies the following:

$$\begin{cases} \phi(\omega) \in F(\omega, \phi(\omega)) \text{ and } \rho(\omega) \in T(\omega, \phi(\omega)) \\ \sup_{y \in F(\omega, \phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y] \leq 0. \end{cases}$$

□

Note that if $T: \Omega \times X \to 2^{E^*}$ is such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact values, then by Lemma 2 of Kim and Tan in [19, p. 140] or Theorem 1 of Aubin in [3, p. 67], the condition (ii) of Theorem 10 is satisfied. Thus Theorem 10 is a stochastic version of Theorem 3 of Shih and Tan in [33, p. 340]. Recall that for a topological vector space E, the strong topology on its dual space E^* is the topology on E^* generated by the family $\{U(B; \epsilon): B$ is a non-empty bounded subset of E and $\epsilon > 0\}$ as a base for the neighborhood system at zero, where $U(B; \epsilon): = \{f \in E^*: \sup_{x \in B} |Re\langle f, x\rangle| < \epsilon\}$.

Now if we impose the upper semicontinuity condition to correspondence T, then we have the following:

Theorem 11. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex and Polish bounded subset of a locally convex Hausdorff topological vector space E. Suppose*

- (i) *$F: \Omega \times X \to 2^X$ is random continuous with non-empty compact and convex values;*
- (ii) *$T: \Omega \times X \to 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;*
- (iii) *$f: \Omega \times X \times X \to \mathbb{R}$ is such that*
  - *(a) for each fixed $(\omega, y) \in \Omega \times X, x \mapsto f(\omega, x, y)$ is lower semicontinuous on X;*

(*b*) *for each fixed* $(\omega,x)\in\Omega\times X, y\mapsto f(\omega,x,y)$ *is* 0-*diagonally concave;*

(*iv*) $\{(\omega,x)\in\Omega\times X: sup_{y\in F(\omega,x)} inf_{u\in T(\omega,x)} Re\langle u,x-y\rangle+f(\omega,x,y)>0\}\in\Sigma\otimes\mathcal{B}(X)$;

(*v*) *for each* $\omega\in\Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $x\in X\backslash K(\omega)$ *there exists* $y\in co(X_0(\omega)\cup\{x\})$ *with* $y\in co(F(x)\cap\{z\in X: sup_{u\in T(\omega,z)} Re\langle u,x-z\rangle+f(\omega,x,z)>0\})$.

Then,

(*a*) *for each fixed* $\omega\in\Omega$, *the set* $\{x\in X: sup_{y\in F(\omega,x)}[inf_{u\in T(\omega,x)} Re\langle u,x-y\rangle+f(\omega,x,y)]>0\}$ *is open in* X;

(*b*) $\{(\omega,x)\in\Omega\times X: x\in F(\omega,x)\}\in\Sigma\otimes\mathcal{B}(X)$;

(*c*) *there exists a measurable mapping* $\phi:\Omega\to X$ *such that* $\phi(\omega)\in F(\omega,\phi(\omega))$ *and*

$$inf_{u\in T(\omega,\phi(\omega))}[Re\langle u,\phi(\omega)-y\rangle+f(\omega,\phi(\omega),y)]\le 0$$

for all $y\in F(\omega,\phi(\omega))$ *and* $\omega\in\Omega$.

Proof. (*a*) Fix $\omega\in\Omega$. Since X is a bounded subset of the locally convex Hausdorff topological vector space E, and E^* is equipped with the strong topology, the function $\psi_1: E^*\times X\times X\to\mathbb{R}\cup\{-\infty,+\infty\}$, defined by

$$\psi_1(u,x,y)=Re\langle u,x-y\rangle$$

for each $(u,x,y)\in E^*\times X\times X$, is continuous. Since $T(\omega,\cdot): X\to 2^{E^*}$ is upper semicontinuous with non-empty strongly compact values, by Theorem 1 of Aubin [3, p. 67], the function $\psi_2: X\times X\to\mathbb{R}\cup\{-\infty,+\infty\}$ defined by

$$\psi_2(x,y)=\inf_{u\in T(\omega,x)} Re\langle u,x-y\rangle$$

for each $(x,y)\in X\times X$, is also lower semicontinuous. Thus the mapping $(x,y)\mapsto inf_{u\in T(\omega,x)} Re\langle u,x-y\rangle+f(\omega,x,y)$ is lower semicontinuous by (*iii*). As $F(\omega,\cdot): X\to 2^X$ is lower semicontinuous with non-empty values, by Proposition III-19 in [4, p. 118], the mapping $x\mapsto sup_{y\in F(\omega,x)} inf_{u\in T(\omega,x)}[Re\langle u,x-y\rangle+f(\omega,x,y)]$ is lower semicontinuous from X to $\mathbb{R}\cup\{-\infty,+\infty\}$ for each fixed $\omega\in\Omega$, so that the set

$$\Sigma(\omega)=\{x\in X: \sup_{y\in F(\omega,x)}\ \inf_{u\in T(\omega,x)}[Re\langle u,x-y\rangle+(\omega,x,y)]>0\}$$

is open in X.

(*b*) Since F is random continuous with closed values, by Theorem 3.5 in [17, p. 57] and Lemma 2.5 of Tan and Yuan [37], the set $\{(\omega,x)\in\Omega\times X: x\in F(\omega,x)\}\in\Sigma\otimes\mathcal{B}(X)$.

Thus all hypotheses of Theorem 10 are satisfied, the conclusion follows. □

If both correspondences T and F are measurable, we have the following:

Theorem 12. *Let* (Ω,Σ) *be a measurable space with* Σ *a Suslin family and* X *a non-empty convex and Polish bounded subset of a locally convex Hausdorff topological vector space* E. *Suppose that*

(*i*) $F:\Omega\times X\to 2^X$ *is measurable such that for each* $\omega\in\Omega$, $F(\omega,\cdot)$ *is continuous with non-empty compact and convex values;*

(*ii*) $T:\Omega\times X\to 2^{E^*}$ *is measurable such that for each* $\omega\in\Omega$, $T(\omega,\cdot)$ *is upper semicontinuous with non-empty strongly compact and convex values;*

(*iii*) *$f: \Omega \times X \times X \to \mathbb{R}$ is measurable such that*
(*a*) *for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X;*
(*b*) *for each fixed $(\omega, x) \in \Omega \times X$, $f(\omega, x, x) = 0$ and $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave;*

(*iv*) *for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \backslash K(\omega)$ there exists $y \in co(X_0(\omega) \cup \{x\})$ with $y \in co(F(\omega) \cap \{z \in X: sup_{u \in T(\omega, z)} Re\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.*

Then there exist measurable maps $\phi: \Omega \to X$ and $\rho: \Omega \to E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

Proof. By Theorem 10 and Theorem 11, it remains to prove that $\{(\omega, x) \in \Omega \times X: sup_{y \in F(\omega, x)} inf_{u \in T(\omega, x)} Re\langle u, x - y\rangle + f(\omega, x, y) > 0\} \in \Sigma \otimes \mathcal{B}(X)$.

Since T and F are measurable, by Theorem 4.2 (*e*) of Wagner [44], there exist two countable families of measurable maps $p_n: \Omega \times X \to X$ and $q_n: \Omega \times X \to E^*$ such that $F(\omega, x) = cl\{p_n(\omega, x): n = 1, 2, \ldots\}$ and $T(\omega, x) = cl\{q_n(\omega, x): n = 1, 2, \ldots\}$ for each $(\omega, x) \in \Omega \times X$. We define $g_0: E^* \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$g_0(u, x, y) = Re\langle u, x - y\rangle$$

for each $(u, x, y) \in E^* \times X \times X$. Then g_0 is continuous and is measurable. Therefore the function $g_0': \Omega \times E^* \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g_0'(\omega, u, x, y) = Re\langle u, x - y\rangle + f(\omega, x, y)$$

for each $(\omega, u, x, y) \in \Omega \times E^* \times X \times X$, is also measurable since f is measurable. Now fix any $x \in \mathbb{N}$, note that $p_n: \Omega \times X \to X$ is measurable and f is measurable. For each $j \in \mathbb{N}$, the function $g_j^n: \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g_j^n(\omega, x) = Re\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))$$

for each $(\omega, x) \in \Omega \times X$, is measurable by Lemma 3 in [28, p. 55]. Therefore the mapping $g_n: \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g_n(\omega, x) = \inf_{j \in \mathbb{N}} g_j^n(\omega, x) = \inf_{j \in \mathbb{N}} [Re\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))]$$

for each $(\omega, x) \in \Omega \times X$, is measurable. Note that $g: \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g(\omega, x) = \sup_{n \in \mathbb{N}} g_n(\omega, x)$$

for each $(\omega, x) \in \Omega \times X$, is also measurable. Since for each $(\omega, x) \in \Omega \times X$, the mapping $y \mapsto f(\omega, x, y)$ is lower semicontinuous, the set

$$\begin{aligned} &\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [Re\langle u, x - y\rangle + f(\omega, x, u)] > 0\} \\ &= \{(\omega, x) \in \Omega \times X: \sup_{n \in \mathbb{N}} \inf_{j \in \mathbb{N}} [Re\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))] > 0\} \\ &= \{(\omega, x): g(\omega, x) > 0\} \in \Sigma \otimes \mathcal{B}(X). \end{aligned}$$

Therefore, we have

$$\{(\omega,x)\in\Omega\times X\colon \sup_{y\in F(\omega,x)}\ \inf_{u\in T(\omega,x)} Re\langle u,x-y\rangle + f(\omega,x,y)>0\}\in\Sigma\otimes\mathfrak{B}(X). \qquad \square$$

Corollary 13. *Let (Ω,Σ) be a measurable space with Σ a Suslin family and X a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

(i) *$F:\Omega\times X\to 2^X$ is measurable such that for each $\omega\in\Omega$, $F(\omega,\cdot)$ is continuous with non-empty compact and convex values;*

(ii) *$T:\Omega\times X\to 2^{E^*}$ is measurable such that for each $\omega\in\Omega$, $T(\omega,\cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;*

(iii) *$f:\Omega\times X\times X\to\mathbb{R}$ is measurable such that*
(a) *for each fixed $(\omega,y)\in\Omega\times X$, $x\mapsto f(\omega,x,y)$ is lower semicontinuous on X;*
(b) *for each fixed $(\omega,x)\in\Omega\times X$, $f(\omega,x,x)=0$ and $y\mapsto f(\omega,x,y)$ is lower semicontinuous and concave.*

Then there exist measurable maps $\phi:\Omega\to X$ and $\rho:\Omega\to E^*$ such that $\phi(\omega)\in F(\omega,\phi(\omega))$, $\rho(\omega)\in T(\omega,\phi(\omega))$ and

$$\sup_{y\in F(\omega,\phi(\omega))}[Re\langle\rho(\omega),\phi(\omega)-y\rangle + f(\omega,\phi(\omega),y)]\le 0$$

for all $\omega\in\Omega$.

By allowing f to be zero in Corollary 13, we have the following:

Corollary 14. *Let (Ω,Σ) be a measurable space with Σ a Suslin family and X a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

(i) *$F:\Omega\times X\to 2^X$ is measurable such that for each $\omega\in\Omega$, $F(\omega,\cdot)$ is continuous with non-empty compact and convex values;*

(ii) *$T:\Omega\times X\to 2^{E^*}$ is measurable such that for each $\omega\in\Omega$, $T(\omega,\cdot)$ is upper semicontinuous with non-empty strongly compact and convex values.*

Then there exist measurable map $\phi:\Omega\to X$ and $\rho:\Omega\to E^*$ such that $\phi(\omega)\in F(\omega,\phi(\omega))$, $\rho(\omega)\in T(\omega,\phi(\omega))$ and

$$\sup_{y\in F(\omega,\phi(\omega))} Re\langle\rho(\omega),\phi(\omega)-y\rangle\le 0$$

for all $\omega\in\Omega$.

Theorem 11 is also a stochastic version of Theorem 4 of Shih and Tan in [33, p. 341] (and its improvements due to Kim [18] and due to Shih and Tan [33, Theorem 2, p. 69-70] (with $M=0$)).

Theorem 11 generalizes a theorem of Tan [36, p. 326] in the following ways:

(1) the correspondence T is upper semicontinuous instead of being continuous, and

(2) the function f need not be random continuous.

In the case where $F(x)=X$ and $T(x)=0$ for each $x\in X$, Theorem 11 also improves Theorem 9.2.3 of Zhang [47, p. 304] with weaker continuity and measurability conditions. We also remark that our arguments used in proving the existence of solutions for generalized random quasi-variational inequalities in this section are different from those used by Tan [36] and Zhang [47], etc.

Quasi-variational inequalities and generalized quasi-variational inequalities have many applications in mathematical economics, game theory and optimization and other applied science (e.g., see [3-4], [7], [15] and [25]). Random quasi-variational inequalities and generalized random quasi-variational inequalities will also have many applications in random mathematical economics, random game theory and related fields.

Acknowledgement

The author would like to express his thanks to Professors K.K. Tan and J. Dshalalow during the writing of this paper.

References

[1] Aliprantis, C., Brown, D. and Burkinshaw, O., *Existence and Optimality of Competitive Equilibria*, Springer-Verlag 1989.

[2] Arrow, K.J. and Debreu, G., Existence of an equilibrium for a competitive economy, *Econometrica* **22** (1954), 265-290.

[3] Aubin, J.P., *Mathematical Methods of Game and Economics Theory* (Revised Edition, ed.), North Holland 1982.

[4] Aubin, J.P. and Ekeland, I., *Applied Nonlinear Analysis*, John Wiley & Sons 1984.

[5] Bharucha-Reid, A.T., Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.* **82** (1976), 641-657.

[6] Borglin, A. and Keiding, H., Existence of equilibrium actions of equilibrium, "A note on the 'new' existence theorems", *J. Math. Economics* **3** (1976), 313-316.

[7] Border, K.C., *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press 1985.

[8] Bocsan, G., A general random fixed point theorem and application to random equations, *Rev. Roum. Math. Pures et Appl.* **26** (1981), 375-379.

[9] Castaing, C. and Valadier, M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. vol. **580**, Springer-Verlag 1977.

[10] Debreu, G., *Existence of Competitive Equilibrium*, Handbook of Mathematical Economics (ed. by K.J. Arrow and M.D. Intriligator), vol. **II**, North-Holland 1982, 697-743.

[11] Ding, X.P. and Tan, K.K., On equilibria of non-compact generalized games, *J. Math. Anal. Appl.* **177** (1993), 226-238.

[12] Ding, X.P., Kim, W.K. and Tan, K.K., Equilibria of non-compact generalized games with $\mathcal{L}^*$-majorized preference correspondences, *J. Math. Anal. Appl.* **164** (1992), 508-517.

[13] Engl, H.W., Random fixed point theorems for multivalued mappings, *Pacific J. Math.* **76** (1978), 351-360.

[14] Hans, O., Reduzierende zufallige transformationen, *Czech. Math. J.* **7** (1957), 154-158.

[15] Hildenbrand, W. and Sonnenschein, H., (eds.), *Handbook of Mathematical Economics*, vol. **IV**, North-Holland 1991.

[16] Itoh, S., Random fixed point theorem with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), 261-273.

[17] Himmelberg, J., Measurable relations, *Fund. Math.* **87** (1975), 53-72.

[18] Kim, W.K., Remark on a generalized quasi-variational inequality, *Proc. Amer. Math. Soc.* **103** (1988), 667-668.

[19] Kim, W.K. and Tan, K.K., A variational inequality in non-compact sets and its applications, *Bull. Austral. Math. Soc.* **46** (1992), 139-148.

[20] Kneser, H., Sur un théorème fondamental de la théorie des jeux, *C.R. Acad. Sci. Paris* **234** (1952), 2418-2420.

[21] Kucia, A. and Nowak, A., Some results and counterexamples on random fixed points, *Trans. 10th Prague Conf. on Information Theory, Stat. Decision, Funct. and Random Processes Prague* 1986, Academia, Prague 1988, 75-82.

[22] Leese, S.J., Multifunctions of Souslintype, *Bull. Austral. Math. Soc.* **11** (1974), 395-411.

[23] Lin, T.C., Random approximations and random fixed point theorems for non-self-maps, *Proc. Amer. Math. Soc.* **103** (1988), 1129-1135.

[24] Liu, Z.S. and Chen, S.Z., On fixed point theorems of random set-valued maps, *Kexue Tongbao* (*Bulletin of Chinese Science*) **28** (1988), 433-435.

[25] Mas-Colell, A. and Zame, W.R., Equilibrium theory in infinite dimensional spaces, *Handbook of Mathematical Economics* (ed. by W. Hildenbrand and H. Sonnenschein), vol. **IV** (1991), 1826-1898.

[26] Nowak, A., Random fixed points of multifunctions, *Prace Nauk. Uniw. Slask. Katowice, Prace Mat.* **11** (1981), 36-41.

[27] Papageorgiou, N.S., Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), 507-514.

[28] Rybinski, L.E., Random fixed points and viable random solutions of functional-differential inclusions, *J. Math. Anal. Appl.* **142** (1989), 53-61.

[29] Saint-Beuve, M.F., On the existence of von Neumann-Aumann's theorem, *J. Funct. Anal.* **17** (1974), 112-129.

[30] Saks, S., *Theory of the Integral*, 2nd Edition, Dover, New York 1968.

[31] Sarbadhikar, H. and Srivastava, S.M., Random versions of extension theorems of Dugundji and fixed point theorems, *Boll. U.M.I.* **7-B (7)** (1993), 631-642.

[32] Sehgal, V.M. and Singh, S.P., On random approximations and fixed point theorems for set valued mappings, *Proc. Amer. Math. Soc.* **95** (1985), 91-94.

[33] Shih, M.H. and Tan, K.K., Generalized quasi-variational inequalities in locally convex topological vector spaces, *J. Math. Anal. Appl.* **108** (1985), 333-343.

[34] Spacek, A., Zufallige gleichungen, *Czech. Math. J.* **5** (1955), 462-466.

[35] Tan, K.K., Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems, *J. London Math. Soc.* **28** (1983), 555-562.

[36] Tan, N.X., Random quasi-variational inequality, *Math. Nachr.* **125** (1986), 319-328.

[37] Tan, K.K., and Yuan, X.Z., Some random fixed point theorems, *Fixed Point Theory and Applications* (ed. by K.K. Tan), World Scientific, Singapore 1995, 334-345.

[38] Tan, K.K. and Yuan, X.Z., On deterministic and random fixed points, *Proc. Amer. Math. Soc.* **119** (1993), 849-856.

[39] Tan, K.K. and Yuan, X.Z., Approximation method and equilibria of abstract economics, *Proc. Amer. Math. Soc.* (in press) (1994).

[40] Tan, K.K. and Yuan, X.Z., Equilibria of generalized games, *Nonlinear Times and Digest* **1** (1994), 61-70.

[41] Tarafdar, E. and Mehta, G., A generalized version of the Gale-Nikaido-Debreu theorem, *Comm. Math. Univ. Carolinae* **28** (1987), 655-659.

[42] Toussaint, S., On the existence of equilibria in economies with infinitely many commodities and without ordered preferences, *J. Econ. Theory* **33** (1984), 98-115.

[43] Tulcea, C.I., On the approximation of upper semicontinuous correspondences and the equilibrium of the generalized games, *J. Math. Anal. Appl.* **136** (1988), 267-289.

[44] Wagner, D.H., Survey of measurable selection theorems, *SIAM J. Control and Optim.* **15** (1977), 859-903.

[45] Xu, H.K., Some random fixed point theorems for condensing and nonexpansive operators, *Proc. Amer. Math. Soc.* **110** (1990), 395-400.

[46] Yannelis, N.C. and Prabhakar, N.D., Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Economics* **12** (1983), 233-245.

[47] Zhang, S.S. (Chang), *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific Literature Press, Shanghai, China, 1991.

[48] Zhou, J.X. and Chen, G., Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, *J. Math. Anal. Appl.* **132** (1988), 213-225.