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Research Article

The Packing Measure of the Trajectory of a One-Dimensional Symmetric Cauchy Process

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Let $X_t = \{X(t),\ t \geq 0\}$ be a one-dimensional symmetric Cauchy process. We prove that, for any measure function, $\varphi, \varphi - p(X[0,\tau])$ is zero or infinite, where $\varphi - p(E)$ is the φ-packing measure of E, thus solving a problem posed by Rezakhanlou and Taylor in 1988.

1. Introduction

Let $X_t = \{X(t),\ t \geq 0\}$ be a strictly stable Levy process taking values in R^n (n-dimensional Euclidean space) of index $\alpha \in (0,2]$, that is, a Markov process with stationary independent increment whose characteristic function is given by

$$E\left[e^{i(u,X_t)}\right] = e^{-t\psi_\alpha(u)}. \tag{1.1}$$

Here, u and X_t are points in R^n, (u,x) is the ordinary inner product in R^n, and $\|x\|^2 = (x,x)$. The Levy exponent $\psi_\alpha(u)$ is of the form

$$\psi_\alpha(u) = |u|^\alpha \int_{S_n} w_\alpha(u,y)\mu(dy), \tag{1.2}$$

where

$$\begin{aligned} w_\alpha(u,y) &= \left[1 - i\,\mathrm{sgn}(u,y)\tan\left(\frac{\pi\alpha}{2}\right)\right]\left(\left|\frac{u}{\|u\|},y\right|\right)^\alpha \quad \text{if } \alpha \neq 1, \\ w_1(u,y) &= \left|\left(\frac{u}{\|u\|},y\right)\right| + \frac{2i}{\pi}(u,y)\log\left|(u,y)\right|. \end{aligned} \tag{1.3}$$

$\mu(dy)$ is an arbitrary finite measure on the unit sphere S_n in R^n, not supported on a diametrical plane. If in (1.2) μ is the uniform distribution on S_n, X_t is called the isotropic stable Levy process with index α. In this case, $\psi_\alpha(u) = \lambda|u|^\alpha$ for some $\lambda > 0$. When $\alpha = 1$, μ must also have the origin as its center of mass, that is,

$$\int_{S_n} y\mu(dy) = 0, \tag{1.4}$$

and the resulting process is the symmetric Cauchy process.

If $\int_{S_n} y\mu(dy) \neq 0$ for $\alpha = 1$, we have the strictly asymmetric Cauchy process. When $\alpha = 2$, we obtain the standard Brownian motion.

We assume that our process has been defined so that the strong Markov property is valid and all sample paths are right continuous and have left limits everywhere.

It is well known that the sample paths X_t of strictly stable Levy processes determine trajectories in R^n that are random fractal sets.

We are interested in the range of the processes, that is, the random set R_τ generated by X_t and defined by

$$R_\tau = X([0,\tau]) = \{x \in R^n : x = X(t) \text{ for some } t \in [0,\tau]\}. \tag{1.5}$$

The Hausdorff and packing measures serve as useful tools for analyzing fine properties of Levy processes.

The problem of determining the exact Hausdorff measure of the range of those processes for $\alpha \in (0,2]$ has been completely solved. See, for example, [1].

The study of the exact packing measure of the range of a stochastic process has a more recent history, starting with the work of Taylor and Tricot [2].

The packing measure of the trajectory was found in [2] by Taylor and Tricot for transient Brownian motion. The corresponding problem for the range of strictly stable processes,$\alpha < n$, was solved by Taylor [3].

Further results on the asymmetric Cauchy process and subordinators have been established by Rezakhanlou and Taylor [4] and Fristedt and Taylor [5], respectively.

For the critical cases, $\alpha = n$, the only known result is due to Le Gall and Taylor [6]. They proved that if $X(t)$ is a planar Brownian motion, $\alpha = n = 2$, $\varphi - p[X([0,t])]$ is either zero or infinite for any measure function φ. Hence, the packing measure problem of the symmetric Cauchy process on the line remained open.

The main objective of this paper is to show that for $\alpha = n = 1$, a similar result to that of planner Brownian motion holds for the packing measure of the range of a one-dimensional symmetric Cauchy process with different criteria on φ.

2. Preliminaries

In this section, we start by recalling the definition and properties of packing measure and packing dimension introduced by Taylor and Tricot [2].

Let Φ be the class of functions:

$$\varphi : (0,\delta) \longrightarrow (0,\infty) \tag{2.1}$$

which are right continuous and monotone increasing with $\varphi(0_+) = 0$ and for which there is a finite constant $k > 0$ with

$$\frac{\varphi(2s)}{\varphi(s)} \leq k \quad \text{for } 0 < s < \frac{\delta}{2}. \tag{2.2}$$

The inequality (2.2) is a weak smoothness condition usually called a doubling property. A function φ in Φ is often called a measure function:

$$\varphi - P(E) = \limsup_{\varepsilon \to 0} \left\{ \sum_i \varphi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \ \varepsilon \right\}, \tag{2.3}$$

where $\overline{B}(x_i, r_i)$ denotes the closure of the open ball $B(x_i, r_i)$ which is centered at x and has radius r.

A sequence of closed balls satisfying the condition on the right side of (2.3) is called a ε-packing of E.

$\varphi - P$ is a φ-packing premeasure. The φ-packing measure on R^n, denoted by $\varphi - p$, is obtained by defining

$$\varphi - p(E) = \inf \left\{ \sum_n \varphi - P(E) : E \subseteq \bigcup_n E_n \right\}. \tag{2.4}$$

It is proved in [2] that $\varphi - p(E)$ is a metric outer measure, and hence every Borel set in R^n is $\varphi - p$ measurable.

We can see that for any $E \subset R^n$,

$$\varphi - p(E) \leq \varphi - P(E). \tag{2.5}$$

This gives a way to determine the upper bound of $\varphi - p(E)$. Using the function $\varphi(s) = s^\alpha$, $\alpha > 0$ gives the fractal index

$$\dim_p(E) = \inf \left\{ \alpha > 0 : s^\alpha - p(E) = 0 \right\} = \sup \left\{ \alpha > 0 : s^\alpha - p(E) = \infty \right\}, \tag{2.6}$$

called the packing dimension of E.

In order to calculate the packing measure, we will use the following density theorem of Taylor and Tricot [2], which we will call Lemma 2.1.

Lemma 2.1. *For a given $\varphi \in \Phi$, there exists a finite constant $k > 0$ such that for any Borel measure μ on R^n with $0 < \|\mu\| = \mu(R^n) < \infty$ and any Borel set $E \subseteq R^n$,*

$$k^{-1} \mu(E) \inf_{x \in E} \left\{ D^{\varphi}_{-\mu}(x) \right\}^{-1} \leq \varphi - p(E) \leq k \|\mu\| \sup_{x \in E} \left\{ D^{\varphi}_{-\mu}(x) \right\}^{-1}, \tag{2.7}$$

where

$$D^{\varphi}_{-\mu}(x) = \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{\varphi(2r)} \tag{2.8}$$

is the lower φ-density of μ at x.

One then uses the sample path X_t to define the random measure

$$\mu(E) = \left| \left\{ t \in [0, \tau] : X(t) \in E \right\} \right| \tag{2.9}$$

known as the occupation measure of the trajectory; $|\cdot|$ denotes the Lebesgue measure.

This gives a Borel measure with $\mu(R) = \|u\| = \tau$, and it is concentrated on $X[0,\tau]$ and spreads evenly on it.

If

$$x = X(t_0), \quad 0 < t_0 < \tau, \tag{2.10}$$

then

$$\mu(B(x,r)) = \int_0^\tau I_{B(x,r)}(X(t))\,dt = T(x,r) \tag{2.11}$$

is the sojourn time of X_t in the ball $B(x,r)$ up to the time τ. Define $\tau = \inf\{t > 0 : |x(t)| > 1\}$; then by a result in [7] about τ one has $E^0\tau = 1$, where E^0 is the associated expectation for the process started at 0. Denote $\int_0^\infty$ by $\int_{0+}$. If $x = 0$, one denotes $T(x,r)$ by $T(r)$.

In [8], we exhibited a measure function φ satisfying the following criteria.

Theorem 2.2. *Suppose $\varphi = rh(r)$, where $h(r)$ is a monotone nondecreasing function and*

$$T(r) = \int_0^\tau I_{B(0,r)}(X(t))\,dt; \tag{2.12}$$

then

$$\liminf_{r\downarrow 0} \frac{T(r)}{\varphi(r)} = \begin{cases} 0 & \text{if } \displaystyle\int_{0+} \frac{h(s)}{s\ln(1/s)} = \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{2.13}$$

where X_t is a one-dimensional symmetric Cauchy process.

For any $t_0 \geq 0$, $X(t+t_0) - X(t_0)$ is also a symmetric Cauchy process on the line since the finite-dimensional distribution of $X(t+t_0) - X(t_0)$ is independent of t_0; see, for example, [1] for the strong Markov property of Cauchy processes.

The following corollary is then immediate.

Corollary 2.3. *Let X_t, $t \geq 0$, be a one-dimensional symmetric Cauchy process. Then, for any $t_0 \geq 0$ with probability one,*

$$\liminf_{r\downarrow 0} \frac{T(X(t_0),r)}{\varphi(r)} = \begin{cases} 0 & \text{if } \displaystyle\int_{0+} \frac{h(s)}{s\ln(1/s)} = \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{2.14}$$

where φ is as defined in Theorem 2.2.

One will also need an estimate for the small ball probability of the sojourn time $T(r)$, taken from [8, Theorem 3.1].

Lemma 2.4. *Suppose $\varphi(r) = rh(r)$, where $h(r)$ is a monotone increasing function. For T defined in* (2.12)*, then for any fixed constant c_1 and $a_k = \rho^{-k}$, $\rho > 1$,*

$$P\{T(a_{k+1}) < c_1\varphi(a_k)\} \leq c_2 \frac{h(a_k)}{k}. \tag{2.15}$$

In the next section, we will use the above results and some known techniques to calculate the packing measure of the trajectory of the one-dimensional symmetric Cauchy process.

3. The measure of the trajectory

In this section, we proceed to the main result.

Theorem 3.1. *Let $X(t) = \{X(t)\colon t \geq 0\}$ be a one-dimensional symmetric Cauchy process. If $\varphi(r) = rh(r)$, where h is a nondecreasing function, then with probability one,*

$$\varphi - p(X([0,\tau])) = \begin{cases} 0 & \textit{if } \int_{0+} \frac{h(s)}{s\ln(1/s)} < \infty, \\ \infty & \textit{otherwise}, \end{cases} \tag{3.1}$$

where $\varphi - p(X([0,\tau]))$ is the φ-packing measure of $X([0,\tau])$.

Proof. In order to apply the density Lemma 2.1, we have to calculate

$$\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\varphi(2r)}. \tag{3.2}$$

But by Corollary 2.3, for each fixed $t_0 \in (0,\tau)$ with probability one,

$$\liminf_{r\downarrow 0} \frac{\mu(B(X(t_0),r))}{\varphi(r)} = \liminf_{r\downarrow 0} \frac{T(X(t_0),r)}{\varphi(r)} = 0 \quad \text{if} \int_{0+} \frac{h(s)}{s\ln(1/s)} = \infty. \tag{3.3}$$

Then a Fubini argument gives

$$\left|\left\{t \in (0,\tau) : \liminf_{r\downarrow 0} \frac{\mu(B(X(t),r))}{\varphi(r)} = 0 \text{ a.s.}\right\}\right| = \tau < \infty \tag{3.4}$$

so that if $E = \{X(t_0)\colon t_0 \in (0,\tau)\}$, then $E \subseteq X([0,\tau])$ and $\mu(E) = \tau < \infty$ a.s. Using an application of the inequality of the density Lemma 2.1, we have

$$\varphi - p(E) = \infty, \tag{3.5}$$

and thus $\varphi - pX([0,t]) = \infty$ with probability one if $\int_{0+}(h(s)/s\ln(1/s)) = \infty$.

In order to prove the upper bound, we use density Lemma 2.1 in the other direction, as well as a "bad-point" argument similar to that in [3].

For each point $x \in R$, let $V_k(x)$ denote a semidyadic interval with length 2^{-k} whose complement is at distance 2^{-k-2} from a dyadic interval of length 2^{-k-2} which contains x.

Let

$$\Gamma_E = \{V_k(x) : \; k = 1,2,\ldots,x \in E\}. \tag{3.6}$$

We use the intervals in Γ_E to replace the balls $B(x,r)$ in (2.3) with length 2^{-k} replacing $2r = \operatorname{diam} B(x,r)$.

This gives a new premeasure $\varphi - P^{xx}(E)$ comparable to $\varphi - P$ as follows. There exist positive finite constants k_1, k_2 such that, for all Borel sets $E \subset R$,

$$k_1\varphi - P^{xx}(E) \leq \varphi - P(E) \leq k_2\varphi - P^{xx}(E), \tag{3.7}$$

where

$$\varphi - p^{xx}(E) = \inf\left\{\sum_i \varphi - P^{xx}(E_i) : E \subset \cup E_i\right\}. \tag{3.8}$$

For

$$\int_{0+}^{\infty} \frac{h(s)}{s\ln(1/s)} < \infty, \tag{3.9}$$

let

$$G = \left\{t_0 \in (0,\tau) : \liminf \frac{\mu(B(X(t_0),r))}{\varphi(2r)} = \infty\right\} \tag{3.10}$$

be the set of "good" points. A Fubini argument tells us that $|G| = \tau < \infty$ a.s.; then using the density lemma in the other direction, we have

$$\varphi - P(X(G)) = 0. \tag{3.11}$$

Let

$$[0,\tau] \setminus G = \bigcup_{i=1}^{\infty} G_j, \tag{3.12}$$

where

$$G_j = \left\{t \in (0,\tau) : \liminf \frac{\mu(B(X(t),r))}{\varphi(2r)} \leq j\right\} \tag{3.13}$$

is the set of "bad" points.

For $t \in G_j$, by monotonicity, we have for a positive constant c,

$$\mu(B(X(t),2^{-k})) \leq cj\varphi(2^{-k}), \tag{3.14}$$

for infinitely many k.

For fixed j, we can only get a contribution to $\varphi - P^{xx}(X(G_j))$ from semidyadic intervals of length 2^{-k} if the dyadic interval of length 2^{-k-2} is entered by $X(t)$ at time $t \leq \tau$ and the restarted process leaves the interval of length 2^{-k-2} in less than $j\varphi(2^{-k})$.

Thus, if S_k is a semidyadic interval of length 2^{-k}, then S_k is bad if $X(s)$ enters inside dyadic interval of length 2^{-k-2} but spends less than $j\varphi(2^{-k})$ in S_k; otherwise it is "good". Any $t \in G_j$ will be in infinitely many such bad S_k.

The probability that S_k is bad given that it is entered is at most

$$P\{T(2^{-k}) \le cj\varphi(2^{-k})\} \le \frac{ch(2^{-k})}{k}, \tag{3.15}$$

by Lemma 2.4.

Let $N_k(\tau)$ be the number of intervals of length 2^{-k} that are entered by the time τ, and let $B_k(\tau)$ denote the number of those that are bad; then

$$EB_k(\tau) \le EN_k(\tau)\frac{h(2^{-k})}{k}. \tag{3.16}$$

Leaving out the nonoverlapping requirement, we have, for a positive constant c_3, $E\varphi - p^{xx}(X(G_j)) \le c_3\sum_{k=k_0}^{\infty} EB_k(\tau)\varphi(2^{-k})$. Now, by [1, Lemma 4.1], $EN_k(s) \le c_2 2^m$, for a positive constant c_2.

Thus, using (3.16), we have

$$E\varphi - p^{xx}(X(G_j)) \le c_3\sum_{k=k_0}^{\infty}\frac{(h(2^{-k}))^2}{k} \longrightarrow 0 \quad \text{a.s. as } k_0 \longrightarrow \infty \tag{3.17}$$

since $\sum((h(2^{-k}))^2/k) < \infty$ if $\sum(h(2^{-k})/k) < \infty$ for $h(2^{-k})$ sufficiently small.

It follows that $\varphi - p^{xx}X(G_j) = 0$ a.s., and from (3.8), $\varphi - p^{xx}X(G_j) = 0$ a.s. So $\varphi - p^{xx}X(\bigcup_{j=1}^{\infty} G_j) \le \sum \varphi - p^{xx}X(G_j) = 0$.

By (3.12), $G\cup\bigcup_{j=1}^{\infty} G_j = [0,\tau]$, and therefore $\varphi - pX[0,\tau] = 0$ if $\int_{0+}(h(s)/s\ln(1/s))ds < \infty$.

This completes the proof. □

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