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INITIAL VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE IN BANACH SPACES[1]

DAJUN GUO[2]

Shandong University
Department of Mathematics
Jinan 250100, PEOPLE'S REPUBLIC OF CHINA

ABSTRACT

This paper investigates the extremal solutions of initial value problems for first order integro-differential equations of Volterra type in Banach spaces by means of establishing a comparison result.

Key words: Integro-differential equation in Banach space, Kuratowski measure of noncompactness, upper and lower solutions, monotone iterative technique.

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1. INTRODUCTION

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ iff $y - x \in P$. P is said to be normal if there exists a positive constant c such that $\theta \leq x \leq y$ implies $\| x \| \leq c \| y \|$, where θ denotes the zero element of E, and P is said to be regular if every nondecreasing and bounded in order sequence in E has a limit, i.e. $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ implies $\| x_n - x \| \to 0$ as $n \to \infty$ for some $x \in E$. The regularity of P implies the normality of P. For details on cone theory, see [1]. This paper investigates the initial value problem (IVP) for integro-differential equation of Volterra type in E:

$$u' = f(t, u, Tu), \quad t \in J; \quad u(0) = u_0, \tag{1}$$

where $J = [0, a]$ $(a > 0)$, $u_0 \in E$, $f \in C(J \times E \times E, E)$,

$$(Tu)(t) = \int_0^t k(t,s)u(s)ds, \; t \in J, \; u \in C(J, E),$$

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$k \in C(D, R_+)$, $D = \{(t,s) \in J \times J: t \geq s\}$ and R_+ denotes the set of non-negative real numbers. After establishing a comparison result which is based on some idea in [2] for PBVP's in the scalar case, we obtain the existence of minimal and maximal solutions for IVP (1) by means of lower and upper solutions and the monotone iterative technique. In the special case where f does not contain Tu, our result becomes the main result in [3] with weaker conditions. Finally, an example of infinite system for scalar integro-differential equations of Volterra type is given.

2. COMPARISON RESULT

We first prove a fundamental comparison lemma:

Lemma 1: *Assume that $p \in C^1(J, E)$ satisfies*

$$p' \leq -Mp - NTp, \ \ t \in J; \ \ p(0) \leq \theta, \tag{2}$$

where M and N are non-negative constants. Then $p(t) \leq \theta$ for $t \in J$ provided that $Nk_0a(e^{Ma} - 1) \leq M$ in case of $M > 0$ and $Nk_0a^2 \leq 1$ in case of $M = 0$, where $k_0 = max\{k(t,s): (t,s) \in D\}$.

Proof: Let $P^* = \{g \in E^*: g(x) \geq 0$ for all $x \in P\}$. For any $g \in P^*$, let $m(t) = g(p(t))$. Then $m \in C^1(J, R)$ (R denotes the set of real numbers) and $m'(t) = g(p'(t))$, $g((Tp)(t)) = (Tm)(t)$. By (2), we have

$$m' \leq -Mm - NTm, \ \ t \in J; \ \ m(0) \leq 0. \tag{3}$$

Let $v(t) = m(t)e^{Mt}$, then (3) reduces to

$$v'(t) \leq -N \int_0^t k^*(t,s)v(s)ds, \quad t \in J; \ \ v(0) \leq 0, \tag{4}$$

where $k^*(t,s) = k(t,s)e^{M(t-s)}$. We now show that

$$v(t) \leq 0, \ \ t \in J. \tag{5}$$

Assume that (5) is not true, i.e. there exists an $0 < t_0 \leq a$ such that $v(t_0) > 0$. Let $min\{v(t): 0 \leq t \leq t_0\} = -b$. Then $b \geq 0$. If $b = 0$, then $v(t) \geq 0$ for $0 \leq t \leq t_0$, so (4) implies that $v'(t) \leq 0$ for $0 \leq t \leq t_0$. Consequently,

$v(t_0) \leq v(0) \leq 0$, which contradicts $v(t_0) > 0$. If $b > 0$, then there exists an $0 \leq t_1 < t_0$ such that $v(t_1) = -b < 0$, and so, there is a t_2 with $t_1 < t_2 < t_0$ such that $v(t_2) = 0$. By the mean value theorem, there exists a t_3 satisfying $t_1 < t_3 < t_2$ and

$$v'(t_3) = \frac{v(t_2) - v(t_1)}{t_2 - t_1} > \frac{b}{a}. \tag{6}$$

On the other hand, (4) implies that

$$v'(t_3) \leq -N \int_0^{t_3} k^*(t_3, s)v(s)ds \leq Nb \int_0^{t_3} k^*(t_3, s)ds \leq Nbk_0 \int_0^{t_3} e^{M(t_3 - s)}ds$$

$$= \begin{cases} M^{-1}Nbk_0(e^{Mt_3} - 1) \leq M^{-1}Nbk_0(e^{Ma} - 1), & \text{if } M > 0; \\ Nbk_0 t_3 \leq Nbk_0 a, & \text{if } M = 0. \end{cases} \tag{7}$$

It follows from (6) and (7) that $M < Nk_0 a(e^{Ma} - 1)$ if $M > 0$ and $1 < Nk_0 a^2$ if $M = 0$. This contradicts the hypotheses. Hence (5) holds, and therefore, $m(t) \leq 0$ for $t \in J$. Since $g \in P^*$ is arbitrary, we get $p(t) \leq \theta$ for $t \in J$, and the lemma is proved. □

We need also the following known lemma (see [4], Corollary 3.1 (b)):

Lemma 2: *Let H be a countable set of strongly measurable functions $x: J \rightarrow E$ such that there exists a $z \in L(J, R_+)$ such that $\| x(t) \| \leq z(t)$ for almost all $t \in J$ and all $x \in H$. Then $\alpha(H(t)) \in L(J, R_+)$ and*

$$\alpha(\{\int_J x(t)dt: x \in H\}) \leq 2 \int_J \alpha(H(t))dt, \tag{8}$$

where $H(t) = \{x(t): x \in H\}$ $(t \in J)$ and α denotes the Kuratowski measure of noncompactness in E.

Corollary: *If $H \subset C(J, E)$ is countable and bounded, then $\alpha(H(t)) \in L(J, R_+)$ and (8) holds.*

3. MAIN THEOREMS

Let us list some conditions for convenience.

(H_1) There exist $v_0, w_0 \in C^1(J, E)$ satisfying $v_0(t) \leq w_0(t)$ for $t \in J$ and

$$v_0' \leq f(t, v_0, Tv_0), \quad t \in J; \quad v_0(0) \leq u_0,$$

$$w_0' \geq f(t, w_0, Tw_0), \quad t \in J; \quad w_0(0) \geq u_0.$$

(H_2) There exist nonnegative constants M and N such that

$$f(t,u,v) - f(t,\bar{u},\bar{v}) \geq -M(u-\bar{u}) - N(v-\bar{v}),$$

$$\text{for } t \in J,\ v_0(t) \leq \bar{u} \leq u \leq w_0(t),\ (Tv_0)(t) \leq \bar{v} \leq (Tw_0)(t),$$

and $N_{k_0} a(e^{Ma} - 1) \leq M$ in the case of $M > 0$ and $Nk_0 a^2 < 1$ in the case of $M = 0$.

(H_3) For any $r > 0$, there exist constants $c_r \geq 0$ and $c_r^* \geq 0$ such that

$$\alpha(f(t,B,B^*)) \leq c_r \alpha(B) + c_r^* \alpha(B^*), \quad t \in J, \quad B \subset B_r,\ B^* \subset B_r,$$

where $B_r = \{x \in E\colon \| x \| \leq r\}$.

In the following, we define the conical segment $[v_0, w_0] = \{u \in C(J,E)\colon v_0(t) \leq u(t) \leq w_0(t) \text{ for } t \in J\}$.

Theorem 1: *Let cone P be normal. Assume that conditions (H_1), (H_2) and (H_3) are satisfied. Then there exist monotone sequences $\{v_n\}$, $\{w_n\} \subset C^1(J,E)$ which converge uniformly and monotonically on J to the minimal and maximal solutions $\bar{u}, u^* \in C^1(J,E)$ of IVP (1) in $[v_0, w_0]$ respectively. That is, if $u \in C^1(J,E)$ is any solution of IVP (1) satisfying $v_0(t) \leq u(t) \leq w_0(t)$ for $t \in J$, then*

$$\begin{aligned} v_0(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq \bar{u}(t) \leq u(t) \leq u^*(t) \leq \ldots \\ \leq w_n(t) \leq \ldots \leq w_1(t) \leq w_0(t), \quad t \in J. \end{aligned} \tag{9}$$

Proof: For any $h \in [v_0, w_0]$, consider the IVP of a linear integro-differential equation in E:

$$u' + Mu = -NTu + g(t),\ t \in J;\ u(0) = u_0, \tag{10}$$

where $g(t) = f(t, h(t), (Th)(t)) + Mh(t) + N(Th)(t)$. It is easy to see that $u \in C^1(J,E)$ is a solution of IVP (10) if and and only if $u \in C(J,E)$ is a solution of the following integral equation

$$u(t) = e^{-Mt}\{u_0 + \int_0^t [g(s) - N(Tu)(s)]e^{Ms} ds\}, \quad t \in J. \tag{11}$$

Consider operator $F\colon C(J,E) \rightarrow C(J,E)$ defined by

$$(Fu)(t) = e^{Mt}\{u_0 + \int_0^t [g(s) - N(Tu)(s)]e^{Ms}ds\}.$$

It is easy to get $\| Fu - Fv \|_c \leq Nk_0 a^2 \| u - v \|_c$ for $u, v \in C(J, E)$, where $\| \cdot \|_c$ denotes the norm in $C(J, E)$. It is easy to see that $M > 0$ and $Nk_0 a(e^{Ma} - 1) \leq M$ imply $Nk_0 a^2 < 1$, and so, by (H_2), we conclude that $Nk_0 a^2 < 1$ in any case. Hence, the Banach fixed point theorem implies that F has a unique fixed point u in $C(J, E)$, and this u is the unique solution of IVP (10) in $C^1(J, E)$. Let $u = Ah$. Then operator A maps $[v_0, w_0]$ into $C(J, E)$, and we shall show that (a) $v_0 \leq Av_0$, $Aw_0 \leq w_0$ and (b) A is nondecreasing in $[v_0, w_0]$. To prove (a), we set $v_1 = Av_0$ and $p = v_0 - v_1$. By (10), we have

$$v_1' + Mv_1 = -NTv_1 + f(t, v_0, Tv_0) + Mv_0 + NTv_0, \quad v_1(0) = u_0,$$

and so, from (H_1) and (H_2) we get $p' \leq -Mp - NTp$, $p(0) \leq \theta$, which implies by virtue of Lemma 1 that $p(t) \leq \theta$ for $t \in J$, i.e. $v_0 \leq Av_0$. Similarly, we can show that $Aw_0 \leq w_0$. To prove (b), let $h_1, h_2 \in [v_0, w_0]$ such that $h_1 \leq h_2$ and let $p = u_1 - u_2$, where $u_1 = Ah_1$ and $u_2 = Ah_2$. It is easy to see from (10) and (H_2) that $p' \leq -Mp - NTp$, $p(0) = \theta$, and so, Lemma 1 implies that $p(t) \leq \theta$ for $t \in J$, i.e. $Ah_1 \leq Ah_2$, and (b) is proved.

Let $v_n = Av_{n-1}$ and $w_n = Aw_{n-1}$ $(n = 1, 2, 3, \ldots)$. By (a) and (b) just proved, we have

$$v_0(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq w_n(t) \leq \ldots \leq w_1(t) \leq w_0(t), \quad t \in J, \tag{12}$$

and consequently, the normality of P implies that $V = \{v_n : n = 0, 1, 2, \ldots\}$ is a bounded set in $C(J, E)$. Since (H_3) implies that $f(t, B_r, B_r)$ is bounded for any $r > 0$, we see that there is a positive constant c_0 such that

$$\| f(t, v_{n-1}(t), (Tv_{n-1})(t)) + Mv_{n-1}(t) - NT(v_n - v_{n-1}))(t) \| \leq c_0,$$
$$t \in J \quad (n = 1, 2, 3, \ldots). \tag{13}$$

From the definition of v_n and (11), we have

$$v_n(t) =$$
$$e^{-Mt}(u_0 + \int_0^t [f(s, v_{n-1}(s), (Tv_{n-1})(s)) + Mv_{n-1}(s) - N(T(v_n - v_{n-1}))(s)]e^{Ms}ds)$$
$$t \in J \quad (n = 1, 2, 3, \ldots). \tag{14}$$

It follows from (13) and (14) that V is equicontinuous on J, so the function $m(t) = \alpha(V(t))$ is continuous on J, where $V(t) = \{v_n(t): n = 0, 1, 2, \ldots\} \subset E$. By applying the Corollary of Lemma 2 to (14), we get

$$m(t) \leq 2\int_0^t \alpha(\{e^{-M(t-s)}[f(s, v_{n-1}(s), (Tv_{n-1})(s)) + Mv_{n-1}(s)$$

$$- N(T(v_n - v_{n-1}))(s)]: \ n = 1, 2, 3, \ldots\})ds$$

$$\leq 2\int_0^t [\alpha(f(s, V(s), (TV)(s))) + M\alpha(V(s)) + 2N\alpha((TV)(s))]ds, \tag{15}$$

where

$$(TV)(t) = \{\int_0^t k(t,s)v_n(s)ds: \ n = 0, 1, 2, \ldots\}.$$

The Corollary of Lemma 2 also implies that

$$\alpha((TV)(t)) \leq 2\int_0^t \alpha(\{k(t,s)v_n(s): n = 0, 1, 2, \ldots\})ds$$

$$\leq 2k_0 \int_0^t m(s)ds, \quad t \in J. \tag{16}$$

On the other hand, (H_3) implies that there exist constants $c \geq 0$ and $c^* \geq 0$ such that

$$\alpha(f(t, V(t), (TV)(t))) \leq c\alpha(V(t)) + c^*\alpha((TV)(t))$$

$$\leq cm(t) + 2k_0c^* \int_0^t m(s)ds, \quad t \in J. \tag{17}$$

It follows from (15) and (17) that

$$m(t) \leq 2(c+M)\int_0^t m(s)ds + 4k_0(c^* + 2N)\int_0^t ds \int_0^s m(s')ds'$$

$$= 2(c+M)\int_0^t m(s)ds + 4k_0(c^*+2N)\int_0^t (t-s)m(s)ds \leq \bar{c}\int_0^t m(s)ds, \ \ t \in J, \tag{18}$$

where $\bar{c} = 2(c+M) + 4k_0a(c^* + 2N) = const.$ Let

$$y(t) = \int_0^t m(s)ds \quad \text{and} \quad z(t) = y(t)e^{-\bar{c}\,t}, \quad t \in J.$$

Then $y'(t) = m(t)$ and (18) implies that $z'(t) = (y'(t) - \bar{c}\,y(t))e^{-\bar{c}\,t} \leq 0$ for $t \in J$. Hence $z(t) \leq z(0) = y(0) = 0$ for $t \in J$, and consequently, $m(t) = 0$ for $t \in J$. Thus, by the Ascoli-Arzela theorem (see [5] Theorem 1.1.5), V is relatively compact in $C(J,E)$, so there exists a subsequence of $\{v_n\}$ which converges uniformly on J to some $\bar{u} \in C(J,E)$. Since, by (12), $\{v_n\}$ is nondecreasing and P is normal, we see that $\{v_n\}$ itself converges uniformly on J to $\bar{u}$. Now we have

$$\begin{aligned} &f(t, v_{n-1}(t), (Tv_{n-1})(t)) + Mv_{n-1}(t) - N(T(v_n - v_{n-1}))(t) \\ &\quad \to f(t, \bar{u}(t), (T\bar{u})(t)) + M\bar{u}(t) \text{ as } n \to \infty,\ t \in J, \end{aligned} \tag{19}$$

and, by (13),

$$\begin{aligned} &\| f(t, v_{n-1}(t), (Tv_{n-1})(t)) + Mv_{n-1}(t) - N(T(v_n - v_{n-1}))(t) \\ &- f(t, \bar{u}(t), (T\bar{u})(t)) - M\bar{u}(t) \| \leq 2c_0,\ t \in J \quad (n = 1, 2, 3, \ldots). \end{aligned} \tag{20}$$

Observing (19) and (20) and taking the limit as $n \to \infty$ in (14), we get

$$\bar{u}(t) = e^{-Mt}(u_0 + \int_0^t [f(s, \bar{u}(s), (T\bar{u})(s)) + M\bar{u}(s)]e^{Ms}ds), \quad t \in J,$$

which implies that $\bar{u} \in C^1(J,E)$ and $\bar{u}$ is a solution of IVP (1). In the same way, we can show that $\{w_n\}$ converges uniformly on J to some u^* and u^* is a solution of IVP (1) in $C^1(J,E)$.

Finally, let $u \in C^1(J,E)$ be any solution of IVP (1) satisfying $v_0(t) \leq u(t) \leq w_0(t)$ for $t \in J$. Assume that $v_{k-1}(t) \leq u(t) \leq w_{k-1}(t)$ for $t \in J$, and set $p = v_k - u$. Then we have, by (10) and (H_2),

$$\begin{aligned} p' = v_k' - u' &= -Mp - NTp - M(u - v_{k-1}) - NT(u - v_{k-1}) \\ &- (f(t, u, Tu) - f(t, v_{k-1}, Tv_{k-1})) \leq -Mp - NTp, \quad p(0) = \theta, \end{aligned}$$

which implies by virtue of Lemma 1 that $p(t) \leq \theta$ for $t \in J$, i.e. $v_k(t) \leq u(t)$ for $t \in J$. Similarly, one can show that $u(t) \leq w_k(t)$ for $t \in J$. Consequently, by induction, we have $v_n(t) \leq u(t) \leq w_n(t)$ for $t \in J$ $(n = 0, 1, 2, \ldots)$, and by taking limits, we get $\bar{u}(t) \leq u(t) \leq u^*(t)$ for $t \in J$. Hence, (9) holds and the theorem is proved. □

Remark 1: In the special case where f does not contain Tu, by setting $N = c_r^* = 0$ in conditions (H_2) and (H_3), Theorem 1 becomes Theorem 3.1 in [3] and, in this case, condition (H_3) becomes "for any $r > 0$, there exists a non-negative constant c_r such that $\alpha(f(t,B)) \leq c_r\alpha(B)$ for $t \in J$ and $B \subset B_r$", which is weaker than condition $(A1)$ of Theorem 3.1 in [3] (condition $(A1)$ is "there exists a constant $L > 0$ such that $\alpha(f(J \times B)) \leq L\alpha(B)$ for any bounded $B \subseteq E$").

Theorem 2: *Let cone P be regular. Assume that conditions (H_1) and (H_2) are satisfied. Then the conclusions of Theorem 1 hold.*

Proof: The proof is almost the same as that of Theorem 1. The only difference is that, instead of using condition(H_3), the conclusion $m(t) = \alpha(V(t)) = 0 \quad (t \in J)$ is implied directly by (12) and the regularity of P. □

Remark 2: The condition that P is regular will be satisfied if E is weakly complete (reflexive, in particular) and P is normal (see [1] Theorem 1.2.1 and Theorem 1.2.2, and [6] Theorem 2.2).

4. AN EXAMPLE

Consider the IVP of an infinite system for scalar integro-differential equations of Volterra type:

$$u_n' = \frac{1}{4n}(t - u_n)^3 + tu_{n+1}^3$$

$$+ \frac{1}{6n}[(t^2 - \int_0^t e^{-ts}u_n(s)ds)^2 + (\int_0^t e^{-tS}u_{2n}(s)ds)^2] \quad 0 \leq t \leq 1;$$

$$u_n(0) = 0, \quad (n = 1,2,3,\ldots). \tag{21}$$

Evidently, $u_n(t) \equiv 0$ $(n = 1,2,3,\ldots)$ is not a solution of IVP (21).

Conclusion: IVP (21) *has minimal and maximal continuously differentiable solutions satisfying* $0 \leq u_n(t) \leq \frac{t}{n}$ *for* $0 \leq t \leq 1$ $(n = 1,2,3,\ldots)$.

Proof: Let $J = [0,1]$ $(a = 1)$, $E = c_0 = \{u = (u_1,\ldots,u_n,\ldots): u_n \to 0\}$ with norm $\| u \| = \sup_n | u_n |$ and $P = \{u = (u_1,\ldots,u_n,\ldots) \in c_0\colon\ u_n \geq 0,\ n = 1,2,3,\ldots\}$. Then P is a normal cone in E and IVP (21) can be regarded as an IVP of form (1) in E. In this situation, $u_0 = (0,\ldots,0,\ldots)$, $k(t,s) = e^{-ts}$, $u = (u_1,\ldots,u_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$ and $f = (f_1,\ldots,f_n,\ldots)$, in which

$$f_n(t,u,v) = \frac{1}{4n}(t-u_n)^3 + tu_{n+1}^3 + \frac{1}{6n}[(t^2-v_n)^2 + v_{2n}^2]. \tag{22}$$

It is clear that $f \in C(J \times E \times E, E)$. Let $v_0(t) = (0,\ldots,0,\ldots)$ and $w_0(t) = (t,\ldots,\frac{t}{n},\ldots)$. Then $v_0, w_0 \in C^1(J,E)$, $v_0(t) \le w_0(t)$ for $t \in J$, and we have

$$v_0(0) = w_0(0) = (0,\ldots,0,\ldots) = u_0,$$

$$v_0'(t) = (0,\ldots,0,\ldots) \text{ and } w_0'(t) = (1,\ldots,\tfrac{1}{n},\ldots) \text{ for } t \in J, \ ,$$

$$f_n(t,v_0(t),(Tv_0)(t)) = \frac{1}{4n}t^3 + \frac{1}{6n}t^4 \ge 0, \ \ t \in J \ \ (n = 1,2,3,\ldots),$$

$$f_n(t,w_0(t),(Tw_0)(t)) = \frac{1}{4n}(t-\tfrac{t}{n})^3 + t(\frac{t}{n+1})^3$$

$$+\frac{1}{6n}[(t^2 - \tfrac{1}{n}\int_0^t se^{-ts}ds)^2 + (\frac{1}{2n}\int_0^t se^{-ts}ds)^2]$$

$$\le \frac{1}{4n} + \frac{1}{(n+1)^3} + \frac{1}{6n}[(1+\tfrac{1}{n}\int_0^1 sds)^2 + (\frac{1}{2n}\int_0^1 sds)^2]$$

$$\le \frac{1}{4n} + \frac{1}{4n} + \frac{1}{6n}[(1+\tfrac{1}{2})^2 + (\tfrac{1}{4})^2] < \tfrac{1}{n}, \ \ t \in J \ \ (n = 1,2,3,\ldots).$$

Consequently, v_0 and w_0 satisfy condition (H_1). On the other hand, for $u = (u_1,\ldots,u_n,\ldots)$, $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$ and $\bar{v} = (\bar{v}_1,\ldots,\bar{v}_n,\ldots)$ satisfying $t \in J$, $v_0(t) \le \bar{u} \le u \le w_0(t)$ and $(Tv_0)(t) \le \bar{v} \le v \le (Tw_0)(t)$, i.e. $t \in J$ and

$$0 \le \bar{u}_n \le u_n \le \tfrac{t}{n}, \ 0 \le \bar{v}_n \le v_n \le \frac{1}{n}\int_0^t se^{-ts}ds \le \frac{t^2}{2n} \ \ (n = 1,2,3,\ldots),$$

we have, by (22),

$$f_n(t,u,v) - f_n(t,\bar{u},\bar{v}) \ge \frac{1}{4n}[(t-u_n)^3 - (t-\bar{u}_n)^3] + \frac{1}{6n}[(t^2-v_n)^2 - (t^2-\bar{v}_n)^2]$$

$$\ge -\frac{3}{4n}(u_n - \bar{u}_n) - \frac{1}{3n}(v_n - \bar{v}_n) \ge -\tfrac{3}{4}(u_n - \bar{u}_n) - \tfrac{1}{3}(v_n - \bar{v}_n), \ t \in J \ \ (n = 1,2,3,\ldots)$$

(since $\frac{\partial}{\partial s}(t-s)^3 = -3(t-s)^2 \ge -3$ for $0 \le s \le t$, $0 \le t \le 1$ and

$$\frac{\partial}{\partial s}(t^2-s)^2 = -2(t^2-s) \text{ for } 0 \le s \le t^2, \ 0 \le t \le 1.$$

Consequently, condition (H_2) is satisfied for $M = \frac{3}{4}$ and $N = \frac{1}{3}$ because

$$Nk_0a(e^{Ma} - 1) = \tfrac{1}{3}(e^{\frac{3}{4}} - 1) < \tfrac{3}{4} = M.$$

From (22), we see that $f = f^{(1)} + f^{(2)}$, where $f^{(1)} = (f_1^{(1)},\ldots,f_n^{(1)},\ldots)$ and $f^{(2)} = (f_1^{(2)},\ldots,f_n^{(2)},\ldots)$ with

$$f_n^{(1)}(t,u,v) = \frac{1}{4n}(t-u_n)^3 + \frac{1}{6n}[(t^2-v_n)^2 + v_{2n}^2] \tag{23}$$

and

$$f_n^{(2)}(t,u,v) = tu_{n+1}^3. \tag{24}$$

Let $r > 0$ be arbitrarily given and $t \in J$ be fixed, and let $\{u^{(m)}\}$, $\{v^{(m)}\} \subset B_r$, where $u^{(m)} = (u_1^{(m)},\ldots,u_n^{(m)},\ldots)$ and $v^{(m)} = (v_1^{(m)},\ldots,v_n^{(m)},\ldots)$. By virtue of (23), we have

$$\begin{aligned} |\, f_n^{(1)}(t,u^{(m)},v^{(m)})\,| &\leq \frac{1}{4n}(1+ \|\, u^{(m)} \,\|)^3 + \frac{1}{6n}[(1+ \|\, v^{(m)} \,\|)^2 + (\|\, v^{(m)} \,\|)^2] \\ &\leq \frac{1}{4n}(1+r)^3 + \frac{1}{6n}(1+2r+2r^2), \quad (n,m = 1,2,3,\ldots). \end{aligned} \tag{25}$$

Therefore, $\{f_m^{(1)}(t,u^{(m)},v^{(m)})\}$ is bounded, and so, by the diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$f_n^{(1)}(t,u^{(m_i)},v^{(m_i)}) \rightarrow w_n \quad (n = 1,2,3,\ldots). \tag{26}$$

From (25), we have

$$|\, w_n \,| \leq \frac{1}{4n}(1+r)^3 + \frac{1}{6n}(1+2r+2r^2), \quad (n = 1,2,3,\ldots), \tag{27}$$

and so $w = (w_1,\ldots,w_n,\ldots) \in c_0 = E$. For any $\epsilon > 0$, (25) and (27) imply that there exists a positive integer n_0 such that

$$|\, f_n^{(1)}(t,u^{(m_i)},v^{(m_i)})\,| < \epsilon, \quad |\, w_n \,| < \epsilon,\ n > n_0 \quad (i = 1,2,3,\ldots). \tag{28}$$

By (26) we know that there is a positive integer i_0 such that

$$|\, f_n^{(1)}(t,u^{(m_i)},v^{(m_i)}) - w_n \,| < \epsilon, \quad i > i_0 \quad (n = 1,2,\ldots,n_0). \tag{29}$$

It follows from (28) and (29) that

$$\|\, f^{(1)}(t,u^{(m_i)},v^{(m_i)}) - w \,\| = \sup_n |\, f_n^{(1)}(t,u^{(m_i)},v^{(m_i)}) - w_n \,| \leq 2\epsilon, \quad i > i_0.$$

Hence $\|\, f^{(1)}(t,u^{(m_i)},v^{(m_i)}) - w \,\| \rightarrow 0$ as $i \rightarrow \infty$, and we have proved that

$$\alpha(f^{(1)}(t,B,B^*)) = 0, \quad t \in J,\ B \subset B_r,\ B^* \subset B_r. \tag{30}$$

On the other hand, (24) implies that, for any $t \in J$ and $u, v, \bar{u}, \bar{v} \in B_r$,

$$\begin{aligned} |\, f_n^{(2)}(t,u,v) - f_n^{(2)}(t,\bar{u},\bar{v})\,| &= |\, t(u_{n+1}^3 - \bar{u}_{n+1}^3\,| \\ &= |\, t(u_{n+1} - \bar{u}_{n+1})(u_{n+1}^2 + u_{n+1}\bar{u}_{n+1} + \bar{u}_{n+1}^2)\,| \\ &\leq 3r^2 \,|\, u_{n+1} - \bar{u}_{n+1} \,| \leq 3r^2 \,\|\, u - \bar{u} \,\|, \quad (n = 1,2,3,\ldots), \end{aligned}$$

and so

which implies that
$$\| f^{(2)}(t,u,v) - f^{(2)}(t,\bar{u},\bar{v}) \| \leq 3r^2 \| u - \bar{u} \|,$$

$$\alpha(f^{(2)}(t,B,B^*)) \leq 3r^2\alpha(B), \quad t \in J, \quad B \subset B_r, \quad B^* \subset B_r. \tag{31}$$

It follows from (30) and (31) that

$$\alpha(f(t,B,B^*)) \leq 3r^2\alpha(B), \quad t \in J, \quad B \subset B_r, \quad B^* \subset B_r,$$

i.e. condition (H_3) is satisfied for $c_r = 3r^2$ and $c_r^* = 0$. Finally, our conclusion follows from Theorem 1. □

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