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Research Article

The Rothe's Method to a Parabolic Integrodifferential Equation with a Nonclassical Boundary Conditions

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This paper is devoted to prove, in a nonclassical function space, the weak solvability of parabolic integrodifferential equations with a nonclassical boundary conditions. The investigation is made by means of approximation by the Rothes method which is based on a semidiscretization of the given problem with respect to the time variable.

1. Introduction

The purpose of this paper is to study the solvability of the following equation:

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)k'(s,v(x,s))ds + g(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{1.1}$$

with the initial condition

$$v(x,0) = V_0(x), \quad x \in (0,1), \tag{1.2}$$

and the integral conditions

$$\begin{aligned} &\int_0^1 v(x,t)dx = E(t), \quad t \in [0,T], \\ &\int_0^1 xv(x,t)dx = G(t), \quad t \in [0,T], \end{aligned} \tag{1.3}$$

where v is an unknown function, E, G, and V_0 are given functions supposed to be sufficiently regular, while k' and a are suitably defined functions satisfying certain conditions to be specified later and T is a positive constant.

Since 1930, various classical types of initial boundary value problems have been investigated by many authors using Rothe time-discretization method; see, for instance, the monographs by Rektorys [1] and Kačur [2] and references cited therein. The linear case of our problem, that is, $\int_0^t a(t-s)k'(s,v(x,s))ds = 0$, appears, for instance, in the modelling of the quasistatic flexure of a thermoelastic rod (see [3]) and has been studied, firstly, by the second author with a more general second-order parabolic equation or a 2m-parabolic equation in [3–5] by means of the energy-integrals method and, secondly, by the two authors via the Rothe method [6–8]. For other models, we refer the reader, for instance, to [9–12], and references therein.

The paper is organized as follows. In Section 2, we transform problem (1.1)–(1.3) to an equivalent one with homogeneous integral conditions, namely, problem (2.3). Then, we specify notations and assumptions on data before stating the precise sense of the desired solution. In Section 3, by the Rothe discretization in time method, we construct approximate solutions to problem (2.3). Some a priori estimates for the approximations are derived in Section 4, while Section 5 is devoted to establish the existence and uniqueness of the solution.

2. Preliminaries, Notation, and Main Result

It is convenient at the beginning to reduce problem (1.1)–(1.3) with inhomogeneous integral conditions to an equivalent one with homogeneous conditions. For this, we introduce a new unknown function u by setting

$$u(x,t) = v(x,t) - R(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{2.1}$$

where

$$R(x,t) = 6(2G(t) - E(t))x - 2(3G(t) - 2E(t)). \tag{2.2}$$

Then, the function u is seen to be the solution of the following problem:

$$\begin{gathered}
\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^t a(t-s)k(s,u(x,s))ds + f(x,t), \quad (x,t) \in (0,1) \times [0,T], \\
u(x,0) = U_0(x), \quad x \in (0,1), \\
\int_0^1 u(x,t)dx = 0, \quad t \in [0,T], \\
\int_0^1 xu(x,t)dx = 0, \quad t \in [0,T],
\end{gathered} \tag{2.3}$$

where

$$f(x,t) = g(x,t) - \frac{\partial R(x,t)}{\partial t},$$
$$U_0(x) = V_0(x) - R(x,0), \tag{2.4}$$
$$k(s,u(x,s)) = k'(s,u(x,s)) - R(x,t).$$

Hence, instead of looking for the function v, we search for the function u. The solution of problem (1.1)–(1.3) will be simply given by the formula $v(x,t) = u(x,t) + R(x,t)$.

We introduce the function spaces, which we need in our investigation. Let $L_2(0,1)$ and $L_2(0,T;L_2(0,1))$ be the standard function spaces. We denote by $C_0(0,1)$ the linear space of continuous functions with compact support in $(0,1)$. Since such functions are Lebesgue integrable, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,v)) = \int_0^1 \Im_x u \Im_x v \, dx, \tag{2.5}$$

where

$$\Im_x u = \int_0^x u(\zeta,\cdot) d\zeta. \tag{2.6}$$

The bilinear form (2.5) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2.1. We denote by $B_2^1(0,1)$ a completion of $C_0(0,1)$ for the scalar product (2.5), which is denoted by $(\cdot,\cdot)_{B_2^1(0,1)}$, called the Bouziani space or the space of square integrable primitive functions on $(0,1)$. By the norm of function u from $B_2^1(0,1)$, we understand the nonnegative number

$$\|u\|_{B_2^1(0,1)} = \sqrt{(u,u)_{B_2^1(0,1)}} = \|\Im_x u\|, \tag{2.7}$$

where $\|v\|$ denotes the norm of v in $L_2(0,1)$.

For $u \in L_2(0,1)$, we have the elementary inequality

$$\|u\|_{B_2^1(0,1)} \le \frac{1}{\sqrt{2}} \|u\|. \tag{2.8}$$

We denote by $L_2(0,T;B_2^1(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,v)_{L_2(0,T;B_2^1(0,1))} = \int_0^T (u(\cdot,t),v(\cdot,t))_{B_2^1(0,1)} dt. \tag{2.9}$$

Since the space $B_2^1(0,1)$ is a Hilbert space, it can be shown that $L_2(0,T;B_2^1(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\le\tau\le T}\|u(\cdot,\tau)\|_{B_2^1(0,1)} \tag{2.10}$$

is denoted $C(0,T;B_2^1(0,1))$. Let V be the set which we define as follows:

$$V=\left\{v\in L_2(0,1);\int_0^1 v(x)dx=\int_0^1 xv(x)dx=0\right\}. \tag{2.11}$$

Since V is the null space of the continuous linear mapping l: $L_2(0,1)\to\mathbb{R}^2$, $\varphi\to l(\varphi)=(\int_0^1\varphi(x)dx,\int_0^1 x\varphi(x)dx)$, it is a closed linear subspace of $L_2(0,1)$, consequently V is a Hilbert space endowed with the inner product $(\cdot,\cdot)$. Strong or weak convergence is denoted by $\to$ or $\rightharpoonup$, respectively. The letter C will stand for a generic positive constant which may be different in the same discussion.

Lemma 2.2 (Gronwall's lemma). (a_1) *Let $x(t)\ge 0$, $h(t)$, $y(t)$ be real integrable functions on the interval $[a,b]$. If*

$$y(t)\le h(t)+\int_a^t x(s)y(s)ds,\quad \forall t\in(a,b), \tag{2.12}$$

then

$$y(t)\le h(t)+\int_a^t h(s)x(s)\exp\left(\int_a^t x(\tau)d\tau\right)ds,\quad \forall t\in(0,T). \tag{2.13}$$

In particular, if $x(t)\equiv C$ is a constant and $h(t)$ is nondecreasing, then

$$y(t)\le h(t)e^{c(t-a)},\quad \forall t\in(0,T). \tag{2.14}$$

(a_2) *Let $\{a_i\}_i$ be a sequence of real nonnegative numbers satisfying*

$$a_i\le A+Bh\sum_{k=1}^{i-1}a_k,\quad \forall i=1,2,\ldots, \tag{2.15}$$

where A, B, and h are positive constants, such that $Bh<1$. Then

$$a_i\le A\exp[B(i-1)h], \tag{2.16}$$

takes place for all $i=1,2,\ldots$.

In the sequel, we make the following assumptions.

(H_1) Functions $f : [0,T] \to L_2(0,1)$ and $a : [0,T] \to \mathbb{R}$ are Lipschitz continuous, that is,

$$\begin{aligned} &\exists l_1 \in \mathbb{R}^+; \quad \left\| f(t) - f(t') \right\| \le l_1 \left| t - t' \right|, \quad \forall t \in [0,T], \\ &\exists l_2 \in \mathbb{R}^+; \quad \left| a(t) - a(t') \right| \le l_2 \left| t - t' \right|, \quad \forall t \in [0,T]. \end{aligned} \tag{2.17}$$

(H_2) The mapping $k : [0,T] \times V \to L_2(0,1)$ is Lipschitz continuous in both variables, that is,

$$\exists l_3 \in \mathbb{R}^+; \quad \left\| k(t,u) - k(t',u') \right\| \le l_3 \left[\left| t - t' \right| + \left\| u - u' \right\| \right], \tag{2.18}$$

for all $t, t' \in I$, $u, u' \in V$, and satisfies

$$\exists l_4, l_5 \in \mathbb{R}^+; \quad \| k(t,u) \|_{B_2^1(0,1)} \le l_4 \| u \|_{B_2^1(0,1)} + l_5, \tag{2.19}$$

for all $t \in I$ and all $u \in V$, where l_4 and l_5 are positive constants.

(H_3) $U_0 \in H^2(0,1)$ and

$$\int_0^1 U_0(x)dx = \int_0^1 x U_0(x)dx = 0. \tag{2.20}$$

We will be concerned with a weak solution in the following sense.

Definition 2.3. A function $u : I \to L_2(0,1)$ is called a weak solution to problem (2.3) if the following conditions are satisfied:

(i) $u \in L^\infty(I,V) \cap C(I, B_2^1(0,1))$,

(ii) u is strongly differentiable a.e. in I and $du/dt \in L^\infty(I, B_2^1(0,1))$,

(iii) $u(0) = U_0$ in V,

(iv) the identity

$$\begin{aligned} &\left(\frac{du}{dt}(t), v \right)_{B_2^1(0,1)} + (u(t), v) \\ &\quad = \left(\int_0^t a(t-s) k(s, u(s)) ds, v \right)_{B_2^1(0,1)} + \left(f(t), v \right)_{B_2^1(0,1)} \end{aligned} \tag{2.21}$$

holds for all $v \in V$ and a.e. $t \in [0,T]$.

To close this section, we announce the main result of the paper.

Theorem 2.4. *Under assumptions* (H_1)–(H_3), *problem* (2.3) *admits a unique weak solution* u, *in the sense of Definition (2.3).*

3. Construction of an Approximate Solution

In order to solve problem (2.3) by the Rothe method, we proceed as follows. Let n be a positive integer, we divide the time interval $I = [0, T]$ into n subintervals $I_j^n := [t_{j-1}^n, t_j^n]$, $j = 1, \ldots, n$, where $t_j^n := jh_n$ and $h_n := T/n$. Then, for each $n \geq 1$, problem (2.3) may be approximated by the following recurrent sequence of time-discretized problems. Successively, for $j = 1, \ldots, n$, we look for functions $u_j^n \in V$ such that

$$\frac{u_j^n - u_{j-1}^n}{h_n} - \frac{d^2 u_j^n}{dx^2} = h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n\right) + f_j^n, \tag{3.1}$$

$$\int_0^1 u_j^n(x)dx = 0, \tag{3.2}$$

$$\int_0^1 x u_j^n(x)dx = 0, \tag{3.3}$$

starting from

$$u_0^n = U_0, \qquad \delta u_0^n = \frac{d^2}{dx^2} U_0 + f(0), \tag{3.4}$$

where $u_j^n(x) := u(x, t_j^n)$, $\delta u_j^n := (u_j^n - u_{j-1}^n)/h_n$, $f_j^n(x) := f(x, t_j^n)$. For this, multiplying for all $j = 1, \ldots, n$, (3.1) by $\Im_x^2 v := \int_0^x (\int_0^\xi v(\tau)d\tau)d\xi$ and integrating over $(0,1)$, we get

$$\int_0^1 \delta u_j^n(x) \Im_x^2 v\, dx - \int_0^1 \frac{d^2 u_j^n}{dx^2}(x) \Im_x^2 v\, dx = h_n \int_0^1 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n\right) \Im_x^2 v\, dx + \int_0^1 f_j^n \Im_x^2 v\, dx. \tag{3.5}$$

Noting that, using a standard integration by parts, we have

$$\Im_1^2 v = \int_0^1 (1 - \xi) v(\xi) d\xi = \int_0^1 v(\xi) d\xi - \int_0^1 \xi v(\xi) d\xi = 0, \quad \forall v \in V. \tag{3.6}$$

Carrying out some integrations by parts and invoking (3.6), we obtain for each term in (3.5)

$$\begin{gathered}
\int_0^1 \delta u_j^n \Im_x^2 v\, dx = -\left(\delta u_j^n, v\right)_{B_2^1(0,1)}, \\
\int_0^1 \frac{d^2 u_j^n}{dx^2}(x) \Im_x^2 v\, dx = \left(u_j^n, v\right), \qquad (3.7) \\
h_n \int_0^1 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n(x)\right) \Im_x^2 v\, dx = -h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) \left(k\left(t_i^n, u_i^n\right), v\right)_{B_2^1(0,1)},
\end{gathered}$$

and for the last one

$$\int_0^1 f_j^n(x)\Im_x^2 v(x)dx = -\left(f_j^n, v\right)_{B_2^1(0,1)}. \tag{3.8}$$

By virtue of (3.7) and (3.8), (3.5) becomes

$$\left(\delta u_j^n, v\right)_{B_2^1(0,1)} + \left(u_j^n, v\right) = h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)(k(t_i^n, u_i^n), v)_{B_2^1(0,1)} + \left(f_j^n, v\right)_{B_2^1(0,1)}, \tag{3.9}$$

or

$$\begin{aligned} &\left(u_j^n, v\right)_{B_2^1(0,1)} + h_n\left(u_j^n, v\right) \\ &\quad = h_n^2 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)(k(t_i^n, u_i^n), v)_{B_2^1(0,1)} + h_n\left(f_j^n, v\right)_{B_2^1(0,1)} + \left(u_{j-1}^n, v\right)_{B_2^1(0,1)}. \end{aligned} \tag{3.10}$$

Let $\eta(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $L_j(\cdot) : V \to \mathbb{R}$ be two functions defined by

$$\begin{aligned} \eta(u, v) &= (u, v)_{B_2^1(0,1)} + h_n(u, v), \\ L_j(v) &= h_n^2 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)(k(t_i^n, u_i^n), v)_{B_2^1(0,1)} + h_n\left(f_j^n, v\right)_{B_2^1(0,1)} + \left(u_{j-1}^n, v\right)_{B_2^1(0,1)}. \end{aligned} \tag{3.11}$$

It is easy to see that the bilinear form $\eta(\cdot,\cdot)$ is *continuous* on V and V-*elliptic*, and the form $L_j(\cdot)$ is *continuous* for each $j = 1, \ldots, n$. Then, Lax-Milgram lemma guarantees the existence and uniqueness of u_j^n, for all $j = 1, \ldots, n$.

4. A Priori Estimates

Lemma 4.1. *There exists $C > 0$ such that, for all $n \geq 1$ and all $j = 1, \ldots, n$, the solution u_j of the discretized problem* (3.1)–(3.4) *satisfies the estimates*

$$\left\| u_j^n \right\| \leq C, \tag{4.1}$$

$$\left\| \delta u_j^n \right\|_{B_2^1(0,1)} \leq C. \tag{4.2}$$

Proof. Testing the difference $(3.9)_{j-1}$-$(3.9)_j$ with $v = \delta u_j^n (\in V)$, taking into account assumptions (H_1)–(H_3) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left\|\delta u_j^n\right\|_{B_2^1(0,1)} + \left\|u_j^n - u_{j-1}^n\right\|_{B_2^1(0,1)} \\ &\quad \leq \left\|\delta u_{j-1}^n\right\|_{B_2^1(0,1)} + \frac{C_1}{3} h_n^2 \sum_{i=0}^{j-2} \|u_i^n\|_{B_2^1(0,1)} + \frac{C_1}{3} h_n + \frac{C_1}{3} h_n \left\|u_{j-1}^n\right\|_{B_2^1(0,1)},\end{aligned} \tag{4.3}$$

where

$$C_1 := 3\max\{l_2\zeta, Tl_2\zeta + M_1\zeta + l_1\}, \quad M_1 := \max_{t\in I}|a(t)|, \quad \zeta := \max\{l_4, l_5\}. \tag{4.4}$$

Multiplying the left-hand side of the last inequality with $(1 - (C_1/3)h_n)(< 1)$ and adding the terme

$$\frac{2}{3} C_1 h_n \left[\left\|u_j^n - u_{j-1}^n\right\|_{B_2^1(0,1)} - \left\|\delta u_j^n\right\|_{B_2^1(0,1)}\right] (< 0), \tag{4.5}$$

we get

$$\begin{aligned}&(1 - C_1 h_n)\left[\left\|\delta u_j^n\right\|_{B_2^1(0,1)} + \left\|u_j^n\right\|_{B_2^1(0,1)}\right] \\ &\quad \leq \left[\left\|u_{j-1}^n\right\|_{B_2^1(0,1)} + \left\|\delta u_{j-1}^n\right\|_{B_2^1(0,1)}\right] + C_1 h_n^2 \sum_{i=0}^{j-2} \|u_i^n\|_{B_2^1(0,1)} + C_1 h_n.\end{aligned} \tag{4.6}$$

Applying the last inequality recursively, it follows that

$$\begin{aligned}&(1 - C_1 h_n)^j \left[\left\|\delta u_j^n\right\|_{B_2^1(0,1)} + \left\|u_j^n\right\|_{B_2^1(0,1)}\right] \\ &\quad \leq \left[\|u_0^n\|_{B_2^1(0,1)} + \|\delta u_0^n\|_{B_2^1(0,1)} + C_1 T\right] + TC_1 h_n \sum_{i=0}^{j-2} \|u_i^n\|_{B_2^1(0,1)},\end{aligned} \tag{4.7}$$

or, by virtue of Lemma 2.2, there exists $n_0 \in \mathbb{N}^*$ such that

$$\left\|\delta u_j^n\right\|_{B_2^1(0,1)} + \left\|u_j^n\right\|_{B_2^1(0,1)} \leq C_2, \quad \forall n \geq n_0, \tag{4.8}$$

where

$$\begin{aligned}C_2 :=& \left(\exp(TC_1) + 1\right)\left[\|\delta u_0^n\|_{B_2^1(0,1)} + \|u_0^n\|_{B_2^1(0,1)} + TC_1\right] \\ &\times \exp\left[\left(\exp(TC_1) + 1\right)TC_1\right],\end{aligned} \tag{4.9}$$

and so our proof is complete. □

We address now the question of convergence and existence.

5. Convergence and Existence

Now let us introduce the Rothe function $u^n(t) : I \to V$ obtained from the functions u_j by piecewise linear interpolation with respect to time

$$u^n(t) = u^n_{j-1} + \delta u^n_j \left(t - t^n_{j-1}\right), \quad \text{in } I^n_j, \tag{5.1}$$

as well the step functions $\widetilde{u}_n(t)$, $\widehat{u}_n(t)$, $\widetilde{f}^n(t)$, and $\widetilde{k}(t, \widetilde{u}_n(t))$ defined as follows:

$$\widetilde{u}_n(t) = \begin{cases} u^n_0, & \text{for } t = 0, \\ u^n_j, & \text{in } \widetilde{I}^n_j := \left(t^n_{j-1}, t^n_j\right], \end{cases} \qquad \widehat{u}_n(t) = \begin{cases} u^n_0, & \text{for } t = 0, \\ u^n_{j-1}, & \text{in } \widetilde{I}^n_j, \end{cases} \tag{5.2}$$

$$\widetilde{f}^n(t) = \begin{cases} f(0), & \text{for } t = 0, \\ f^n_j, & \text{in } \widetilde{I}^n_j, \end{cases} \tag{5.3}$$

$$\widetilde{k}_n(t) = \begin{cases} 0, & \text{for } t = 0, \\ h_n \displaystyle\sum_{i=0}^{j-1} a\left(t^n_j - t^n_i\right) k\left(t^n_i, u^n_i\right), & \text{in } \widetilde{I}^n_j = \left(t^n_{j-1}, t^n_j\right]. \end{cases} \tag{5.4}$$

Corollary 5.1. *There exist $C > 0$ such that the estimates*

$$\|u^n(t)\| \le C, \quad \|\widetilde{u}_n(t)\| \le C, \quad \forall t \in I, \tag{5.5}$$

$$\left\| \frac{du^n}{dt}(t) \right\|_{B^1_2(0,1)} \le C, \quad \textit{for a.e. } t \in I, \tag{5.6}$$

$$\|\widetilde{u}_n(t) - u^n(t)\|_{B^1_2(0,1)} \le Ch_n, \quad \|\widehat{u}_n(t) - u^n(t)\|_{B^1_2(0,1)} \le Ch_n, \quad \forall t \in I, \tag{5.7}$$

$$\left\| \widetilde{k}_n(t) \right\| \le C, \quad \forall t \in I, \tag{5.8}$$

hold for all $n \in \mathbb{N}^*$.

Proof. For the inequalities (5.5), (5.6), and (5.7) see [6, Corollary 4.2.], whereas for the last inequality, assumption (H_2) and estimate (4.1) guarantee the desired result. □

Proposition 5.2. *The sequence $(u^n)_n$ converges in the norm of the space $C(I, B^1_2(0,1))$ to some function $u \in C(I, B^1_2(0,1))$ and the error estimate*

$$\|u^n - u\|_{C(I, B^1_2(0,1))} \le C\sqrt{h_n} \tag{5.9}$$

takes place for all $n \ge n_0$.

Proof. By virtue of (5.2), (5.3), and (5.4) the variational equation (3.9) may be rewritten in the form

$$\left(\frac{du^n}{dt}(t), v\right)_{B_2^1(0,1)} + (\widetilde{u}_n(t), v) = \left(\widetilde{k}_n(t), v\right)_{B_2^1(0,1)} + \left(\widetilde{f}^n(t), v\right)_{B_2^1(0,1)}, \tag{5.10}$$

for a.e. $t \in [0,T]$. In view of (5.10), using (5.6) and (5.8) with the fact that

$$\left\|\widetilde{f}^n(t)\right\|_{B_2^1(0,1)} \leq M_2 := \max_{t\in I}\|f(t)\|_{B_2^1(0,1)} < \infty, \tag{5.11}$$

we obtain

$$\begin{aligned} |(\widetilde{u}_n(t), v)| &\leq \left(\left\|\widetilde{k}_n(t)\right\|_{B_2^1(0,1)} + \left\|\widetilde{f}^n(t)\right\|_{B_2^1(0,1)} + \left\|\frac{du^n}{dt}(t)\right\|_{B_2^1(0,1)}\right)\|v\|_{B_2^1(0,1)} \\ &\leq C\|v\|_{B_2^1(0,1)}, \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{5.12}$$

Now, for n, m being two positive integers, testing the difference $(5.10)^n$-$(5.10)^m$ with $v = u^n(t) - u^m(t)$ which is in V, with the help of the Cauchy-Schwarz inequality and taking into account that

$$2\left(\frac{d}{dt}u(t), u(t)\right)_{B_2^1(0,1)} = \frac{d}{dt}\|u(t)\|^2_{B_2^1(0,1)}, \quad \text{a.e. } t \in [0,T], \tag{5.13}$$

and, by virtue of (5.12) we obtain after some rearrangements

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^n(t) - u^m(t)\|^2_{B_2^1(0,1)} + \|\widetilde{u}^n(t) - \widetilde{u}^m(t)\|^2 \\ &\quad \leq C\|u^m(t) - \widetilde{u}_m(t)\|_{B_2^1(0,1)} + C\|\widetilde{u}_n(t) - u^n(t)\|_{B_2^1(0,1)} \\ &\qquad + \left\|\widetilde{k}_n(t) - \widetilde{k}_m(t)\right\|_{B_2^1(0,1)}\|u^n(t) - u^m(t)\|_{B_2^1(0,1)} \\ &\qquad + \left\|\widetilde{f}^n(t) - \widetilde{f}^m(t)\right\|_{B_2^1(0,1)}\|u^n(t) - u^m(t)\|_{B_2^1(0,1)}, \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{5.14}$$

To derive the required result, we need to estimate the third and the last terms in the right-hand side, for this, let t be arbitrary but fixed in $(0,T]$, without loss of generality we can suppose that there exist three positive integers p, q and β, such that

$$t \in \left(t^n_{p-1}, t^n_p\right] \cap \left(t^m_{q-1}, t^m_q\right], \quad n = \beta m, \; t^n_p = t^m_q. \tag{5.15}$$

Hence, using (5.4) we can write

$$\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} = h_m \left\| \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} \left(a\left(t_p^n - t_j^n\right) k\left(t_j^n, u_j^n\right) - a\left(t_q^m - t_i^m\right) k\left(t_i^m, u_i^m\right) \right) \right] \right\|_{B_2^1(0,1)}. \tag{5.16}$$

By virtue of assumption (H_1) and the fact that $|a(t_p^n - t_j^n) - a(t_q^m - t_i^m)| \le Ch_n$, there exist $\varepsilon_n \in [0, Ch_n]$ such that

$$\begin{aligned}
&\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} \\
&\le h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} \left\| (Ch_n - \varepsilon_n) k\left(t_j^n, u_j^n\right) \right\|_{B_2^1(0,1)} + \left| a\left(t_q^m - t_i^m\right) \right| \left\| k\left(t_j^n, u_j^n\right) - k\left(t_i^m, u_i^m\right) \right\|_{B_2^1(0,1)} \right].
\end{aligned} \tag{5.17}$$

Therefore, recalling assumptions (H_1), (H_2) and having in mind that $\varepsilon_n \in [0, Ch_n]$, we estimate

$$\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} \le h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C\left(h_n + \left\| u_j^n - u_i^m \right\|_{B_2^1(0,1)} \right) \right], \tag{5.18}$$

from where, we derive for all $s \in (t_i^m, t_{i+1}^m]$

$$\begin{aligned}
\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} \le h_m \sum_{j=0}^{p-1} \Bigg[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C\Big(h_n &+ \|\tilde{u}_n(s) - u^n(s)\|_{B_2^1(0,1)} \\
&+ \|u^n(s) - u^m(s)\|_{B_2^1(0,1)} + \|u^m(s) - \tilde{u}_m(s)\|_{B_2^1(0,1)} \Big) \Bigg].
\end{aligned} \tag{5.19}$$

Taking the supremum with respect to s from 0 to t in the right-hand side, invoking the fact that $s \in (t_i^m, t_{i+1}^m] \subset (t_{j-1}^n, t_j^n]$ and estimate (5.7), we obtain

$$\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} \le h_m \sum_{i=0}^{q-1} \left(Ch_n + C \sup_{0 \le s \le t} \|u^n(s) - u^m(s)\|_{B_2^1(0,1)} \right), \tag{5.20}$$

so that

$$\left\|\tilde{k}_n(t) - \tilde{k}_m(t)\right\|_{B_2^1(0,1)} \le Ch_n + C \sup_{0 \le s \le t} \|u^n(s) - u^m(s)\|_{B_2^1(0,1)}. \tag{5.21}$$

Let $t \in (t_{p-1}^n, t_p^n] \cap (t_{q-1}^m, t_q^m]$, from assumption (H_1) it follows that

$$\begin{aligned}\left\|\widetilde{f}^n(t) - \widetilde{f}^m(t)\right\|_{B_2^1(0,1)} &= \left\|f\left(t_p^n\right) - f\left(t_q^m\right)\right\|_{B_2^1(0,1)} \\ &\leq l_1\left|t_p^n - t_q^m\right| \\ &\leq l_1 h_n.\end{aligned} \tag{5.22}$$

Ignoring the second term in the left-hand side of (5.14) which is clearly positive and using estimates (5.5), (5.7), (5.21), and (5.22) yield

$$\frac{d}{dt}\|u^n(t) - u^m(t)\|_{B_2^1(0,1)}^2 \leq C(h_n + h_m) + C\sup_{0\leq s\leq t}\|u^n(s) - u^m(s)\|_{B_2^1(0,1)}^2, \quad \text{a.e. } t \in [0,T]. \tag{5.23}$$

Integrating this inequality with respect to time from 0 to t and invoking the fact that $u^n(0) = u^m(0) = U_0$, we get

$$\|u^n(t) - u^m(t)\|_{B_2^1(0,1)}^2 \leq C(h_n + h_m) + C\int_0^t \sup_{0\leq \xi\leq t}\|u^n(\xi) - u^m(\xi)\|_{B_2^1(0,1)}^2 d\xi, \tag{5.24}$$

whence

$$\sup_{0\leq s\leq t}\|u^n(s) - u^m(s)\|_{B_2^1(0,1)}^2 \leq C(h_n + h_m) + C\int_0^t \sup_{0\leq \xi\leq t}\|u^n(\xi) - u^m(\xi)\|_{B_2^1(0,1)}^2 d\xi. \tag{5.25}$$

Accordingly, by Gronwall's lemma we obtain

$$\sup_{0\leq s\leq t}\|u^n(s) - u^m(s)\|_{B_2^1(0,1)}^2 \leq C(h_n + h_m)\exp(Ct), \quad \forall t \in [0,T], \tag{5.26}$$

consequently

$$\sup_{0\leq s\leq t}\|u^n(s) - u^m(s)\|_{B_2^1(0,1)} \leq C\sqrt{h_n + h_m} \tag{5.27}$$

takes place for all $n, m \in \mathbb{N}^*$. This implies that $(u^n(t))_n$ is a Cauchy sequence in the Banach space $C(I, B_2^1(0,1))$, and hence it converges in the norm of this latter to some function $u \in C(I, B_2^1(0,1))$. Besides, passing to the limit $m \to \infty$ in (5.27), we obtain the desired error estimate, which finishes the proof. □

Now, we present some properties of the obtained solution.

The limit-function u from Proposition 5.2, possesses the following properties:

(i) $u \in C(I, B_2^1(0,1)) \cap L^\infty(I, V))$,

(ii) u is strongly differentiable a.e. in I and $du/dt \in L^\infty(I, B_2^1(0,1))$,

(iii) $\widetilde{u}_n(t) \rightarrow u(t)$ in $B_2^1(0,1)$ for all $t \in I$,

(iv) $u^n(t), \widetilde{u}_n(t) \rightharpoonup u(t)$ in V for all $t \in I$,

(v) $(du^n/dt)(t) \rightharpoonup (du/dt)(t)$ in $L^2(I, B_2^1(0,1))$.

Proof. On the basis of estimates (5.5) and (5.6), uniform convergence statement from Proposition 5.2, and the continuous embedding $V \hookrightarrow B_2^1(0,1)$, the assertions of the present theorem are a direct consequence of [2, Lemma 1.3.15]. □

Theorem 5.3. *Under Assumptions* (H_1)–(H_3), (2.3) *admits a unique weak solution, namely, the limit function u from Proposition 5.2, in the sense of Definition 2.3.*

Proof. We have to show that the limit function u satisfies all the conditions (i), (ii), (iii), and (iv) of Definition 2.3. Obviously, in light of the properties of the function u listed in Theorem 5.3, the first two conditions of Definition 2.3 are already seen. On the other hand, since $u^n \rightarrow u$ in $C(I, V)$ as $n \rightarrow \infty$ and, by construction, $u^n(0) = U_0$, it follows that $u(0) = U_0$, so the initial condition is also fulfilled, that is, Definition 2.3(iii) takes place. It remains to see that the integral identity (2.21) is obeyed by u. For this, integrating (5.10) over $(0,t)$ and using the fact that $u^n(0) = U_0$, we get

$$(u^n(t) - U_0, v)_{B_2^1(0,1)} + \int_0^t (\widetilde{u}_n(\tau), v) d\tau = \int_0^t \left(\widetilde{k}(\tau, \widetilde{u}_n(\tau)), v\right)_{B_2^1(0,1)} d\tau + \int_0^t \left(\widetilde{f}^n(\tau), v\right)_{B_2^1(0,1)} d\tau, \tag{5.28}$$

consequently, after some rearrangements

$$\begin{aligned}
(u^n(t) - U_0, v)_{B_2^1(0,1)} &+ \int_0^t (\widetilde{u}_n(\tau), v) d\tau \\
&= \int_0^t \left(\int_0^\tau a(\tau - s) k(s, u(s)) ds, v \right)_{B_2^1(0,1)} d\tau + \int_0^t (f(\tau), v)_{B_2^1(0,1)} d\tau \\
&+ \int_0^t \left(\widetilde{k}(\tau, \widetilde{u}_n(\tau)) - \int_0^\tau a(\tau - s) k(s, u(s)) ds, v \right)_{B_2^1(0,1)} d\tau \\
&+ \int_0^t \left(\widetilde{f}^n(\tau) - f(\tau), v \right)_{B_2^1(0,1)} d\tau.
\end{aligned} \tag{5.29}$$

Let $\widehat{s}_n : I \rightarrow I$ and $\widehat{s}_n : I \rightarrow I$ denote the functions

$$\widehat{s}_n(t) = \begin{cases} 0, & \text{for } t = 0, \\ t_{j-1}^n, & \text{in } \widetilde{I}_j^n, \end{cases} \qquad \widetilde{s}_n(t) = \begin{cases} 0, & \text{for } t = 0, \\ t_j^n, & \text{in } \widetilde{I}_j^n. \end{cases} \tag{5.30}$$

To investigate the desired result, we prove some convergence statements. Using (5.2), (5.4), and (5.30) we have for all $t \in (t_{j-1}^n, t_j^n]$

$$\widetilde{k}(t, \widetilde{u}_n(t)) - \int_0^t a(t-s)k(s,u(s))ds$$
$$= \int_0^{t_j^n} \left[a\left(t_j^n - \widehat{s}_n(s)\right) k(\widehat{s}_n(s), \widehat{u}_n(s)) - a(t-s)k(s,u(s)) \right] ds + \int_t^{t_j^n} a(t-s)k(s,u(s))ds. \tag{5.31}$$

Taking into account (5.5), (5.9), and assumptions (H_1), (H_2) it follows that

$$\left\| a\left(t_j^n - \widehat{s}_n(s)\right) k(\widehat{s}_n(s), \widehat{u}_n(s)) - a(t-s)k(s,u(s)) \right\|_{B_2^1(0,1)} \leq C\sqrt{h_n}. \tag{5.32}$$

Thanks to (5.31) and (5.32) we obtain

$$\left\| \widetilde{k}(t, \widetilde{u}_n(t)) - \int_0^t a(t-s)k(s,u(s))ds \right\|_{B_2^1(0,1)} \leq C\sqrt{h_n}. \tag{5.33}$$

On the other hand, in view of the assumed Lipschitz continuity of f, we have

$$\left\| \widetilde{f}^n(\tau) - f(\tau) \right\|_{B_2^1(0,1)} \leq \| f(\widetilde{s}_n(\tau)) - f(\tau) \|_{B_2^1(0,1)}$$
$$\leq l_1 h_n. \tag{5.34}$$

Now, the sequences $\{(\widetilde{u}_n(\tau), v)\}$, $\{(\widetilde{f}^n(\tau), v)_{B_2^1(0,1)}\}$, and $\{(\widetilde{k}(\tau, \widetilde{u}_n(\tau)), v)_{B_2^1(0,1)}\}$ are uniformly bounded with respect to both τ and n, so the Lebesgue theorem of majorized convergence is applicable to (5.29). Thus, having in mind (5.7), (5.9), (5.33), and (5.34), we derive that

$$(u(t) - U_0, v)_{B_2^1(0,1)} + \int_0^t (u(\tau), v)d\tau$$
$$= \int_0^t \left(\int_0^\tau a(\tau - s)k(s,u(s))ds, v \right)_{B_2^1(0,1)} d\tau + \int_0^t (f(\tau), v)_{B_2^1(0,1)} d\tau \tag{5.35}$$

takes place for all $v \in V$ and $t \in [0,T]$. Finally, differentiating (5.35) with respect to t, we get

$$\left(\frac{d}{dt} u(t), v \right)_{B_2^1(0,1)} + (u(t), v)$$
$$= \left(\int_0^t a(t-s)k(s,u(s))ds, v \right)_{B_2^1(0,1)} + (f(t), v)_{B_2^1(0,1)}, \quad \text{a.e. } t \in [0,T]. \tag{5.36}$$

The uniqueness may be argued in the usual manner. Indeed, exploiting an idea in [11], consider u_1 and u_2 two different solutions of (2.3), and define $w = u_1 - u_2$ then, we have

$$\left(\frac{d}{dt}w(t), v\right)_{B_2^1(0,1)} + (w(t), v) = \left(\int_0^t a(t-s)[k(s, u_1(s)) - k(s, u_2(s))]ds, v\right)_{B_2^1(0,1)}. \tag{5.37}$$

Choosing $v = w(t)$ as a test function, with the aid of Cauchy-Schwarz inequality and assumption (H_1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)} + \|w(t)\|^2 \le C\int_0^t \left[\|k(s, u_1(s)) - k(s, u_2(s))\|_{B_2^1(0,1)}\right]ds\|w(t)\|_{B_2^1(0,1)}. \tag{5.38}$$

Let $\xi \in [0, p]$ such that

$$\|w(\xi)\|_{B_2^1(0,1)} = \max_{s\in[0,p]}\|w(s)\|_{B_2^1(0,1)}, \tag{5.39}$$

integrating (5.38) over $(0, p)$, $0 \le p \le T$, using (5.39), and invoking assumption (H_2), we get

$$\int_0^p \left[\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)} + \|w(t)\|^2\right]dt \le Cp^2\|w(\xi)\|^2_{B_2^1(0,1)}, \tag{5.40}$$

consequently, with the fact that $w(0) = 0$

$$\int_0^p \left[\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)} + \|w(t)\|^2\right]dt \le Cp^2\int_0^\xi \frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)}dt. \tag{5.41}$$

Choosing p as a constant verifying the condition

$$\exists \alpha \in \mathbb{N}, \quad T = \alpha p, \qquad Cp^2 \le \frac{1}{2}, \tag{5.42}$$

we have, by virtue of (5.41)

$$\int_0^p \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)}dt + \int_0^p \|w(t)\|^2 dt \le \int_0^\xi \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{B_2^1(0,1)}dt, \tag{5.43}$$

taking into account that $\xi \le p$, we obtain

$$\|w(t)\| = 0, \quad \text{on } [0, p]. \tag{5.44}$$

Following the same lines as for $[0, p]$, we deduce that

$$\|w(t)\| = 0, \quad \text{on } [ip, (i+1)p], \; i = 1, 2, 3, \ldots, \tag{5.45}$$

therefore, we derive $w(t) \equiv 0$, on $[0, T]$, then $u_1 \equiv u_2$. This achives the proof. □

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