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BOUNDEDNESS AND ASYMPTOTIC STABILITY IN THE LARGE OF SOLUTIONS OF AN ORDINARY DIFFERENTIAL SYSTEM $y' = f(t, y, y')$[1]

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ABSTRACT

Differential equations of the form $y' = f(t, y, y')$, where f is not necessarily linear in its arguments, represent certain physical phenomena and solutions have been known for quite some time. The well known Clairut's and Chrystal's equations fall into this category. Earlier existence of solutions of first order initial value problems and stability of solutions of first order ordinary differential system of the above type were established. In this paper we study boundedness and asymptotic stability in the large of solutions of an ordinary differential system of the above type under certain natural hypotheses on f.

Key words: Existence, unique, solution, continuous, differentiable, contraction, system, bounded, stable, uniform, asymptotic, exponential, equi, ultimate, Lyapunov, function.

AMS (MOS) subject classifications: 34-XX, 34DXX, 34D20, 34D40.

1. INTRODUCTION

Differential equations of the form $y' = f(t, y, y')$ where f is not necessarily linear in its arguments represent certain physical phenomena and are known for quite some time. The well known Clairut's and Chrystal's equations fall into this category [1]. A few authors, notably E.L. Ince [2], H.T. Davis [1] et. al. have given some methods of finding solutions of equations of the above type. Apart from these, to the authors knowledge, there does not seem to exist any systematic study of these equations.

In our earlier papers [4,5,6], we studied the initial value problems and stability (in the sense of Lyapunov) of solutions of equations of the above type. In the present paper we study

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the boundedness and asymptotic stability in the large of solutions of this new class of problems.

There is yet another type of stability called "Practical Stability" associated with the systems of the form $y' = g(t,y)$ and a recent book by Professor V. Lakshmikantham et. al. [3] gives a very good account of practical stability. But since practical stability is neither weaker nor stronger than Lyapunov stability, in the present paper we confine ourselves to Lyapunov stability and in a subsequent paper we shall study the practical stability of $y' = f(t,y,y')$.

Before proceeding to the main theorems, we present a few preliminary results under certain natural assumptions. Let $I = [0,\infty)$ and let R^n denote the n-dimensional real space equipped with the box norm $|\cdot|$ given by $|x| = \sum_{i=1}^{n} |x_i|$. Let $G = I \times R^n \times R^n$.

Consider the initial value problem (IVP)

$$y' = f(t,y,y') \quad (\prime = \tfrac{d}{dt}), \tag{1}$$

$$y(t_0) = y_0 \tag{2}$$

where f is an n-vector and $(t_0, y_0) \in I \times R^n$.

Assumption: *Let f satisfy the following conditions:*

(I) *$f(t,y,z)$ is continuous with respect to $(t,y,z) \in G$,*

(II) *for every $(t_0,y_0) \in I \times R^n$ and for every pair of constants $a > 0, b > 0$, there exists a constant $c > 0$ such that if*

$$D = \{(t,y,z) \in G \mid \ |t - t_0| \le a, \ |y - y_0| \le b, \ |z| \le c\},$$

then $|f(t,y,z)| \le c$ for all $(t,y,z) \in D$,

and

(III) *there exist constants $k_1 > 0, 0 \le k_2 < 1$, which may depend upon D, such that*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \le k_1 |y_1 - y_2| + k_2 |z_1 - z_2|$$

for all (t,y_1,z_1), $(t,y_2,z_2) \in D$.

The following local existence and uniqueness result is an immediate consequence of Result 2 [6].

Result 1: If f satisfies conditions $(I)-(III)$, then IVP (1), (2) has a unique solution $y(t,t_0,y_0)$ existing on the interval $[t_0 - r, t_0 + r] \cap I$, where

$$r = min\left(\frac{1-k_2}{k_1}, \frac{b}{c}, a\right).$$

Here, $y(t, t_0, y_0)$ denotes the (continuous) dependence of the solution $y(t)$ on (t_0, y_0).

Below, we present a continuation result.

Result 2 (Continuation of the solution of IVP (1), (2)): Suppose that $f(t, y, z)$ satisfies conditions $(I)-(III)$. Also, suppose that the solution $y(t, t_0, y_0)$, for as long as it exists, is strictly bounded by β for some $\beta > 0$. Then $y(t, t_0, y_0)$ is continuable up to any t.

Proof: Let $\alpha > t_0$ be any number. We shall show that the solution $y(t, t_0, y_0)$ exists on $[t_0, \alpha]$. To this end, by condition (II), we choose a constant $c > 0$ such that on

$$D = \{(t, y, z) \in G \mid \ |t - t_0| \leq \alpha - t_0, |y - y_0| \leq 2\beta, |z| \leq c\},$$

we have $|f(t, y, z)| \leq c$.

Then by Result 1, the solution $y(t, t_0, y_0)$ exists on $[t_0 - r, t_0 + r]$, where

$$r = min\Big(\frac{1-k_2}{k_1}, \frac{2\beta}{c}, \alpha - t_0\Big).$$

Now, if possible, let $t_0 + r \leq \tilde{t} < \alpha$ be such that the solution $y(t, t_0, y_0)$ can be continued only upto $\tilde{t}$. Then we have $|y(\tilde{t}) - y_0| < 2\beta$, and consider the set

$$D_1 = \{(t, y, z) \in G \mid \ |t - \tilde{t}| \leq a, \ |y - y(\tilde{t})| \leq 2\beta - |y_0 - y(\tilde{t})|, |z| \leq c\},$$

where $a_1 > 0$ is such that $\tilde{t} + a_1 \leq \alpha$. Clearly $D_1 \subset D$. Then, by Result 1, the solution $y(t, t_0, y_0)$ can be continued up to $\tilde{t} + r_1$, where

$$r_1 = min\Big(\frac{1-k_2}{k_1}, \frac{2\beta - |y_0 - y(\tilde{t})|}{c}, a_1\Big).$$

This is certainly a contradiction and hence the proof is complete.

Whenever the solution $y(t, t_0, y_0)$ is continuable up to any t, $t \geq t_0$, we say that $y(t, t_0, y_0)$ exists for all future times and write $y(t, t_0, y_0)$ exists for $t \geq t_0$.

Remark 1: In addition to assumption 1, if f has continuous first order partial derivatives with respect to $(t, y, z) \in G$ and that k_1, k_2 in condition (III) denote the upper bounds for $\frac{\partial f}{\partial y_j}$ and $\frac{\partial f}{\partial z_j}$ $(j = 1, 2, \ldots, n)$, respectively, then it can be easily verified that $y(t, t_0, y_0)$ is continuously differentiable with respect to t and that

$$y'' = (E - (\frac{\partial f_i}{\partial z_j}))^{-1}(\frac{\partial f}{\partial t} + (\frac{\partial f_i}{\partial y_j})f),$$

where E is the $(n \times n)$ identity matrix, and $(\frac{\partial f_i}{\partial z_j})$, $(\frac{\partial f_i}{\partial y_j})$ are the Jacobian matrices.

Definition 1: We call a real valued function $V(t,y,z)$ defined on G a Lyapunov function if $V(t,y,z)$ is continuously differentiable with respect to $(t,y,z) \in G$.

Definition 2: The derivative of $V(t,y,z)$ with respect to system (1) is defined by

$$V^*(t,y,z) = \frac{\partial V}{\partial t} + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j} f_j + \sum_{j=1}^{n} \frac{\partial V}{\partial z_j} (E - (\frac{\partial f_i}{\partial z_j}))^{-1} (\frac{\partial f}{\partial t} + (\frac{\partial f_i}{\partial y_j}) f)_j.$$

Along a solution of system (1), we always have $\frac{dV}{dt} = V^*$.

Throughout the work, $a(r), b(r)$ and $c(r)$ denote positive definite functions such that $a(r) \to \infty$ as $r \to \infty$. For the definition of positive definiteness see [6], p. 217.

Result 3: Suppose that $f(t,y,z)$ satisfies the conditions of Remark 1. Also, suppose that there exists a Lyapunov function $V(t,y,z)$ defined on G satisfying the conditions

$$a(|y|) \leq V(t,y,z)$$

and

$$V^*(t,y,z) \leq 0$$

for all $(t,y,z) \in G$ such that $z = f(t,y,z)$. Then all solutions of system (1) are continuable up to any t.

The proof follows along the lines of the proof of Theorem 3.4 [7] and hence is omitted.

In the rest of the work, we assume that the conditions of Result 3 are true. Hence all solutions of (1) are continuable up to any t.

2. BOUNDEDNESS OF SOLUTIONS OF SYSTEM (1)

Definition 3: Solutions of system (1) are:

(B_1) equi-bounded if, for each $\alpha > 0, t_0 \in I$, there exists a positive constant $\beta = \beta(t_0, \alpha)$ such that $|y_0| \leq \alpha$ implies $|y(t,t_0,y_0)| < \beta, t \geq t_0$;

(B_2) uniformly bounded if the β in (B_1) is independent of t_0;

(B_3) ultimately bounded if there exist a $B > 0$ and a $T > 0$ such that for every solution $y(t,t_0,y_0)$ of (1), $|y(t,t_0,y_0)| < B$ for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend upon each solution;

(B_4) equi-ultimately bounded if there exists a $B > 0$ and if, for each $\alpha > 0$, $t_0 \in I$, there exists a $T = T(t_0, \alpha) > 0$ such that $|y_0| \leq \alpha$ implies $|y(t,t_0,y_0)| < B$, $t \geq t_0 + T$;

(B_5) uniform-ultimately bounded if the T in (B_4) is independent of t_0.

We note that the uniform (-ultimately) boundedness of solutions of system (1) implies the equi (-ultimately) boundedness of solutions of (1). Below, we shall show that the converse is also true if f is either periodic in t or autonomous.

Theorem 1: *Let $f(t,y,z)$ be such that $f(t+w,y,z) = f(t,y,z)$ for all $(t,y,z) \in G$, where $w > 0$ is a constant. If the solutions of* (1) *are equi (-ultimately) bounded, then they are uniform (-ultimately) bounded.*

The proof follows, using result 3 [6], along the lines of proof of Theorems 9.2 and 9.3 [7] and hence is omitted.

Theorem 2 (Equi-boundedness of the solutions): *Under the hypotheses of Result* 3, *solutions of system* (1) *are equi-bounded.*

Proof: Let $t_0 \in I$ and $\alpha > 0$ be given. For y_0 with $|y_0| \leq \alpha$, consider the solution $y(t,t_0,y_0)$. Using condition (II), we choose a constant $c > 0$ such that on the set

$$D = \{(t,y,z) \in G \mid 0 \leq t \leq t_0, |y| \leq \alpha, |z| \leq c\},$$

we have $|f(t,y,z)| \leq c$.

Let

$$M = \{z \in R^n \mid |z| \leq c\}.$$

and define a map $F: M \to M$ by

$$F(z) = f(t_0, y_0, z).$$

Clearly, F maps M into itself and, by (III), is a contraction on M. Hence F has a unique fixed point z in M. Consequently,

$$z = y'(t_0, t_0, y_0)$$

and

$$|y'(t_0, t_0, y_0)| \leq c.$$

That is, for $t_0 \in I$ and for all y_0 with $|y_0| \leq \alpha$, we have

$$|y'(t_0, t_0, y_0)| \leq c,$$

where c depends on t_0 and α.

Now, define

$$S = \{(y,z) \in R^n \times R^n \mid |y| \leq \alpha, |z| \leq c\}.$$

Clearly, S is compact and $V(t_0, y, z)$ is continuous on S. Hence there exists a constant $k = k(t_0, \alpha) > 0$ such that

$$V(t_0, y, z) \leq k$$

for all $(y, z) \in S$. Consequently, for all y_0 with $|y_0| \leq \alpha$, we have

$$V(t_0, y(t_0), y'(t_0)) \leq k.$$

Finally, by choosing a constant $\beta = \beta(t_0, \alpha) > \alpha$ sufficiently large such that $k < a(\beta)$ and proceeding along the lines of proof of Result 2, it can be shown that

$$|y(t, t_0, y_0)| < \beta$$

for all $t \geq t_0$. This completes the proof.

Theorem 3: *Let $V(t, y, z)$ be a Lyapunov function defined on G.*

(A) *(Uniform boundedness of solutions): If*

$$a(|y|) \leq V(t, y, z) \leq b(|y|)$$

and

$$V^*(t, y, z) \leq 0$$

for all $(t, y, z) \in G$ satisfying $z = f(t, y, z)$, then solutions of (1) are uniformly bounded.

(B) *(Equi-ultimately boundedness of solutions): If*

$$a(|y|) \leq V(t, y, z)$$

and

$$V^*(t, y, z) \leq -cV(t, y, z)$$

for all $(t, y, z) \in G$ satisfying $z = f(t, y, z)$, where c is a positive constant, then solutions of (1) are equi-ultimately bounded.

(C) *(Uniform-ultimately boundedness of solutions): If*

$$a(|y|) \leq V(t, y, z) \leq b(|y|)$$

and

$$V^*(t, y, z) \leq -c(|y|)$$

for all $(t, y, z) \in G$ *satisfying* $z = f(t, y, z)$, *then solutions of* (1) *are uniform-ultimately bounded.*

Proof: Proof of part (A) is similar to the proof of Result 3 and hence is omitted.

To prove part (B), take any positive constant β. Let $t_0 \in I$ and α be a constant such that $0 < \alpha < \beta$. For $y_0 \in R^n$, consider the solution $y(t, t_0, y_0)$. It can be shown, as in the proof of Theorem 2, that there exists a constant $k = k(t_0, \alpha) > 0$ such that

$$V(t_0, y(t_0), y'(t_0)) \leq k$$

for all y_0 with $|y_0| \leq \alpha$.

Now, choose a constant $M = M(t_0, \alpha)$ such that

$$M(t_0, \alpha) > max(k, a(\beta))$$

and let

$$T = T(t_0, \alpha) = \tfrac{1}{c} ln(M/a(\beta)).$$

Clearly $T > 0$ and we get that

$$|y(t, t_0, y_0)| < \beta$$

for all $t \geq t_0 + T$. Otherwise, by integrating the inequality

$$V^*(t, y, z) \leq -cV(t, y, z)$$

along $y(t, t_0, y_0)$ between t_0 and t_1, where t_1 is such that

$$|y(t_1, t_0, y_0)| = \beta,$$

we arrive at a contradiction that $a(\beta) < a(\beta)$.

To prove part (C), we note that, by Theorem 3(A), solutions of (1) are uniformly bounded. Take two real numbers α, β such that $0 < \alpha < \beta$. There exist constants B_1, B_2 with $\alpha < B_1 < B_2, \beta < B_2$ such that for any $t_0 \in I$ and y_0 with $|y_0| \leq \alpha$ (β), we have

$$|y(t, t_0, y_0)| < B_1 \ (B_2)$$

for all $t \geq t_0$. Now, define

$$k_1 = inf\{a(r) \mid \alpha \leq r \leq B_2\},$$

$$k_2 > max(sup\{b(r) \mid 0 \leq r \leq \beta\}, a(\alpha)),$$

and

$$k_3 = inf\{c(r) \mid \alpha \leq r \leq B_2\}.$$

Let

$$T = (k_2 - k_1)/k_3.$$

Clearly $T > 0$ and is independent of t_0. It can be proved as in part (B), that there exists a $t_1 \in [t_0, t_0 + T]$ such that

$$\mid y(t_1, t_0, y_0) \mid \leq \alpha.$$

Consequently,

$$\mid y(t, t_0, y_0) \mid \leq B_1$$

for all $t \geq t_0 + T$. For $0 < \beta < \alpha$, T can be assigned any positive value and the proof is complete.

The following corollary follows immediately from Theorem 3 (C).

Corollary 1: *If we replace in Theorem* 3(C), *the condition*

$$V^*(t, y, z) \leq -c(\mid y \mid)$$

by

$$V^*(t, y, z) \leq -cV(t, y, z)$$

for all $(t, y, z) \in G$ *satisfying* $z = f(t, y, z)$, *where* c *is a positive constant, then solutions of* (1) *are uniform-ultimately bounded.*

3. ASYMPTOTIC STABILITY IN THE LARGE OF SOLUTIONS OF SYSTEM (1)

In addition to the assumptions made earlier, in this section we also assume that $f(t, 0, 0) = 0$, $t \in I$. Thus $y \equiv 0$ is a solution of system (1). It is quite easy to verify that the study of stability of solutions of $y' = f(t, y, y')$ with $f(t, 0, 0) \neq 0$ is equivalent to the study of stability of the zero solution of an equivalent system and thus $f(t, 0, 0) = 0, t \in I$ is not a severe restriction on f (see [6]). Also, for the definitions of stability and uniform stability refer to [6].

Definition 4: The solution $y(t) \equiv 0$ of system (1) is

(S_1) asymptotically stable in the large, if it is stable and every solution of (1) tends to zero as $t \to \infty$;

(S_2) equi-asymptotically stable in the large, if it is stable, and for each $\alpha > 0$, $\epsilon > 0$, $t_0 \in I$, there exists a $T = T(t_0, \epsilon, \alpha) > 0$ such that $|y_0| < \alpha$ implies $|y(t, t_0, y_0)| < \epsilon$, $t \geq t_0 + T$;

(S_3) uniform-asymptotically stable in the large, if it is uniformly stable, and for each $\alpha > 0$, $\epsilon > 0$, there exists a $T = T(\epsilon, \alpha) > 0$ such that $t_0 \in I$ and $|y_0| \leq \alpha$ implies $|y(t, t_0, \epsilon)| < \epsilon$, $t \geq t_0 + T$, and the solutions of (1) are uniformly bounded;

(S_4) exponential-asymptotically stable in the large, if there exists a $c > 0$ and for each $\alpha > 0$, there exists a constant $k = k(\alpha) > 0$ such that $|y_0| \leq \alpha$ implies

$$|y(t, t_0, y_0)| \leq ke^{-c(t - t_0)} |y_0|, t \geq t_0.$$

We note that the uniform-asymptotic stability in the large implies the asymptotic stability in the large. The next theorem shows that the converse is also true if f is either periodic in t or autonomous.

Theorem 4: *Let $f(t, y, z)$ be such that $f(t + w, y, z) = f(t, y, z)$, for all $(t, y, z) \in G$, where w is a positive constant. If the zero solution of (1) is asymptotically stable in the large, then it is uniform-asymptotically stable in the large.*

The proof of this theorem follows, using Result 3 [6] and Theorem 1, along the lines of the proof of Theorem 7.4 [7] and hence is omitted.

Theorem 5: *Let $V(t, y, z)$ be a Lyapunov function defined on G.*

(A) *(Asymptotically stable in the large of the zero solution): Suppose that*

(i) $V(t, 0, 0) = 0$, $t \in I$,

(ii) $a(|y|) \leq V(t, y, z)$,

and

(iii) $V^*(t, y, z) \leq -c(|y|)$,

for all $(t, y, z) \in G$ such that $z = f(t, y, z)$. Then the zero solution of (1) is asymptotically stable in the large.

(B) *(Equi-asymptotically stable in the large of the zero solution): If condition (iii) of part (A) is replaced by*

$$V^*(t, y, z) \leq -cV(t, y, z),$$

where c is a positive constant, then the zero solution of (1) is equi-asymptotically stable in the large.

(*C*) (*Uniform-asymptotically stable in the large of the zero solution*): *Suppose that condition* (*iii*) *of part* (*A*) *is true and*

$$a(|y|) \leq V(t,y,z) \leq b(|y|)$$

for all $(t,y,z) \in G$ *such that* $z = f(t,y,z)$. *Then the zero solution of* (1) *is uniform-asymptotically stable in the large.*

Proof: Part (A): Stability of the zero solution of (1) follows from Theorem 2 [6], and for $t_0 \in I$, $y_0 \in R^n$, the solution $y(t,t_0,y_0) \to 0$ as $t \to \infty$ can be established along similar lines of proof of Theorem 8.5 [7].

Part (B): Again, stability of the zero solution follows from Theorem 2 [6]. Also, by Theorem 2, solutions of (1) are equi-bounded.

Now, let $t_0 \in I$ and α be a positive constant. Then there exists a constant $\beta = \beta(t_0,\alpha) > 0$ such that $|y_0| \leq \alpha$ implies $|y(t,t_0,y_0)| < \beta$, $t \geq t_0$. Also, as in the proof of Theorem 2, there exists a constant $k = k(t_0,\alpha) > 0$ such that

$$V(t_0,y_0,y'(t_0)) \leq k$$

for all y_0 with $|y_0| \leq \alpha$. Let ϵ be such that $0 < \epsilon < \beta$. Let

$$k_1 = inf\{a(r) \mid \epsilon \leq r \leq \beta\},$$

and choose a constant $N = N(t_0,\epsilon,\alpha)$ such that

$$N > max(k_1,k).$$

Let

$$T = T(t_0,\epsilon,\alpha) = \frac{1}{c} ln(N/k_1).$$

Then

$$|y(t,t_0,y_0)| < \epsilon$$

for all $t \geq t_0 + T$. Otherwise, integrating the inequality in condition (*iii*) along $y(t,t_0,y_0)$ from t_0 to t_1, where t_1 is such that

$$|y(t_1,t_0,y_0)| \geq \epsilon,$$

we get a contradiction that $k_1 < k_1$.

Part (C): Uniform stability of the zero solution of (1) follows from Theorem 3 [6]. Also, by Theorem 3(A), solutions of (1) are uniformly bounded.

Now, let $t_0 \in I$ and α be a positive constant. Then there exists a constant $\beta = \beta(\alpha) > 0$ such that $|y_0| \leq \alpha$ implies $|y(t, t_0, y_0)| < \beta$, $t \geq t_0$. Let ϵ be such that $0 < \epsilon < \beta$. Let

$$k_1 = inf\{a(r) \mid \epsilon \leq r \leq \beta\},$$

$$k_2 = inf\{c(r) \mid \epsilon \leq r \leq \beta\},$$

and

$$M > max(sup\{b(r) \mid 0 \leq r \leq \alpha\}, k_1).$$

Let

$$T = T(\epsilon, \alpha) = (M - k_1)/k_2.$$

Clearly $T > 0$, and we get that

$$|y(t, t_0, y_0)| < \epsilon$$

for all $t \geq t_0 + T$. Otherwise, proceeding as in part (B), we get a contradiction that $k_1 < k_1$. This completes the proof.

The following corollary is an immediate consequence of Theorem 5 (C).

Corollary 2: *If the condition $V^*(t, y, z) \leq -c(|y|)$ in Theorem 5 (C) is replaced by $V^*(t, y, z) \leq -cV(t, y, z)$, where c is a positive constant, then the zero solution of (1) is uniform asymptotically stable in the large.*

Finally, we end this section by presenting a theorem on the exponential-asymptotical stability of the zero solution.

Theorem 6: *Suppose that $V(t, y, z)$ is a Lyapunov function defined on G and satisfies the following conditions:*

(*i*) *For each $\alpha > 0$, there exists a constant $k = k(\alpha) > 0$ such that*

$$|y| \leq V(t, y, z) \leq k|y|$$

and

(*ii*) $V^*(t, y, z) \leq -cV(t, y, z)$, *where c is a positive constant, for all $(t,y,z) \in G$ such that $z = f(t,y,z)$. Then the zero solution of (1) is exponential-asymptotically stable in the large.*

4. EXAMPLES

Example 1: Let $g(y)$ be a continuously differentiable function defined on R such that $yg(y) > 0$ for $y \neq 0$, and that $|g(y)| \leq M$, $|g'(y)| \leq k$ for all $y \in R$.

Consider the following second order nonlinear ordinary differential equation:

$$u'' + w^2 u + \epsilon e^{\sin u''} g(u') = 0, \tag{3}$$

where w is a positive constant and ϵ is such that $0 < \epsilon e M < 1$.

Equation (3) is equivalent to the system

$$Y' = F(t, Y, Y') \tag{4}$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -(w^2 y_1 + \epsilon e^{\sin y_2'} g(y_2) \end{pmatrix}$ and $y_1 = u$.
Let $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$.

Clearly, $F(t,Y,Z)$ is a continuously differentiable real valued function defined on $G = I \times R^2 \times R^2$ and that

$$\left|\frac{\partial F}{\partial y_1}\right| \leq w^2, \left|\frac{\partial F}{\partial y_2}\right| \leq 1 + \epsilon e k, \frac{\partial F}{\partial z_1} = 0, \left|\frac{\partial F}{\partial z_2}\right| \leq \epsilon e M$$

for all $(t,Y,Z) \in G$.

Now let $(t_0, Y_0) \in I \times R^2$ and $a > 0$, $b > 0$ be arbitrary. Choose a constant c such that

$$(1 + w^2)(b + |Y_0|) + Me\epsilon < c.$$

Define

$$D = \{(t,Y,Z) \in G \mid |t - t_0| \leq a, |Y - Y_0| \leq b, |z| \leq c\}.$$

Then we can easily verify that F satisfies the conditions of Result 3. Also, it is easy to check that the Lyapunov function

$$V(t,Y,Z) = \frac{w^2 y_1^2}{2} + \frac{y_2^2}{2}$$

satisfies the hypotheses of Theorem 3 (A) with

$$a(r) = \alpha(r/\sqrt{2})^2, b(r) = \beta(r/\sqrt{2})^2$$

where $\alpha = min(1, w^2)$ and $\beta = max(1, w^2)$. Therefore, by Theorem 3 (A), solutions of (4) and hence solutions of (3) are uniformly bounded.

Example 2: Let

$$g(y) = \begin{cases} sin\ y, & -\pi/2 \leq y \leq \pi/2 \\ 1, & \pi/2 < y \\ -1, & y < -\pi/2. \end{cases}$$

Clearly, g is continuously differentiable on R, $g(0) = 0, yg(y) > 0$ for $y \neq 0$ and $|g(y)| \leq 1$.

Consider the differential equation

$$y' = -\alpha g(y)(cos\ y')^2 \tag{5}$$

where $0 < \alpha < \frac{1}{2}$. We can easily verify that

$$f(t, y, z) = -\alpha g(y)(cos\ z)^2$$

satisfies the conditions of Result 3 on $D = I \times R \times R$, and that $f(t, 0, 0) = 0$ for all $t \in I$. Also, it is easy to check that the Lyapunov function

$$V(t, y, z) = y^2$$

satisfies the hypotheses on Theorem 5 (C) with

$$a(r) = b(r) = r^2, \text{ and } c(r) = 2\alpha(cos\ 1)^2 rg(r).$$

Hence by Theorem 5 (C), the zero solution of (5) is uniform-asymptotically stable.

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The authors dedicate the work to the Chancellor of the Institute Bhagawan Sri Sathya Sai Baba.

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