

Two New Algorithms for Discrete Boundary Value Problems*

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ABSTRACT

We propose two new methods of constructing the solutions of linear multi-point discrete boundary value problems. These methods are applied to solve some continuous two-point boundary value problems which are known to be numerically unstable.

Key words: Adjoint Identities, Discrete Systems, Multi-point Boundary Value Problems.

AMS Subject Classification: 65Q05, 65L10.

1. INTRODUCTION

The purpose of this paper is to provide two new algorithms to compute the solution of the linear discrete system

$$u(k+1) = A(k)u(k) + b(k), \quad k \in N(k_1, k_r) \quad (1.1)$$

satisfying the multi-point boundary conditions

$$\sum_{i=1}^r \sum_{p=1}^n \alpha_{qp}^i u_p(k_i) = l_q, \quad 1 \leq q \leq n \quad (1.2)$$

where $A(k)$ is a given nonsingular $n \times n$ matrix with elements $a_{qp}(k)$, $1 \leq q, p \leq n$; $b(k)$ is a given $n \times 1$ vector with components $b_q(k)$, $1 \leq q \leq n$; $u(k)$ is an unknown $n \times 1$ vector with components $u_q(k)$, $1 \leq q \leq n$; $0 \leq k_1 < k_2 < \dots < k_r$ ($r \geq 2$) where each k_i is a positive integer; $N(k_1, k_r)$ is the discrete interval $[k_1, k_1+1, \dots, k_r]$; α_{qp}^i , l_q , $1 \leq q, p \leq n$, $1 \leq i \leq r$ are given constants.

In particular we shall also consider the system (1.1) together with the implicit separated boundary conditions

$$\sum_{p=1}^n \alpha_{i(s_i),p} u_p(k_i) = l_{i,j(s_i)}; \quad 1 \leq i \leq r \leq (2 \leq r \leq n), \quad (1.3)$$

where $s_1 = 1, 2, \dots, \beta_1; \dots; s_r = 1, 2, \dots, \beta_r$ and $\sum_{i=1}^r \beta_i = n$. The subscript $i(s_i)$ allows the possibility that at the same point k_i several boundary conditions are prescribed.

Motivated by the work of Angel and Kalaba [4] on two-point boundary value problems for difference equations, recently in [1, 3, 7, 12], we have discussed several new methods of constructing the solutions of linear as well as nonlinear multi-point discrete boundary value problems. In this paper we shall develop adjoint identities which are in terms of solutions of (1.1) and its adjoint system. These identities lead to the backward-forward and forward-backward methods, which seem to be new processes for computing the solutions of (1.1) and (1.2). However, the application of adjoint equations to solve discrete problems is not new, e.g., Clenshaw [6] used adjoint difference equations to sum the Chebyshev series. To demonstrate the usefulness of the proposed methods we solve some continuous two-point boundary value problems which are known to be unstable.

2. BACKWARD-FORWARD AND FORWARD-BACKWARD METHODS

The adjoint system of the difference system (1.1) is defined as

$$v(k) = A^T(k) v(k+1), \quad k \in N(k_1, k_r) \quad (2.1)$$

where $A^T(k)$ is an $n \times n$ matrix with elements $a_{qp}(k)$, $1 \leq p, q \leq n$ and $v(k)$ is an $n \times 1$ vector with components $v_q(k)$, $1 \leq q \leq n$.

We multiply the q th equation of (1.1) by $v_q(k+1)$ and sum over all n equations to obtain

$$\sum_{q=1}^n u_q(k+1)v_q(k+1) = \sum_{q=1}^n v_q(k+1) \sum_{p=1}^n a_{qp}(k)u_p(k) + \sum_{q=1}^n b_q(k)v_q(k+1). \quad (2.2)$$

Next we multiply the q th equation of (2.1) by $u_q(k)$ and sum over all n equations to get

$$\sum_{q=1}^n u_q(k)v_q(k) = \sum_{q=1}^n u_q(k) \sum_{p=1}^n a_{qp}(k)v_p(k+1). \quad (2.3)$$

On subtracting (2.3) from (2.2), we find

$$\sum_{q=1}^n [u_q(k+1)v_q(k+1) - u_q(k)v_q(k)] = \sum_{q=1}^n b_q(k)v_q(k+1). \quad (2.4)$$

Let $k_0 \in N(k_1, k_r)$ be fixed. In (2.4) letting $k = l$ and summing from k_0 to $k-1 \in N(k_1, k_r)$ results in

$$\sum_{q=1}^n [u_q(k)v_q(k) - u_q(k_0)v_q(k_0)] = \sum_{l=k_0+1}^k \sum_{q=1}^n v_q(l) b_q(l-1), \quad (2.5)$$

for all $k_0 \leq k \in N(k_1, k_r)$

and, similarly

$$\sum_{q=1}^n [u_q(k)v_q(k) - u_q(k_0)v_q(k_0)] = - \sum_{t=k+1}^{k_0} \sum_{q=1}^n v_q(t) b_q(t-1), \quad (2.6)$$

for all $k_0 \geq k \in N(k_1, k_r)$.

Equations (2.5) and (2.6) will be referred to as adjoint identities. We compute backward solutions once for each $u_q(k_i)$, $2 \leq i \leq r$ appearing in (1.2) with the conditions

$$v_q^{p(i)}(k_i) = \alpha_{pq}^i; \quad 2 \leq i \leq r, \quad 1 \leq p, q \leq n \quad (2.7)$$

where $v_q^{p(i)}(k_i)$ is the q th component at k_i for the p th backward solution. Substituting (2.7) into the adjoint identity (2.5) with $k_0 = k_1$, we obtain

$$\sum_{q=1}^n \alpha_{pq}^i u_q(k_i) - \sum_{q=1}^n v_q^{p(i)}(k_1) u_q(k_1) = \sum_{t=k_1+1}^{k_i} \sum_{q=1}^n v_q^{p(i)}(t) b_q(t-1), \quad 2 \leq i \leq r. \quad (2.8)$$

Summing $(r-1)$ equations (2.8) and making use of (1.2), we get

$$\sum_{q=1}^n [\alpha_{pq}^i + \sum_{i=2}^r v_q^{p(i)}(k_1)] u_q(k_1) = l_p - \sum_{i=2}^r \sum_{t=k_1+1}^{k_i} \sum_{q=1}^n v_q^{p(i)}(t) b_q(t-1), \quad 1 \leq p \leq n. \quad (2.9)$$

If the matrix $[\alpha_{pq}^i + \sum_{i=2}^r v_q^{p(i)}(k_1)]$ is nonsingular, then the system (2.9) provides

the unknowns $u_q(k_1)$, $1 \leq q \leq n$. The solution of the problem (1.1) and (1.2) is obtained by computing the solution of (1.1) with these values of $u_q(k_1)$, $1 \leq q \leq n$. However, to evaluate the summation term in (2.9) we need to store the solutions of (2.1). This can be avoided at the cost of solving another $(r-1)$ system. For this, we denote

$$w_{p(i)}(k) = - \sum_{t=k+1}^{k_i} \sum_{q=1}^n v_q^{p(i)}(t) b_q(t-1); \quad 1 \leq p \leq n, \quad 2 \leq i \leq r,$$

which is equivalent to solving

$$w_{p(i)}(k) = - \sum_{q=1}^n v_q^{p(i)}(k+1) b_q(k) + w_{p(i)}(k+1) \quad (2.10)$$

$$w_{p(i)}(k_i) = 0; \quad 1 \leq p \leq n, \quad 2 \leq i \leq r. \quad (2.11)$$

Thus, at the point k_i , $2 \leq i \leq r$, we solve a system of order $2n$ given by (2.1) and (2.10) subject to the conditions (2.7) and (2.11).

With this adjustment system (2.9) takes the form

$$\sum_{q=1}^n [\alpha_{pq}^1 + \sum_{i=2}^n v_q^{p(i)}(k_1)] u_q(k_1) = l_p + \sum_{i=2}^r w_{p(i)}(k_1), \quad 1 \leq p \leq n. \quad (2.12)$$

This method of constructing the solution of (1.1) and (1.2) is called the backward-forward process and requires $(r-1)n$ backward solutions of the adjoint system (2.1) satisfying (2.7), $(r-1)$ backward solutions of (2.10) satisfying (2.11), and 1 forward solution of (1.1) with the obtained values of $u_q(k_1)$, $1 \leq q \leq n$, from the system (2.12), i.e., a total of $(r-1)(n+1) + 1$ solutions of n th order systems. In particular, if $r=2$ then we need $(n+2)$ solutions. Similar to the backward-forward process we have the forward-backward process. For this we solve (2.1) forward once for each $u_q(k_i)$, $1 \leq i \leq r-1$, appearing in (1.2) with the conditions

$$v_{-q}^{p(i)}(k_j) = \alpha_{pq}^1; \quad 1 \leq i \leq r-1, \quad 1 \leq p, q \leq n, \quad (2.13)$$

where $v_{-q}^{p(i)}(k_j)$ is the q th component at k_j for the p th forward solution.

Substituting (2.13) into the adjoint identity (2.6) with $k_0 = k_r$, we obtain

$$\sum_{q=1}^n \alpha_{pq}^i u_q(k_j) - \sum_{q=1}^n v_q^{p(i)}(k_r) u_q(k_r) = - \sum_{t=k_j+1}^{k_r} \sum_{q=1}^n v_q^{p(i)}(t) b_q(t-1), \quad 1 \leq i \leq r-1. \quad (2.14)$$

Summing $(r-1)$ equations (2.14) and making use of (1.2), we get

$$\sum_{q=1}^n [\alpha_{pq}^r + \sum_{i=1}^{r-1} v_{-q}^{p(i)}(k_r)] u_q(k_r) = l_p + \sum_{i=1}^{r-1} \sum_{t=k_i+1}^{k_r} \sum_{q=1}^n v_{-q}^{p(i)}(t) b_q(t-1), \quad 1 \leq p \leq n. \quad (2.15)$$

We introduce

$$w_{-p(i)}(k) = \sum_{t=k_1+1}^k \sum_{q=1}^n v_{-q}^{p(i)}(t) b_q(t-1); \quad 1 \leq p \leq n, 1 \leq i \leq r-1,$$

which is equivalent to solving

$$w_{-p(i)}(k) = - \sum_{q=1}^n v_{-q}^{p(i)}(k+1) b_q(k) + w_{-p(i)}(k+1) \quad (2.16)$$

$$w_{-p(i)}(k_j) = 0; \quad 1 \leq p \leq n, 1 \leq i \leq r-1. \quad (2.17)$$

Thus, the system (2.15) is the same as

$$\sum_{q=1}^n [\alpha_{pq}^r + \sum_{i=1}^{r-1} v_{-q}^{p(i)}(k_r)] u_q(k_r) = l_p + \sum_{i=1}^{r-1} w_{-p(i)}(k_r), \quad 1 \leq p \leq n. \quad (2.18)$$

The solution of the problem (1.1) and (1.2) is obtained by solving backward the system (1.1) with the obtained values of $u_q(k_r)$, $1 \leq q \leq n$ from the system (2.18).

Next we shall consider the system (1.1) together with the implicit separated conditions (1.3). We compute $(n - \beta_1)$ solutions of (2.1) backward with the conditions

$$v_q^{i(s_i)}(k_i) = \alpha_{i(s_i),q}; \quad 2 \leq i \leq r, 1 \leq s_i \leq \beta_i, 1 \leq q \leq n, \quad (2.19)$$

where $v_q^{i(s_i)}(k_i)$ is the q th component at k_i for the s_i th backward solution.

Substituting (2.19) into (2.5) with $k_0 = k_1$ and using (1.3), we obtain

$$\sum_{q=1}^n v_q^{i(s_i)}(k_1) u_q(k_1) = l_{i,i(s_i)} - \sum_{t=k_1+1}^{k_i} \sum_{q=1}^n v_q^{i(s_i)}(t) b_q(t-1); \quad (2.20)$$

$$2 \leq i \leq r, 1 \leq s_i \leq \beta_i.$$

We introduce

$$w_{i(s_i)}(k) = - \sum_{t=k+1}^{k_i} \sum_{q=1}^n v_q^{i(s_i)}(t) b_q(t-1); \quad 2 \leq i \leq r, 1 \leq s_i \leq \beta_i$$

which is equivalent to solving

$$w_{i(s_i)}(k) = - \sum_{q=1}^n v_q^{i(s_i)}(k+1) b_q(k) + w_{i(s_i)}(k+1) \quad (2.21)$$

$$w_{i(s_i)}(k_i) = 0; \quad 2 \leq i \leq r, \quad 1 \leq s_i \leq \beta_i. \quad (2.22)$$

Thus, the system (2.20) can be written as

$$\sum_{q=1}^n v_q^{i(s_i)}(k_1) u_q(k_1) = l_{i,i(s_i)} + w_{i(s_i)}(k_1); \quad 2 \leq i \leq r, \quad 1 \leq s_i \leq \beta_i. \quad (2.23)$$

System (2.23) together with (1.3) for $i = 1$, i.e.,

$$\sum_{q=1}^n \alpha_{1(s_1), q} u_q(k_1) = l_{1,1(s_1)}, \quad 1 \leq s_1 \leq \beta_1, \quad (2.24)$$

forms a system of n equations in n unknowns $u_q(k_1)$, $1 \leq q \leq n$. The solution of (1.1) and (1.3) is obtained by solving forward the system (1.1) with these values of $u_q(k_1)$, $1 \leq q \leq n$.

In practice we couple the adjoint system (2.1) with the equation (2.21) and solve this system of $(n+1)$ equations from the point k_i , $2 \leq i \leq r$ to k_1 with the conditions (2.19) and (2.22).

Similarly, in the forward-backward process for (1.1) and (1.3), the unknowns $u_q(k_r)$, $1 \leq q \leq n$ are computed from the system

$$\sum_{q=1}^n v_q^{i(s_i)}(k_r) u_q(k_r) = l_{i,i(s_i)} + w_{-i(s_i)}(k_r); \quad 1 \leq i \leq r-1, \quad 1 \leq s_i \leq \beta_i, \quad (2.25)$$

$$\sum_{q=1}^n \alpha_{r(s_r), q} u_q(k_r) = l_{r,r(s_r)}, \quad 1 \leq s_r \leq \beta_r, \quad (2.26)$$

where $v_q^{i(s_i)}(k)$ is the q th component of the s_i th forward solution from the point k_i of the adjoint system (2.1) satisfying

$$v_{-q}^{i(s_i)}(k_i) = \alpha_{i(s_i), q}; \quad 1 \leq i \leq r-1, \quad 1 \leq s_i \leq \beta_i, \quad 1 \leq q \leq n \quad (2.27)$$

and $w_{-i(s_i)}(k)$ is the forward solution of the initial value problem

$$w_{-i(s_i)}(k) = - \sum_{q=1}^n v_q^{i(s_i)}(k+1) b_q(k) + w_{-i(s_i)}(k+1) \quad (2.28)$$

$$w_{-i(s_i)}(k_i) = 0; \quad 1 \leq i \leq r-1, \quad 1 \leq s_i \leq \beta_i. \quad (2.29)$$

The solution of (1.1) and (1.3) is obtained by solving backward the system (1.1) with the obtained values of $u_q(k_r)$, $1 \leq q < r$.

3. NUMERICAL EXAMPLES

We shall construct an approximate solution of the continuous boundary value problem

$$\begin{aligned} y'' &= f(t)y + g(t) \\ y(\alpha) &= A, y(\beta) = B \end{aligned} \quad (3.1)$$

by employing the discrete variable method due to Nomerov. Let $h = \frac{\beta - \alpha}{K+1}$, $t_k = \alpha + kh$, $k \in N(0, K+1)$, $f_k = f(t_k)$, $g_k = g(t_k)$, and $u(k)$ be the approximation to the true solution of $y(t)$ at t_k , satisfies the second order difference equation,

$$\begin{aligned} (-1 + \frac{1}{12}h^2f_{k-1})u(k-1) + (2 + \frac{10}{12}h^2f_k)u(k) + (-1 + \frac{1}{12}h^2f_{k+1})u(k+1) \\ = -\frac{1}{12}h^2(g_{k-1} + 10g_k + g_{k+1}), k \in N(1, K) \end{aligned} \quad (3.2)$$

together with the boundary conditions

$$u(0) = A, u(K+1) = B. \quad (3.3)$$

Theorem 3.1. (see [12]) *The discrete boundary value problem (3.2) and (3.3) has a unique solution provided*

(a) $f(t) \geq 0$ on $[\alpha, \beta]$ and $\frac{1}{12}h^2f_k < 1$, $k \in N(0, K+1)$; or

$$(b) \frac{1}{12}h^2 \frac{10+2\cos\frac{\pi}{K+1}}{4\sin^2\frac{\pi}{2K+1}} \max_{\alpha \leq t \leq \beta} |f(t)| \leq 1. \quad [12]$$

If $\frac{1}{12}h^2 \max_{\alpha \leq t \leq \beta} f(t) < 1$, then in system form the problem (3.2) and (3.3) can be written

as

$$u_1(k+1) = u_2(k) \quad (3.4)$$

$$u_2(k+1) = -\frac{c_0(k)}{c_2(k)}u_1(k) + \frac{c_1(k)}{c_2(k)}u_2(k) + d(k), k \in N(0, K-1)$$

$$u_1(0) = A, u_2(K) = B \quad (3.5)$$

where $c_0(k) = 1 - \frac{1}{12}h^2 f_k$, $c_1(k) = 2 + \frac{10}{12}h^2 f_{k+1}$, $c_2(k) = 1 - \frac{1}{12}h^2 f_{k+2}$ and $d(k) = \frac{1}{c_2(k)} \times \frac{1}{12}h^2 [g_k + 10g_{k+1} + g_{k+2}]$.

Applying the backward-forward process for the boundary value problem (3.4) and (3.5), we note that (2.1), (2.21), (2.19) and (2.22) reduce to

$$\begin{aligned} v_1^2(k) &= -\frac{c_0(k)}{c_2(k)} v_2^2(k+1) \\ v_2^2(k) &= v_1^2(k+1) + \frac{c_1(k)}{c_2(k)} v_2^2(k+1) \end{aligned} \tag{3.6}$$

$$w_2(k) = -v_2^2(k+1) d(k) + w_2(k+1)$$

$$v_1^2(K) = 0, v_2^2(K) = 1, w_2(K) = 0. \tag{3.7}$$

Further, the system (2.23) and (2.24) takes the form

$$v_1^2(0)u_1(0) + v_2^2(0)u_2(0) = B + w_2(0)$$

$$u_1(0) = A,$$

which easily determines

$$u_1(0) = A, u_2(0) = \frac{B + w_2(0) - v_1^2(0)A}{v_2^2(0)}. \tag{3.8}$$

The solution of (3.4) and (3.5) is obtained by recursing forward the system (3.4) with the initial values (3.8).

Similarly, applying the forward-backward process we find that (2.1), (2.28), (2.27) and (2.29) reduce to

$$v_{-1}^1(k+1) = v_{-2}^1(k) + \frac{c_1(k)}{c_0(k)} v_{-1}^1(k) \tag{3.9}$$

$$v_{-2}^1(k+1) = -\frac{c_2(k)}{c_0(k)} v_{-1}^1(k)$$

$$w_{-1}(k+1) = w_{-1}(k) + v_{-2}^1(k+1)d(k) \tag{3.10}$$

$$v_{-1}^1(0) = 1, v_{-2}^1(0) = 0, w_{-1}(0).$$

Further, the system (2.25) and (2.26) becomes

$$v_{-1}^1(K)u_1(K) + v_{-2}^1(K)u_2(K) = A + w_{-1}(K)$$

$$u_2(K) = B$$

which gives

$$u_1(K) = \frac{A + w_{-1}(K) - v_{-2}^1(K)B}{v_1^1(K)}, \quad u_2(K) = B. \quad (3.11)$$

The solution of (3.4) and (3.5) is obtained by recursing backward the system (3.4) with the final values (3.11).

Example 3.1. For the discrete analogue (3.4) and (3.5) of the boundary value problem

$$y'' = \frac{2}{t^2} y(t) - \frac{1}{t}; \quad y(2) = 0, \quad y(3) = 0,$$

both the methods discussed in this section work equally well. The errors obtained, as calculated from the exact solution $y(t) = \frac{1}{36} (19t - 5t^2 - \frac{36}{t})$ and approximate solution $u_1(k)$ with $h = \frac{1}{256}$, are presented in Table 1.

Table 1

t	Backward-Forward Method	Forward-Backward Method
2.000	0.00000000D 00	0.64884418D-14
2.125	0.10546078D-12	0.10817389D-12
2.250	0.15874801D-12	0.15646512D-12
2.375	0.17716384D-12	0.16960044D-12
2.500	0.16946167D-12	0.15705492D-12
2.625	0.14502982D-12	0.12977119D-12
2.750	0.10855900D-12	0.93355879D-13
2.875	0.65239480D-12	0.49092674D-13
3.000	0.16924973D-13	0.00000000D-00

Example 3.2. For the discrete analogue (3.4) and (3.5) of the boundary value problem

$$y'' = 400y; \quad y(0) = 1, \quad y(5) = e^{-100},$$

we apply both the methods. The errors obtained, as calculated from the exact solution $y(t) = e^{-20t}$ and approximate solution $u_1(k)$ with $h = \frac{5}{1024}$, are presented in Table 2.

Table 2

t	Backward-Forward Method	Forward-Backward Method
0.0000		0.16875390D-13
0.3125		0.22852524D-08
0.6250		0.88231446D-11
0.9375		0.25548999D-13
1.2500		0.65761522D-16
1.5625		0.15868691D-18
1.8750		0.36760514D-21
2.1875		0.82791852D-24
2.5000	Fails	0.18265803D-26
2.1825		0.39668934D-29
3.1250		0.85087791D-32
3.4375		0.18068378D-34
3.7500		0.38051077D-37
4.0625		0.79577134D-40
4.3750		0.16543683D-42
4.6875		0.34217867D-45
5.0000		0.00000000D-00

Example 3.3. The boundary value problem

$$y'' = (2m + 1 + t^2)y; \quad y(0) = \beta, \quad y(\infty) = 0, \quad (3.12)$$

where $m \geq 0$ and β are known constants, is known as Holt's problem. This problem is a typical example wherein usual shooting methods fail [2,5,7,9-11]. Replacing the boundary condition $y(\infty) = 0$ by $y(T) = 0$ (T finite) Holt [8] used finite difference methods (however, for $m = 0$, $\beta = 1$, $T = 12$; $m = 1$, $\beta = \pi^{-1/2}$, $T = 8$; $m = 2$, $\beta = \frac{1}{4}$, $T = 8$, the results are unsatisfactory [8,11]), whereas Osborne [9] used a multiple shooting method and Roberts and Shipman [10] used a multi-point approach. In [2] we have formulated a new shooting method which gives accurate solutions of (3.12) for several different values of m and β up to $T = 18$. (This value of T has been chosen in view of restricted computer capabilities.) For the same and several other values of m and β accurate solutions of (3.12) up to $T = 18$ have also been obtained in [5]. Here the error estimates in the solution of (3.12) when approximating $y(\infty) = 0$ by an appropriate boundary condition at T are also available. For the discrete analogue (3.4) and (3.5) of (3.12) (replacing $y(\infty) = 0$ by $y(18) = 0$) with $m = 0$, $\beta = 1$ and $h = \frac{1}{60}$, we apply both the methods of this section. The numerical solution $u_1(k)$ is shown in Table 3.

Table 3

t	Backward-Forward Method	Forward-Backward Method
0.0		0.10000000D-01
1.0		0.25934255D-00
2.0		0.34564046D-01
3.0		0.19885232D-02
4.0		0.45958196D-04
5.0		0.41255769D-06
6.0		0.14129840D-08
7.0		0.18272052D-11
8.0		0.88629857D-15
9.0	Fails	0.16054990D-18
10.0		0.10827935D-22
11.0		0.27128207D-27
12.0		0.25206584D-32
13.0		0.86750117D-38
14.0		0.11047255D-43
15.0		0.52013857D-50
16.0		0.90485908D-57
17.0		0.58130757D-64
18.0		0.00000000D-00

Example 3.4. For the discrete analogue (3.4) and (3.5) of the boundary value problem

$$y'' = (\sin 2t)y + \cos 2t; \quad y(-1) = y(1) = 0,$$

we apply both the methods. The numerical solution $u_1(k)$ for $h = \frac{1}{540}$ is shown in Table 4.

Table 4

t	Backward-Forward Method	Forward-Backward Method
-1.0	0.00000000D-00	0.22781405D-13
-0.9	-0.54219742D-01	-0.54219742D-01
-0.8	-0.11017210D-00	-0.11017210D-00
-0.7	-0.16531803D-00	-0.16531803D-00
-0.6	-0.21715198D-00	-0.21715198D-00
-0.5	-0.26336748D-00	-0.26336748D-00
-0.4	-0.30200211D-00	-0.30200211D-00
-0.3	-0.33154697D-00	-0.33154697D-00
-0.2	-0.35101159D-00	-0.35101159D-00
-0.1	-0.35994166D-00	-0.35994166D-00
0.0	-0.35839390D-00	-0.35839390D-00
0.1	-0.34687723D-00	-0.34687723D-00
0.2	-0.32627260D-00	-0.32627260D-00
0.3	-0.29774424D-00	-0.29774424D-00
0.4	-0.26265376D-00	-0.26265376D-00
0.5	-0.22248554D-00	-0.22248554D-00
0.6	-0.17878818D-00	-0.17878818D-00
0.7	-0.13313322D-00	-0.13313322D-00
0.8	-0.87088932D-01	-0.87088932D-01
0.9	-0.42204687D-01	-0.42204687D-01
1.0	-0.58914799D-12	0.00000000D-00

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