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ON REVELATION TRANSFORMS THAT CHARACTERIZE PROBABILITY DISTRIBUTIONS[1]

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ABSTRACT

A characterization of exponential, geometric and of distributions with almost-lack-of-memory property, based on the "revelation transform of probability distributions" and "relevation of random variables" is discussed. Known characterizations of the exponential distribution on the base of relevation transforms given by Grosswald et al. [4], and Lau and Rao [7] are obtained under weakened conditions and the proofs are simplified. A characterization the class of almost-lack-of-memory distributions through the relevation is specified.

Key words: Relevation, characterization, convolution, geometric distribution, exponential distribution, almost-lack-of-memory distributions, failure rate function.

AMS (MOS) subject classifications: Primary 60E05, 60E10. Secondary 60G55, 60K10.

1. INTRODUCTION

The concept of relevation was originally introduced by Krakowski [6]. We present here the distribution of relevation for completeness and a better understanding of the properties discussed later on.

Definition 1: Let Y and Z be two independent non-negative random variables. The relevation of Y and Z is a new random variable X defined by the survival function

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$$\bar{F}_X(t) = P\{X \geq t\} = P\{Y \geq t\} + \int_0^{t-0} P\{Z \geq t \mid Z \geq X\} dP\{Y \geq X\}, t > 0. \quad (1)$$

We will denote the revelation of Y and Z by the sign "#", and express it by the equation

$$X = Y \# Z. \quad (2)$$

Interpretations of revelation were given by Krakowski [6], Grosswald et al. [4], Baxter [1], Lau and Rao [7]. Moreover, Grosswald et al. [4] and Lau and Rao [7] obtained characterizations of the exponential distribution under various special conditions imposed to the distributions of Y and Z. Applications of the multiple relevation transforms

$$X = Y_1 \# \ldots \# Y_n, \quad n = 2, 3, \ldots,$$

where $\{Y_n\}$ is a sequence of i.i.d. random variables, were considered from reliability point of view by Baxter [1]. Thus we omit the detailed descriptions of those aspects.

We state the principal results, concerning the characterization of the exponential distribution as follows:

Theorem (general): *Under specific requirements for the distributions of Y and Z the equation in distribution*

$$Y \# Z \stackrel{d}{=} Y + Z \quad (3)$$

takes place if and only if Z has exponential distribution.

Grosswald et al. [4] assumed in their characterization theorem that $F_Z(t) = F_Y(t)$ and also $F_Y(t)$ was required to be expressed in the form of power series of T. Lau and Rao [7] required $F_Z(t)$ to have continuous derivative, and the r.v. Y to have values arbitrarily close to zero.

We show first that when $P\{Y = c\} = 1$, $c > 1$, equation (3) holds true iff Z has a distribution of the form called by Chukova and Dimitrov [2] the almost-lack-of-memory (ALM) distribution. We also show that (3) is true for two coprime (incommensurable) constants $Y = c_1$ and $Y = c_2$; $c_1 > 0$ and $c_2 > 0$ iff Z has either an exponential or geometric distribution. A characterization of a geometric distribution for discrete-valued Y and Z is obtained under the condition $P\{Y=1\}>0$. Another characterization of the geometric distribution is obtained in the case Y and Z have identical distributions.

2. CHARACTERIZATION OF ALM DISTRIBUTIONS

The ALM distributions were introduced by Chukova and Dimitrov [2]. They considered a single server queueing systems with non-reliable server. They considered the equation in the distribution between the blocking time of the server (with constant life times, instantaneous restarts and repeated different services after an interruption), and the required service time, if it has the ALM distribution. Chukova et al. [3] gave several other equivalent properties for the distributions of this class. Here we recall its definition.

Definition 2: A random variable Z is said to possess the almost-lack-of-memory property iff there exists some $c>0$ such that for any $n=0,1,2,\ldots$ and all $t>0$, the relation

$$P\{Z \geq nc+t \mid Z \geq nc\} = P\{Z \geq t\} \tag{4}$$

holds true. We will write in that case $Z \in ALM(c)$.

The next results is taken from Chukova et al. [3]. It gives a short equivalent property of an ALM r.v. and reveals the explicit form of its distribution. We will be using the abbreviations "c.d.f." for the "cumulative distribution functions" $F_X(x)=P\{X<x\}$ and the "p.d.f." for the corresponding "probability density function".

Theorem 1: *$Z \in ALM(c)$ iff either*

(*i*) *equation* (4) *holds true for $n=1$, some $c>0$ and all $t \geq 0$;*

or

(*ii*) *the c.d.f. $F_Z(t)$ has the form*

$$F_Z(t) = 1-\alpha^{[t/c]}\{1-(1-\alpha)G(t-[t/c]C)\}, \tag{5}$$

where $\alpha \in (0,1)$ is a certain parameter and $G(u)$ is a certain c.d.f. with its support in the interval $[0,c)$, for some $c>0$.

Now we formulate the following characterization property of the ALM distributions through the relevation transform.

Theorem 2: *Assume* $P\{Z=0\}<1$. *Then, for some positive constant* $c>0$, *the equation in distribution*

$$c\# Z \overset{d}{=} c+Z \tag{6}$$

takes place iff $Z \in ALM(c)$.

Proof: Substituting the c.d.f. $F_Y(t)=U(t-c)$ for $t \geq 0$ into (1) (where $U(t)$ is defined as $U(t)=0$ for $t<0$ and $U(t)=1$ for $t \geq 0$), one gets

$$\bar{F}_X(t)=1-U(t-c)+\frac{1-F_Z(t)}{1-F_Z(c)}U(t-c). \tag{7}$$

We notice here the from $P\{Z+c=c\}=P\{Z=0\}<1$, (6) holds true only when $P\{Z \geq c\}=1-F_Z(c)>0$. Equation (6) also shows that for any $t>0$ the right-hand side of (7) coincides with

$$P\{c+Z \geq t\}=1-F_Z(t-c)U(t-c). \tag{8}$$

This implies the identity

$$U(t-c)[(1-F_Z(t-c))-\frac{1-F_Z(t)}{1-F_Z(c)}]=0$$

for all $t \geq 0$. For $t<c$ (9) is trivial. Considering the case $t \geq c$, we see that (9) and (6) take place when$P\{Z=0\}=1$. If $F_Z(0)<1$, then the inequality $1-F_Z(t)>0$ must hold for any $t>0$. Since (9) is equivalent to the equation

$$1-F_Z(t-c)=\frac{1-F_Z(t)}{1-F_Z(c)}, \quad t>c, \tag{10}$$

by setting here $t=2c, 3c, \ldots$ we obtain

$$1-F_Z(kc)=[1-F_Z(c)\}^k, \ k=1,2,\ldots.$$

Substituting $t-c=x$, ($x>0$ for $t>c$), one gets from (10) that

$$1-F_Z(x+c)=[1-F_Z(x)][1-F_Z(c)] \text{ for all } x \geq 0,$$

holds true, and, therefore,

$$1-F_Z(t)=[1-F_Z(t-[t/c]c)][1-F_Z(c)]^{[t/c]}, \quad t \geq 0. \tag{11}$$

Let

$$\alpha=P\{Z \geq c\}=1-F_Z(c). \tag{12}$$

Define G by the equation

$$G(x) = \frac{F_Z(x)}{F_Z(c)} = P\{Z < x \mid Z < c\}, \quad x \in [0, c). \tag{13}$$

G is obviously the conditional distribution of the r.v. Z under the condition that Z occurs before the time c (as it is interpreted by Grosswald et al. [4]. Using (12) and (13) we can rewrite (11) in form (5). Thus we proved that (6) implies (5).

Conversely, if Z has a distribution given by (5) then it satisfies identities (9) for $t > 0$ and (10) for any $t > c$. Either (9) or (10) is equivalent to (6). We observe it by calculating

$$\bar{F}_Z(t) = 1 - F_Z(t) = \alpha^{[t/c]}[\bar{G}(t - [t/c]c)(1-\alpha) + \alpha]; \tag{14}$$

$$\bar{F}_Z(c) = \alpha;$$

$$\bar{F}_Z(t-c) = \alpha^{[t/c]-1}[\bar{G}(t - [t/c]c)(1-\alpha) + \alpha]$$

$$= \alpha^{[(t-c)/c]-1}[\bar{G}(t - c - [(t-c)/c]c)(1-\alpha) + \alpha] \text{ for } t > c,$$

and substituting it in (9) and (10).

Corollary 1: *If (6) holds true for some r.v. Z with $P\{Z = 0\} < 1$, and for some constant $c > 0$, then for any integer $k > 0$ one has*

$$(kc)\# Z \stackrel{d}{=} (kc) + Z. \tag{15}$$

Proof: Using Theorem 1 (i) and Definition 1, one has $Z \in ALM(c)$ iff $Z \in ALM(kc)$ for arbitrary $k = 1, 2, \ldots$. Corollary 1 then follows directly from Theorem 2. □

From Corollary 1, we can see that equation (6) is valid for some $c > 0$ not only for $Z \in ALM(c)$, but also for $Z \in ALM(kc)$ and for all integers $k > 1$.

Then an adjustment of Theorem 1 in view of Corollary 1 might be expressed as follows:

Corollary 2: *The equation $c\# Z \stackrel{d}{=} c + Z$ takes place for some $c > 0$ iff $Z \in ALM(c/m)$ with some integer $m > 0$.* □

According to Corollary 2, equation (6) does not determine uniquely the class of ALM distributions for Z. However, we will use the notation $ALM(c)$ for this class as one which is likely to correspond to (6).

Assume now that Z has a p.d.f. $f_Z(t)$ so that its failure rate function $\lambda_Z(t)$ is defined by

$$\lambda_Z(t) = f_Z(t)/\bar{F}_Z(t). \tag{16}$$

Then the following theorem holds true.

Theorem 3: *If equation (6) is valid for some $c > 0$, then λ_Z is a periodic function with period c, i.e.*

$$\lambda_Z(t+c) = \lambda_Z(t), \quad t \geq 0. \tag{17}$$

Proof: From Theorem 2 we know that the distribution of Z is of form (5). (If Z is continuous r.v. then the p.d.f. is $f_Z(t) = F'_Z(t)$; if Z is discrete, then$f_Z(t) = F_Z(t) - F_Z(-1)$, and all the arguments in (5) including c, are integers). Therefore the p.d.f. f_Z has to be in the form

$$f_Z(t) = \alpha^{[t/c]}(1-\alpha)g(t-[t/c]c), \quad t \geq 0, \tag{18}$$

where g is the p.d.f. corresponding to the c.d.f. G. Substituting (18) and (14) in (16), we obtain

$$\lambda_Z(t) = \frac{(1-\alpha)g(t-[t/c]c)}{(1-\alpha)G(t-[t/c]c)+\alpha} = \lambda_Z\,(t-[t/c]c), \quad t > 0,$$

which includes (17) as a particular case. □

Remark: We would like to mention that class (5) contains exponential distributions (for the continuous case) and many other continuous distributions defined by a p.d.f. of the form (18). The exponential case is obtained when $\alpha = exp\{-\lambda c\}$, $g(t) = \lambda exp\{-\lambda t\}/(1-exp\{-\lambda c\})$ for any given $c > 0$. This gives a negative response to the conjecture of Grosswald et al. [4]. They propose that under the condition of continuity of F_Z (and perhaps its continuous derivative), there is no such F_Y for which the relevation $(Y\#\ Z)$ and the convolution are identical, unless F_Z is exponential.

3. CHARACTERIZATION OF THE GEOMETRIC DISTRIBUTION THROUGH RELEVATION

We introduce a reasonable concept of relevation for non-negative integer-valued random variables as it is used in reliability models. If Y and Z are such random variables which, in addition, are mutually independent we say $X = Y\#Z$ is its relevation iff

$$P\{X=n\} = P\{Y=n\} + \sum_{k=0}^{n-1} P\{Y=k\}\frac{P\{Z=n\}}{P\{Z \geq k\}},\ n = 0,1,2,\ldots \tag{19}$$

Expression (19) is a special case of (1) due to $P\{X=n\} = P\{X \geq n\} - P\{X \geq n+1\}$ and some algebra. Also it is important to define the survival function $P\{X \geq t\}$ for the discrete and continuous case in one equation.

The convolution of the distributions of Y and Z is the distribution of the sum $X = Y + Z$, where Y and Z are independent, i.e.

$$P\{Y+Z=n\} = \sum_{k=0}^{n} P\{Y=k\}P\{Z=n-k\},\ n = 0,1,2,\ldots \tag{20}$$

Theorem 4: *The equality in distribution* (6) *holds true for two independent non-negative integer-valued random variables Y and Z when $P\{Y=1\} > 0$ iff Z has a geometric distribution.*

Proof: Equation (6) means that for any integer $n \geq 0$ the right-hand sides of (19) and (20) coincide. We rewrite the corresponding equalities in the form

$$\sum_{k=0}^{n} P\{Y=k\}[P\{Z=n-k\} - \frac{P\{Z=n\}}{P\{Z \geq k\}}] = 0,\ \ n = 0,1,2,\ldots \tag{21}$$

For $n = 0$ this reduces to the identity. For $n = 1$ equation (21) holds true if and only if

$$P\{Z=1\} = P\{Z=0\}[1 - P\{Z=0\}].$$

Denote $P\{Z=0\} = 1 - \alpha$, and assume that

$$P\{Z=k\} = \alpha^k(1-\alpha),\ \text{for all } k = 0,1,\ldots,n. \tag{22}$$

We use mathematical induction to prove that $P\{Y=1\} > 0$ and (21) imply (22) for any integer $k \geq 0$.

Equation (21), written for $n+1$ and the observation that $P\{Z \geq k\} = \alpha^k$, for $k = 0,1,\ldots,n$ gives that the equation

$$[\sum_{k=1}^{n+1} P\{Y=k\}(1/\alpha)^k][\alpha^{n+1}(1-\alpha) - P\{Z=n+1\}] = 0$$

holds true. The first factor does not equal 0 (since it contains at least one positive term), therefore the second factor equals zero. Thus (22) holds for $k = n+1$. □

Theorem 5: *Let $Z \geq 0$ be an integer-valued r.v. The equalities in distribution*

$$c_1 \# Z \stackrel{d}{=} c_1 + Z; \quad c_2 \# Z \stackrel{d}{=} c_2 + Z$$

take place for two coprime integers c_1 and c_2 iff Z has geometric distribution.

Proof: We prove this theorem as a corollary to Theorem 3. Each of the equations in (25) implies that the failure rate $\lambda_Z(n)$ is a periodic function with periods c_1 and c_2. But from the numbers theory we know that any $n > c_1c_2$ can be decomposed as

$$n = k_1c_1 + k_2c_2$$

with some non-negative integers k_1 and k_2. By (18) then we have

$$\lambda_Z(u) = \lambda_Z(k_1c_1 + k_2c_2) = \lambda_Z(k_1c_1) = \lambda_Z(0)$$

i.e. the failure rate $\lambda_Z(u)$ is a constant. Thus Z has a geometric distribution. □

It might be true analogously to the formulated conjecture in Khalil et al. [5] that equation (6) takes place for some integer r.v. $Y \geq 0$ with either $P\{Y=1\} > 0$, or support $F_Y(t)$ contains at least two coprime numbers iff Z has a geometric distribution. We still have no proof of the necessity of this assertion without additional assumptions, as it will be seen in the remark below.

Theorem 6: *For non-negative i.i.d. integer valued random variables Z_1 and Z_2, the equation in distribution*

$$Z_1 \# Z_2 \stackrel{d}{=} Z_1 + Z_2 \tag{24}$$

holds true iff

either Z degenerates with $P\{Z=\infty\} = 1$ when $P\{Z=0\} = 0$, or $P\{Z=0\} = 1$

or

$$P\{Z = nc\} = \alpha^n(1-\alpha), \quad \text{for } n = 0,1,2,\ldots, \tag{25}$$

when $0 < P\{Z=0\} < 1$ and $c > 0$ is defined by the relation

$$c = inf\{n; n > 0, P\{Z=n\} > 0\}. \tag{26}$$

Proof: We know from the proof of Theorem 4 that (24) holds iff (21) is an identity. It was assumed $P\{Y=k\}=P\{Z=k\}$ for all $k=0,1,\ldots$ Then,

$$\sum_{k=0}^{n} P\{Z=k\}[P\{Z=n-k\}-\frac{P\{Z=n\}}{P\{Z\geq k\}}]=0,\quad n=0,1,2,\ldots \tag{27}$$

Let c be defined as in (28). By letting $n=c$ in (27) we obtain

$$P\{Z=c\}[P\{Z=0\}-\frac{P\{Z=c\}}{P\{Z\geq c\}}]=0. \tag{28}$$

Since $P\{Z=c\}>0$ we have $P\{Z=0\}=1-\alpha>0$ with an $\alpha>0$. Also (26) and (27) give that $P\{Z\geq c\}=\alpha$ and $P\{Z=c\}=\alpha(1-\alpha)$. Equation (26) and (27) for $k=c$ and $n=c+1,\ c+2,\ \ldots,\ 2c-1$ imply that

$$P\{Z=n\}=0 \text{ and for } n=c+1,\ c+2,\ \ldots,\ 2c-1. \tag{29}$$

But $P\{Z=2c\}>0$. If we suppose $P\{Z=2c\}=0$, then (27) for $n=2c$ and (28) and (29) imply that

$$P\{Z=c\}(P\{Z=c\}-\frac{P\{Z=2c\}}{P\{Z\geq c\}})+P\{Z=2c\}(P\{Z=0\}-\frac{P\{Z=2c\}}{P\{Z\geq 2c\}})=0, \tag{30}$$

and therefore $P\{Z=c\}=0$, which contradicts with (29). Thus $P\{Z=2c\}>0$, and the only solution of (32) is $P\{Z=2c\}=\alpha^2(1-\alpha)$.

Suppose that

$$P\{Z=kc\}=\alpha^k(1-\alpha),\text{ for } k=0,1,\ldots,m \text{ and } P\{Z=\nu\}=0,\text{ for } \nu\neq kc,\ \nu<mc. \tag{31}$$

holds. By induction we prove that

$$P\{Z=\nu\}=0 \text{ for } \nu=mc+1,\ldots(m+1)c-1, \text{ and } P\{Z=(m+1)c\}=\alpha^{m+1}(1-\alpha) \tag{32}$$

is true. Substituting $n=mc+1, mc+2,\ldots,(m+1)c-1$ into (27) we obtain

$$P\{Z=c\}(P\{Z=n-c\}-\frac{P\{Z=n\}}{P\{Z\geq c\}})=0,\ n=mc+1,\ldots,(m+1)c-1.$$

Since $P\{Z=c\}>0$, the first part of (32) holds. Also from (27) for $n=(m+1)c$, (31) and the validated part of (32) we have that

$$P\{Z\geq kc\}=1-\sum_{\nu=0}^{kc-1}P\{Z=\nu\}=1-\sum_{\nu=0}^{k-1}\alpha^{\nu}(1-\alpha)=\alpha^k.$$

Like in the proof of Theorem 4, we see that

$$(\alpha^{m+1}(1-\alpha)-P\{Z=(m+1)c\})\sum_{k=0}^{m+1}P\{Z=kc\}\alpha^{-k}=0.$$

The conclusion of the proof is similar to that of Theorem 4. □

4. CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION THROUGH RELEVATION

In analogy to Theorem 5 we formulate an assertion which gives weaker conditions for characterizing the exponential distribution through the equation in distribution of the relevation and the sum of two independent random variables. We need the following concept.

Definition 3: Any two real numbers c_1, and c_2 are said to be incommensurable iff the ratio c_1/c_2 is an irrational real number.

Theorem 7: *Let $Z \geq 0$ be a nondegenerate at zero arbitrary continuous r.v. The equalities in distribution* (23) *take place for two incommensurable numbers c_1 and c_2 iff Z has exponential distribution.*

Proof: Here we also use the result of Theorem 3. Due to (23), the failure rate function $\lambda_Z(t)$ is periodic with periods c_1 and c_2. Denote by T the set of all real numbers c which are periods of $\lambda_Z(t)$, i.e. for which $\lambda_Z(t+c) = \lambda_Z(t)$ for all $t \geq 0$. Obviously, $c_1 \in T$ and $c_2 \in T$.

Moreover we observe that:

(*i*) if $c, d \in T$ then the numbers $2c, 3c, \ldots$ and $2d, 3d, \ldots$ also belong to T;

(*ii*) if $c, d \in T$ and $c < d$, then $d - c \in T$ as well as $c + d \in T$. This is true because if λ_Z is periodic with periods d and $c > 0$, with $c < d$, then

$$\lambda_Z(0) = \lambda_Z(d) = \lambda_Z(d - c + c) = \lambda_Z(d - c)$$

and for any $t \geq 0$ it is true that

$$\lambda_Z(t + (d - c)) = \lambda_Z(t + (d - c) + c) = \lambda_Z(t + d) = \lambda_Z(t).$$

Let $\delta = inf(c, c > 0, c \in T)$. Then $\delta = 0$ implies that the failure rate $\lambda_Z(t)$ is a constant, i.e. Z has exponential distribution.

Since c_1/c_2 is irrational, the sets $\{c_1, 2c_1, 3c_1, \ldots\}$ and $\{c_2, 2c_2, 3c_2, \ldots\}$ are arbitrarily "close" to each other (i.e. for any $\epsilon > 0$ there exist some integers $k > 0$, $m > 0$ such that $\mid kc_1 - mc_2 \mid < \epsilon$). Thus $\delta = 0$. □

The next corollary gives weaker conditions for the characterization of the exponential distribution, if additional information for the distribution of Z is available, e.g. if Z has a distribution from the class NBU (new better than used) or NWU (new worst than used), applicable in reliability theory.

Corollary 3: *For a continuous r.v. $Z \geq 0$ having NBU (or NWU) distribution and for a r.v. $Y \geq 0$ with support $[F_Y(t)]$ containing at least two incommensurable numbers c and d the equation in distribution* (3) *holds iff Z has an exponential distribution.*

Proof: We have to prove only the sufficiency part, since the necessity is a well known fact (Grosswald et al. [4]). The assumed equation in distribution is equivalent to the equation

$$\int_0^t \frac{1}{\bar{F}_Z(x)}[\bar{F}_Z(t-x)\bar{F}_Z(x) - \bar{F}_Z(t)]dF_Y(x) = 0. \tag{33}$$

Since Z is an NBU (or NWU) r.v., the part of the integrand in the square brackets has constantly either positive or negative sign for any $t > 0$. Due to the above assumption about Y, the Lebesgue-Stieltjes measure dF_Y is positive for at least the two incommensurable values $x = c$ and $x = d$. Thus, (33) is true for all $t > max(c, d)$, if and only if $\bar{F}_Z(t-c)\bar{F}_Z(c) = \bar{F}_Z(t)$ and $\bar{F}_Z(t-d)\bar{F}_Z(d) = \bar{F}_Z(t)$, and this is equivalent to (25). Consequently, Z satisfies the conditions of Theorem 5 and therefore Z has an exponential distribution. □

Remark: In the case when Y and Z have discrete distributions the conditions of Corollary 3 can be reformulated to give a characterization of the geometric distribution of Z: For a discrete r.v. Z with an NBU (or NWU) distribution and for a r.v. $Y \geq 0$ with $P\{Y = c_1\} \geq 0$ and $P\{Y \geq c_2\} \geq 0$ for at least two coprime integers c_1 and c_2 the coincidence in distribution (3) holds iff Z has a geometric distribution. The proof of this statement duplicates that of Theorem 7.

Analogously to Theorem 6 we formulate and prove the following theorem, which differs from that in Grosswald et al. [4] by imposing weaker conditions. The method of the proof is also different.

Theorem 8: *For non-negative continuous i.i.d. random variables Z_1 and Z_2 the equation in distribution* (35) *holds true iff Z has an exponential distribution.*

Proof: Equation (24) is equivalent to (33) with $F_Y(x) = F_Z(x)$ for all $x \geq 0$. Throughout the remainder we drop the subscript and write simply $F(x)$. Then, from (24) we obtain the following equation

$$\int_0^t \frac{1-F(t)}{1-F(x)} dF(x) = \int_0^t [1-F(t-x)]dF(x),$$

or, equivalently,

$$[1-f(t)]ln[1-F(t)] = \int_0^t [a-F(t-x)]dF(x). \quad (34)$$

Let

$$\Lambda(t) = -ln[1-F(t)] = \int_0^t \lambda(x)dx,$$

where $\lambda(x) = f(x)/[1-F(x)]$ is the corresponding failure rate function to F, assumed existing. Then, (34) can be rewritten in the form

$$\int_0^t e^{-\Lambda(t-x)} dF_X(x) = \Lambda(t)e^{-\Lambda(t)}. \quad (35)$$

Integrating the left-hand side by parts we get

$$\int_0^t e^{-\Lambda(t-x)} dF_X(x) = e^{-\Lambda(t-x)}e^{-\Lambda(x)}\Lambda(x)\Big|_0^t$$

$$+ \int_0^t \Lambda(x)e^{-[\Lambda(t-x)+\Lambda(x)]}[\lambda(x)-\lambda(t-x)]dx.$$

Since $\Lambda(0) = 0$ we have that the first summand above equals $\Lambda(t)e^{-\Lambda(t)}$ and coincides with the right-hand side of (35). Thus,

$$\int_0^t \Lambda(x)e^{-[\Lambda(t-x)+\Lambda(t)]}[\lambda(x)-\lambda(t-x)]dx = 0.$$

This means that whenever $\Lambda(x) \neq 0$

$$\lambda(x) = \lambda(t-x)$$

for $t > x$. Since there exists at least one positive x_0 for which $\Lambda(x_0) \neq 0$ (and then $\Lambda(x) \neq 0$ for any $x > x_0$) the last equation holds true for any $x > x_0$ and $t > x$ if and only if $\lambda(x)$ is a constant. Thus $F_X(t) = 1 - exp\{-\lambda t\}$, and Z is exponentially distribution. □

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