Research Article

A Stochastic Two Species Competition Model: Nonequilibrium Fluctuation and Stability

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The object of this paper is to study the stability behaviours of the deterministic and stochastic versions of a two-species symmetric competition model. The logistic parameters of the competitive species are perturbed by colored noises or Ornstein-Uhlenbeck processes due to random environment. The Fokker-Planck equation has been used to obtain probability density functions. Here, we have also discussed the relationship between stability behaviours of this model in a deterministic environment and the corresponding model in a stochastic environment.

1. Introduction

In recent years, scientists have become increasingly aware of the fact that most natural phenomena do not follow strictly deterministic laws but rather oscillate randomly about some average behaviour. This is especially true in the ecological models where environmental influences should be taken as stochastic. Many models in theoretical ecology take the Volterra-Lotka model of interacting species as a starting point. In an improved analysis, the influence of the environment has to be taken into account. This is often done by arbitrarily augmenting the deterministic equations with stochastic terms or taking the environmental parameters as time dependent and rapidly varying. In both cases, the corresponding stochastic properties have to be postulated [1]. In 1995, Renshaw mentioned that the most natural phenomena do not follow strictly deterministic laws but rather oscillate randomly about some average so that the deterministic equilibrium is not an absolutely fixed state; instead, it is a "fuzzy" value around which the biological system fluctuates. In fact, randomness or stochasticity plays a vital role in the structure and function of biological systems. The environmental factors are time dependent, randomly varying and should be taken as stochastic. In ecology, we have two types of stochasticity, namely, the demographic

stochasticity and the environmental stochasticity [2, 3]. Both types of stochasticity play a significant part in the realistic dynamical modelling of ecosystems. Several systematic procedures have been developed to obtain the stationary probability density function for the response of general single-degree-of-freedom nonlinear oscillators and nonlinear dynamical systems under parametric and external Gaussian white noise excitations [4–7].

In this paper, we have studied the relationship between stability behaviours of a two-species symmetric competition model in which environmental parameters are prescribed constant (deterministic environment) and the corresponding model in which these parameters have an element of random fluctuation (stochastic environment), where it is assumed that the fluctuations in the environment manifest themselves as random fluctuations in the logistic parameters about some mean value, as they usually do in nature [8, 9].

2. Two-Species Symmetric Competition: Stochastic Differential Equations

Two or more species-populations compete for the same limited food source or in some way the growth of each is inhibited by members both of its own and of the other species. For example, competition may be for territory which is directly related to food resources. Some interesting phenomena have been observed from the study of practical competition models [10, 11]. Here, we will discuss a competition model which demonstrates a fairly general principle which is observed to hold in nature. Assume that the two-species symmetric competition model in a deterministic environment satisfies the following deterministic differential equations:

$$\frac{1}{N_1(t)}\frac{dN_1}{dt} = K_0 - N_1(t) - \alpha N_2(t),$$

$$\frac{1}{N_2(t)}\frac{dN_2}{dt} = K_0 - N_2(t) - \alpha N_1(t), \tag{2.1}$$

where $N_1(t)$, $N_2(t)$ denote the number or density of individuals of two competing species. Thus, it is assumed that the *per capita* growth rate of each population at any instant is a linear function of the sizes of the two competing populations at that instant. Each population would grow logistically if it were alone with a constant environmental parameter $K_0(> 0)$ and a parameter $\alpha(> 0)$ which measures the (symmetric) competition between the two species (i.e., it measures the degree to which the presence of one species affects the growth of the other species), we assume that $\alpha < 1$.

Let us rewrite the system of differential equations (2.1) in the following form:

$$\frac{dN_1}{dt} = N_1(t)\{K_0 - N_1(t) - \alpha N_2(t)\},$$

$$\frac{dN_2}{dt} = N_2(t)\{K_0 - N_2(t) - \alpha N_1(t)\}. \tag{2.2}$$

The deterministic equilibrium populations $N_1^\star$, $N_2^\star$ are given by

$$N_1^\star = N_2^\star = \frac{K_0}{1+\alpha} = N^\star (\text{say}). \tag{2.3}$$

The 2×2 interaction matrix in the neighbourhood of this equilibrium is

$$A = \begin{bmatrix} -N^\star & -\alpha N^\star \\ -\alpha N^\star & -N^\star \end{bmatrix}. \tag{2.4}$$

The two eigenvalues of this interaction matrix are $\lambda_1 = -K_0$, $\lambda_2 = -K_0((1-\alpha)/(1+\alpha))$, and the deterministic stability criterion is satisfied since $\alpha < 1$.

To take into account the random environment, we extend model (2.2) to the form of the following stochastic differential equations:

$$\frac{dN_1}{dt} = N_1(t)\{K_0 + u_1(t) - N_1(t) - \alpha N_2(t)\},$$
$$\frac{dN_2}{dt} = N_2(t)\{K_0 + u_2(t) - N_2(t) - \alpha N_1(t)\}, \tag{2.5}$$

where the fluctuations $u_i(t)$ are colored noises or Ornstein-Uhlenbeck processes which are more realistic noises than white noises. These are extremely useful to model rapidly fluctuating phenomena, because it can be seen by studying their spectra that thermal noises in electrical resistance and climate fluctuations, disregarding the periodicities of astronomical origin and so forth, are colored noises to a very good approximation. These examples support the usefulness of the colored noise idealization in applications to natural systems. The mathematical expectations and correlation functions of the processes $u_i(t)$ are given by

$$\langle u_i(t)\rangle = 0, \qquad \langle u_i(t_1)u_i(t_2)\rangle = \epsilon\delta_0 \exp(-\delta_0|t_1 - t_2|), \quad (\epsilon,\delta_0 > 0), \tag{2.6}$$

where ϵ, δ_0^{-1} are, respectively, the intensity and the correlation time of the colored noise and $\langle\cdot\rangle$ represents the average over the ensemble of the stochastic process. The $u_i(t)$ are the solutions of the stochastic differential equations [12]

$$\frac{du_i}{dt} = -\delta_0 u_i + \delta_0\sqrt{2\epsilon}\frac{dW_i}{dt}, \quad (i = 1,2), \tag{2.7}$$

where $\eta_i = dW_i/dt (i = 1,2)$ are standard white noises characterized by

$$\langle \eta_i(t)\rangle = 0, \qquad \langle \eta_i(t_1)\eta_i(t_2)\rangle = \delta(t_1 - t_2),$$
$$\langle \eta_i(t_1)\eta_j(t_2)\rangle = \frac{\alpha N^\star}{N^\star + \delta_0}\delta(t_1 - t_2), \quad (i \neq j), \tag{2.8}$$

where $\delta(t)$ denotes the Dirac delta function.

Let $y_i = \ln N_i$; therefore, $N_i = e^{y_i}$. Hence,

$$\frac{dy_i}{dt} = K_0 + u_i(t) - e^{y_i} - \alpha e^{y_j}, \quad (i = 1, 2; \; j = 3 - i). \tag{2.9}$$

Put $n_i = y_i - y^\star$; therefore, $y_i = n_i + y^\star$, where $y^\star = \ln N^\star$. Hence,

$$e^{y_i} = e^{n_i + y^\star} = e^{y^\star}\left\{1 + n_i + \frac{n_i^2}{2!} + \cdots\right\} \approx e^{y^\star}(1 + n_i) = N^\star(1 + n_i). \tag{2.10}$$

Therefore,

$$\frac{dn_i}{dt} = -N^\star(n_i + \alpha n_j) + u_i(t), \quad (i = 1, 2; \; j = 3 - i). \tag{2.11}$$

From (2.7) and (2.11), we have

$$\frac{d^2 n_i}{dt^2} + (N^\star + \delta_0)\frac{dn_i}{dt} + \alpha N^\star \frac{dn_j}{dt} + \delta_0 N^\star(n_i + \alpha n_j) = \delta_0 \sqrt{2\epsilon}\frac{dW_i}{dt}, \quad (i = 1, 2; \; j = 3 - i). \tag{2.12}$$

Put, $X_1 = n_1$, $X_2 = \dot{n}_1$, $X_3 = n_2$ and $X_4 = \dot{n}_2$.
    Therefore,

$$\dot{X}_{2i-1} = X_{2i}, \quad \dot{X}_{2i} = -h_1(X_1, X_2, X_3, X_4) + \delta_0\sqrt{2\epsilon}\frac{dW_i}{dt}, \quad (i = 1, 2), \tag{2.13}$$

where,

$$h_1 = (N^\star + \delta_0)X_2 + \alpha N^\star X_4 + \delta_0 N^\star(X_1 + \alpha X_3),$$

$$h_2 = (N^\star + \delta_0)X_4 + \alpha N^\star X_2 + \delta_0 N^\star(X_3 + \alpha X_1). \tag{2.14}$$

3. Stationary Probability Density

The reduced Fokker-Planck equation is given by [13]

$$\frac{\partial}{\partial x_i}\left[A_i(x)p_s(x)\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(x)p_s(x)\right] = 0, \quad (i, j = 1, 2, 3, 4), \tag{3.1}$$

where x_i are the state variables (representing the possible values) of X_i, p_s is the stationary probability density, the drift and diffusion coefficients are

$$A_{2i-1} = x_{2i}, \qquad A_{2i} = -h_i, \qquad B_{2i-1,j} = B_{i,2j-1} = 0,$$

$$B_{2i,2j} = 2\pi \sum_{l,s=1}^{2} k_{ls} f_{js} f_{il} \quad (i, j = 1, 2),$$

(3.2)

where

$$f_{11} = f_{22} = 1, \qquad f_{12} = f_{21} = 0, \qquad k_{11} = k_{22} = \frac{\delta_0^2 \epsilon}{\pi},$$

(3.3)

$$k_{12} = k_{21} = \frac{\epsilon K_0 \delta_0^2 \alpha}{\pi \{K_0 + \delta_0(1 + \alpha)\}}.$$

After some simplifications using the results of Cai and Lin [13], we have

$$p_s(x_1, x_3) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp\left\{-\frac{1}{2(1 - \rho^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2\rho\frac{x_1 x_3}{\sigma_1 \sigma_2} + \frac{x_3^2}{\sigma_2^2}\right)\right\},$$

(3.4)

$$(-\infty < x_1, x_3 < \infty),$$

where

$$\rho = -\alpha, \qquad \sigma_1^2 = \sigma_2^2 = \left(\frac{1 + \alpha}{1 - \alpha}\right)\frac{\epsilon \delta_0}{K_0\{K_0 + \delta_0(1 + \alpha)\}}$$

(3.5)

$$= \left(\frac{\epsilon}{\Lambda}\right)\frac{\delta_0}{\{K_0 + \delta_0(1 + \alpha)\}}, \qquad \Lambda = K_0\left(\frac{1 + \alpha}{1 - \alpha}\right).$$

By integrating over all values of x_3, we obtain

$$p_s(x_1) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\}, \quad (-\infty < x_1 < \infty).$$

(3.6)

Similarly, we have

$$p_s(x_3) = \frac{1}{\sigma_2 \sqrt{2\pi}} \exp\left\{-\frac{x_3^2}{2\sigma_2^2}\right\}, \quad (-\infty < x_3 < \infty).$$

(3.7)

4. Concluding Remarks

In this paper, we have studied the relationship between stability behaviours of a two-species symmetric competition model for interacting species, in which environmental parameters are prescribed constants (deterministic environment), and the corresponding model in which these parameters have an element of random fluctuation (stochastic environment). In the stochastic environment, the environmental parameter K_0 has been perturbed by colored noises or Ornstein-Uhlenbeck processes, which are more realistic noises than white noises.

Now, in the deterministic environment, the maximum of the eigenvalues of the 2×2 interaction matrix is $-\Lambda = -K_0((1 + \alpha)/(1 - \alpha))$. Therefore, the stability determining quantity is Λ, and the deterministic stability criterion is satisfied, since $\Lambda > 0$ (as $\alpha < 1$). In the stochastic environment, whose random fluctuation has intensity ϵ, the stability provided by the population interaction dynamics is again characterized by Λ. It is no longer enough that $\Lambda > 0$, for if $\epsilon \ll \Lambda$, population fluctuations are relatively small; in this case, the probability cloud persists for an appreciable time, and the environment is effectively deterministic. For $\epsilon < \Lambda$, but not much less, populations are likely to undergo significant fluctuations, even though they persist for long times. Finally, if $\epsilon \gg \Lambda$, population exhibit large fluctuations in this case, the interaction dynamics provides an ever-weaker stabilizing influence to offset the randomizing $\sigma_1^2 (= \sigma_2^2)$, which rapidly lead to extinction. These results are in good agreement with those of Samanta [14], Samanta and Maiti [15], and Maiti and Samanta [16].

The correlation time of the colored noise is $\tau_{\text{cor}} = \delta_0^{-1}$. The limiting process for $\tau_{\text{cor}} \to 0$ of the colored noise is the Gaussian white noise, and in this case we obtain the well-known results of May [8]. Now, for $\tau_{\text{cor}} \to \infty$, we see that the population fluctuations are very much small, and in this case, the environment is effectively deterministic.

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