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ON TRANSFORMATIONS OF WIENER SPACE

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ABSTRACT

We consider transformations of the form

$$(T_a x)_t = x_t + \int_0^t a(s,x)ds$$

on the space C of all continuous functions $x = x_t: [0,1] \to \mathbb{R}$, $x_0 = 0$, where $a(s,x)$ is a measurable function $[0,1] \times C \to \mathbb{R}$ which is $\tilde{\mathfrak{C}}_s$-measurable for a fixed s and $\tilde{\mathfrak{C}}_s$ is the σ-algebra generated by $\{x_u, u \leq t\}$. It is supposed that T_a maps the Wiener measure μ_0 on $(C, \tilde{\mathfrak{C}}_1)$ into a measure μ_a which is equivalent with respect to μ_0. We study some conditions of invertibility of such transformations. We also consider stochastic differential equations of the form

$$dy(t) = dw(t) + a(t, y(t))dt, \quad y(0) = 0$$

where $w(t)$ is a Wiener process. We prove that this equation has a unique strong solution if and only if it has a unique weak solution.

Key words: Wiener Space, Invertible Transformation, Girsanov's Theorem, Sets of the Second Category, Stochastic Differential Equation, Weak and Strong Solutions of Stochastic Differential Equations.

AMS (MOS) subject classifications: 60G99, 60H99, 60H10.

1. Introduction

Denote by C the space of continuous functions $x = x_t: [0,1] \to \mathbb{R}$ for which $x_0 = 0$ and by $\tilde{\mathfrak{C}}_t$, $t \in [0,1]$ the σ-algebra of subsets C which is generated by subsets $\{x \in C: x_s < \lambda\}$, $\lambda \in R$, $s \leq t$. Let μ_0 be a Wiener measure and C_t be the completion of $\tilde{\mathfrak{C}}_t$ with respect to the measure μ_0. Note that the measurable space with the measure $\{C, \tilde{\mathfrak{C}}_1, \mu_0\}$ is called the Wiener space. We consider transformations $T_a: C \to C$ of the form

$$T_a(x)_t = x_t + \int_0^t a(s,x)ds, \tag{1}$$

where the function $a: [0,1] \times C \rightarrow R$ satisfies condition

$A1)$ a is $\mathfrak{B}_{[0,1]} \otimes \widetilde{\mathfrak{C}}_1$-measurable, where $\mathfrak{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$, and $a(s,x)$ is $\widetilde{\mathfrak{C}}_s$-measurable for a fixed $s \in [0,1]$.

Such transformations were considered by R. Sh. Liptser and A.N. Shiryaev [3] and M.P. Ershov [2]. They established conditions under which the image μ_a of the measure μ_0 under transformation T_a is an equivalent measure with respect to measure μ_0. If this is true then there exists a function $c(s,x)$ which satisfies condition $A1)$ and

$$\frac{d\mu_a}{d\mu_0}(x) = e(c,x) = exp\left\{ \int_0^1 c(s,x)dx(s) - \frac{1}{2}\int_0^1 c^2(s,x)ds \right\} \tag{2}$$

with

$$\int e(c,x)\mu_0(dx) = 1 \tag{3}$$

(the integral with respect to $dx(s)$ is Ito's integral). A.A. Novikov proved in [4] that the condition

$$\int e^{\frac{1}{2}\int_0^1 c^2(s,x)ds} \mu_0(dx) < \infty \tag{4}$$

implies (3).

We consider the set $\mathbb{A}$ of functions $a(s,x): [0,1] \times C \rightarrow R$ which satisfy condition $A1)$ and condition

$A2)$ $\lim_{r \rightarrow \infty} r^{-1} sup\left\{ \int_0^1 a^2(s,x)ds; x \in U_r \right\} = 0$, where $U_r = \left\{ x \in C: \int_0^1 x_s^2 ds \leq r \right\}$.

Note that if $a \in \mathbb{A}$ then $\int (e(a,x))^k \mu_0(dx) < \infty$ for all integer numbers k, which is a consequence of Novikov's results.

Denote $\mathbb{T} = \{T_a, a \in \mathbb{A}\}$. Note that $\mathbb{T}$ is a semigroup with respect to the product

$$(T_a \cdot T_b)x_s = x_s + \int_0^s b(u,x)du + \int_0^s a(u, T_b x)du. \tag{5}$$

Obviously, T_a is an invertible transformation if there exists a function $c \in \mathbb{T}$ for which $T_a T_c x = x$ (μ_0-*a.s.*). Then, $T_c T_a x = x$ (μ_0-*a.s.*), and we call T_c the *inverse transformation* and denote it T_a^{-1}.

Remark: Below we consider all relations with x as valid (μ_0-*a.s.*).

We denote by $\mathbb{T}_R$ the set of all invertible transformations T_a and by $\mathbb{A}_R$ the subset of those $a \in \mathbb{A}$ for which $T_a \in \mathbb{T}_R$.

The main goal of this article is to formalize the set $\mathbb{T}_R$. Besides, we consider the stochastic differential equation

$$dy(t) = w(t) + a(t, y(\cdot))dt; \quad t \in [0,1], \; y(0) = 0, \tag{6}$$

where $w(t)$ is a Wiener process, $a \in \mathbb{A}$, and describe its weak and strong solutions.

2. Representations of Densities

Denote $\rho_a(x) = \frac{d\mu_a}{d\mu_0}(x)$. We consider $\{C, \mathfrak{C}_1, \mu_0\}$ as a probability space and denote by E_{μ_0} and $E_{\mu_0}(\cdot \mid \cdot)$, respectively, the expectation and the conditional expectation on this space. For $a \in \mathbb{A}$ we define the function $\bar{a}(t,x)$ by the relation

$$\bar{a}(t, T_a x) = E_{\mu_0}(a(t,x)/\sigma(T_a x_s, s \le t). \tag{7}$$

Here $\sigma(T_a x_s, s \le t)$ is the σ-algebra induced by $\{T_a x_s, s \le t\}$. It is easy to verify that we can choose $\bar{a}$ in such a way that $\bar{a} \in \mathbb{A}$.

Theorem 1: *The following equation holds true:*

$$\rho_a(x) = e(\bar{a}, x). \tag{8}$$

Proof: The stochastic process

$$z(t) = x_t + \int_0^t (a(s,x) - \bar{a}(s, T_a x))ds$$

on the probability space $\{C, \mathfrak{C}_1, \mu_0\}$ is a martingale with respect to the filtration $\{\sigma(T_a x_s, s \le t), t \in [0,1]\}$ because

$$z(t) = (T_a x)_t - \int_0^t \bar{a}(s, T_a x)ds. \tag{9}$$

It is easy to verify that $\langle z, z\rangle_t = t$, so, $z(t)$ is a Wiener process. Girsanov's theorem (see [1]) and relation (9) imply that the process $(T_a x)_t$ is a Wiener process on the probability space $\{C, \mathfrak{C}_1, \bar{\mu}\}$, where

$$\frac{d\bar{\mu}}{d\mu_0} = exp\left\{ - \int_0^1 \bar{a}(s, T_a x)dx_s - \frac{1}{2}\int_0^1 \bar{a}^2(s, T_a x)ds \right\}.$$

Therefore, for bounded $\mathfrak{C}_1$-measurable functions $f(x): C \to R$, we have that

$$\int f(x)\mu_0(dx) = \int f(T_a x)\bar{\mu}(dx)$$

$$= \int f(T_a x) exp\left\{ - \int_0^1 \bar{a}(s, T_a x)dT_a x_s + \frac{1}{2}\int_0^1 \bar{a}^2(s, T_a x)ds \right\}\mu_0(dx)$$

$$= \int f(x) e^{-1}(\bar{a}, x)\rho_a(x)\mu_0(dx).$$

Due to the relation

$$\int f(T_a x)\mu_0(dx) = \int f(x)\mu_a(dx) = \int f(x)\rho_a(x)\mu_0(dx),$$

we have that

$$e^{-1}(\bar{a},x)\rho_a(x) = 1 \qquad (\mu_0\text{-}a.s.). \qquad \square$$

Remark: Denote by μ_a^t the restriction of measure μ_a on the σ-algebra $\mathcal{C}_t$ and by

$$\rho_a^t(x) = \frac{d\mu_a^t}{d\mu_0^t}(x). \tag{10}$$

Let

$$e_t(c,x) = exp\left\{\int_0^t c(s,x)dx(s) - \frac{1}{2}\int_0^t c^2(s,x)ds\right\}. \tag{11}$$

Then

$$\rho_a^t(x) = e_t(\bar{a},x). \tag{12}$$

This relation can be proved in the same way as relation (8).

3. The Conditions of Invertibility of T_a

Theorem 2: *The statements*

(*i*) $a(t,x)$ *is* $\sigma(T_a x_s, s \le t)$*-measurable for* $t \in [0,1]$.

(*ii*) $T_a \in \mathbb{T}_R$, *and*

(*iii*) $\rho_a^t(T_a x) = e_t^{-1}(-a,x), \;\; t \in [0,1]$

are equivalent.

Proof: (*ii*)⇒(*i*), since $a(t,x) = a(t, T_a^{-1}(T_a x))$. (*i*) implies that $-a(t,x) = \tilde{a}\,(t, T_a x)$ and

$$(T_{\tilde{a}}\, T_a x)_t = x_t + \int_0^t a(s,x)ds + \int_0^t \tilde{a}\,(s, T_a x)ds = x_t.$$

Thus, (*ii*) is true. (*ii*) implies that $\bar{a}\,(t,x) = a(t, T_a^{-1}x)$ and (*iii*) is a consequence of formula (12). Suppose (*iii*) is true, then the martingale $e_t(-a,x)$ is $\sigma(T_a x_s, s \le t)$-measurable.

Using the representation

$$e_t(-a,x) = 1 - \int_0^t e_s(-a,x)a(s,x)dx_s$$

we can establish $\sigma(T_a x_s, s \le t)$-measurability of $a(t,x)$. Therefore, (*iii*)⇒(*i*). $\square$

Theorem 3: *Let* $a_n \in n = 1,2,\ldots \mathbb{A}$, $a \in \mathbb{A}$ *and let the following conditions be satisfied:*

1) $a_n \in \mathbb{A}_R$, $n = 1,2,\ldots$;

2) $T_{a_n} x \to T_a x$ *in* $C(\mu_0\text{-}a.s.)$;

3) $\lim_{n\to\infty} \int |\, e_t^{-1}(-a_n,x) - e_t^{-1}(-a,x)\,|\, \mu_0(dx) = 0,\; t \in [0,1]$;

and

4) $\lim_{n\to\infty} \int (\rho_{a_n}(x) - \rho_a(x))^2 \mu_0(dx)$.

Then $a \in \mathbb{A}_R$.

Proof: Condition 2) implies the relation

$$\lim_{n\to\infty} \int \mid \phi(T_{a_n}x) - \phi(T_a x) \mid \mu_0(dx) = 0$$

for all bounded continuous functions $\phi: C \to \mathbb{R}$. Using approximations of $\rho_a(x)$ by bounded continuous functions in $L_1(\mu_0)$ we can prove that

$$\lim_{n\to\infty} \int \mid \rho_a(T_{a_n}x) - \rho_a(T_a x) \mid \mu_0(dx). \tag{13}$$

Since

$$\lim_{n\to\infty} \int \mid \rho_a(T_{a_n}x) - \rho_{a_n}(T_{a_n}x) \mid \mu_0(dx)$$

$$= \lim_{n\to\infty} \int \mid \rho_a(x) - \rho_{a_n}(x) \mid \mu_0(dx) = 0$$

because of condition 4), we have that

$$\lim_{n\to\infty} \int \mid \rho_{a_n}(T_{a_n}x) - \rho_a(T_a x) \mid \mu_0(dx) = 0. \tag{14}$$

Besides, conditions 2) and 3) and theorem 2 imply the relation

$$\lim_{n\to\infty} \int \mid \rho_{a_n}(T_{a_n}x) - e^{-1}(-a, x) \mid \mu_0(dx) = 0. \tag{15}$$

It follows from (14) and (15) that $e^{-1}(-a, x) = \rho_a(T_a x)$. In the same way, we prove that statement (*iii*) of theorem 2 holds true for all $t \in [0,1]$. □

4. Topological Properties of $\mathbb{A}_R$

We introduce the distance in $\mathbb{A}$:

$$d(a_1, a_2) = \int \| a_1(\cdot, x) - a_2(\cdot, x) \|_c \, \mu_0(dx)$$

$$+ \left(\int (e^{-1}(-a_1, x) - e^{-1}(-a_2, x))^2 \mu_0(dx) \right)^{1/2},$$

where $\| x \|_c = \sup_{t \in [0,1]} \mid x_t \mid$.

Theorem 4: *Denote by*

$$Q(a) = \int \rho_a^2(x) \mu_0(dx).$$

Then

$$\mathbb{A}_R = \left\{ a: \lim_{d(\tilde{a}, a) \to 0} Q(\tilde{a}) = Q(a) \right\}.$$

Proof: We have

$$\int (\rho_a(x) - \rho_{\tilde{a}}(x))^2 \mu_0(dx) = Q(a) + Q(\tilde{a}) - 2 \int \rho_a(T_{\tilde{a}} x) \mu_0(dx).$$

Let $d(\widetilde{a},a)\to 0$. Then,

$$\lim_{d(\widetilde{a},a)\to 0}\int \rho_a(T_{\widetilde{a}}x)\mu_0(dx)=\int \rho_a(T_a x)\mu_0(dx)=Q(a).$$

Therefore,

$$\limsup_{d(\widetilde{a},a)\to 0}\int (\rho_a(x)-\rho_{\widetilde{a}}(x))^2\mu_0(dx)=\limsup_{d(\widetilde{a},a)}(Q(\widetilde{a})-Q(a)). \tag{16}$$

Introduce the sequence

$$a_n(s,x)=Ea(s,\tfrac{1}{n}w(\cdot)+f_n(x,\cdot)),\quad x\in C, \tag{17}$$

where $w(t)$ is a Wiener process,

$$f_n(x,s)=n\int_{0\vee s-\frac{1}{n}}^{s} x_u du.$$

It is easy to verify that

$$\lim_{n\to\infty} d(a_n,a)=0 \text{ if } a\in \mathbb{A}. \tag{18}$$

a_n can be rewritten in the form

$$a_n(s,x)=Ea(s,\tfrac{1}{n}w(\cdot))exp\left\{\int_0^1 g_n(x,u)dw(u)-\frac{1}{2}\int_0^1 g_n^2(x,u)du\right\}, \tag{19}$$

where

$$g_n(x,u)=n^2(x(u)-x(0\vee u-\tfrac{1}{n})).$$

(18) implies that there exists a constant ℓ_n for which

$$|\,a_n(s,x)-a_n(s,\widetilde{x})\,|\le \ell_n\,\|\,x-\widetilde{x}\,\|_c,\ x,\widetilde{x}\in C. \tag{20}$$

Therefore, $T_{a_n}\in \mathbb{T}_R$. Let $\lim_{\widetilde{a}\to a} Q(\widetilde{a})-Q(a)$, then (16), (18) and theorem 3 imply that $a\in \mathbb{A}_R$.

Now we consider the space $L_2(\mu_0)$ of functions f for which $\int f^2(x)\mu_0(dx)<\infty$. It is a separable Hilbert space. Let $\{\varphi_k, k=1,2,\ldots\}$ be an orthonormal base in $L_2(\mu_0)$. Then,

$$Q(a)=\sum_k q_k^2(a),$$

where for all k,

$$q_k(a)=\int \rho_a(x)\varphi_k(x)\mu_0(dx)=\int \varphi_k(T_a x)\mu_0(dx)$$

are continuous functions. Therefore,

$$\liminf_{d(\widetilde{a},a)\to 0} Q(\widetilde{a})\ge Q(a).$$

If

$$Q(a)<\lim_{n\to\infty} Q(a_n)=\lim_{n\to\infty}\int \rho_{a_n}(T_{a_n}x)\mu_0(dx)$$

$$=\lim_{n\to\infty}\int e^{-1}(a-a_n,x)\mu_0(dx)=\int e^{-1}(-a,x)\mu_0(dx),$$

then, $\int(\rho_a(T_a x)-e^{-1}(-a,x))\mu_0(dx)<0$, and $a\in\mathbb{A}\backslash\mathbb{A}_R$ because of theorem 2. □

Corollary: *Let* $\lambda(t): R_+\to R_+$ *be a decreasing continuous function for which* $\lim_{t\to+\infty}\lambda(t)=0$. *Denote*

$$\mathbb{A}^\lambda=\left\{a\in\mathbb{A}: \int_0^1 a^2(s,x)ds\le r\lambda(r) \text{ for } x\in U_r\right\}.$$

is of second Baire's category.

This follows from the properties of the set of points of continuity of a half-continuous function (*see for example* [5], *p.* 57).

5. Consequences for Stochastic Differential Equations

We recall that $y(t)$ is a weak solution of equation (6) if the stochastic process

$$z(t)=y(t)-\int_0^t a(s,y(\cdot))ds$$

is a Wiener process. Note that the measure μ_z corresponding to the process z is determined by the measure m_y which corresponds to y. It is natural to call a weak solution of equation (6) a measure μ *for which* $\mu T_{-a}^{-1}=\mu_0$.

Theorem 5: *Let* $S^a=\{\mu: \mu T_{-a}^{-1}=\mu_0\}$. *Then,*

1) S^a *is a convex weakly closed set in* $M(C)$, *where* $M(C)$ *is the set of all probability measures on* $\mathfrak{C}_1$.

2) $a\in\mathbb{A}_R$ *if and only if* $S^a=\{\mu^a\}$, *where* μ^a *is the measure for which*

$$\frac{d\mu^a}{d\mu_0}(x)=e(a,x).$$

Proof: Girsanov's theorem implies that $\mu^a\in S^a$ for all $a\in\mathbb{A}$. Let $-a\in\mathbb{A}\backslash\mathbb{A}_R$. Then for a bounded $\mathfrak{C}_1$-measurable function $f: C\to\mathbb{R}$ we have that

$$\int f(x)\mu_0(dx=\int e(a,x)f(T_{-a}x)\mu_0(dx)$$

$$=\int E_{\mu_0}(e(a,x)/\sigma(T_{-a}x_s,s\le 1))\cdot f(T_{-a}x)\mu(dx).$$

Therefore, the measure $\widehat{\mu}$, which is determined by the relation

$$\frac{d\widehat{\mu}}{d\mu_0}(x)=E_{\mu_0}(e(a,x)/\sigma(T_{-a}x_s,s\le 1)),$$

belongs to S^a since

$$\int f(x)\mu_0(dx)=\int f(T_{-a}x)\widehat{\mu}(dx).$$

It may be shown that the equality $\widehat{\mu}=\mu^a$ implies the measurability of $e_t(a,x)$ with respect to the σ-algebra $\sigma((T_{-a}x)_s,s\le t)$ and invertibility of T_{-a}. Thus, $\widehat{\mu}$ and μ^a are two distinct points of S^a. □

$y(t)$ is a strong solution of equation (6) if $y(t)$ is $\sigma(w(s), s \leq t)$-measurable for all $t \in [0,1]$.

Theorem 6: 1) *Let $y(t)$ be a strong solution of equation (6). Then, $T_{-a} \in \mathbb{T}_R$, $y(t) = (T_{-a}^{-1} w)_t$ and $y(t)$ is the unique solution of equation (6).*

2) *Equation (6) has no strong solution if $a \in \mathbb{A} \backslash \mathbb{A}_R$.*

Proof: 1) $y(t)$ may be represented in the form: $y(t) = Y(t, w(\cdot))$, where $Y(t,x)$ is $\mathfrak{B}_{[0,1]} \otimes \mathfrak{C}_1$-measurable function and, for a fixed $t \in [0,1]$, it is $\mathfrak{C}_t$-measurable. Therefore,

$$y(t) = w(t) + \int_0^t a(s, Y(\cdot, w(\cdot))ds. \tag{21}$$

Set

$$b(s,x) = a(s, Y(\cdot, x)).$$

It follows from (21) and (6) that

$$y(t) = (T_b w)_t \text{ and } (T_{-a} T_b w)_t = w(t).$$

Hence,

$$T_{-a} \in \mathbb{T}_R, \ T_b = T_{-a}^{-1}, \text{ and } y(t) = (T_{-a}^{-1} w)_t.$$

This is true for any solution of (6). Therefore, $y(t)$ is unique.

2) follows from 1). □

Corollary: *Equation (6) has a strong solution and then it is unique if and only if this equation has a unique weak solution.*

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