Research Article

Pricing Participating Products under a Generalized Jump-Diffusion Model

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We propose a model for valuing participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. It also nests a number of important and popular models in finance, including the classes of jump-diffusion models and Markovian regime-switching models. The Esscher transform is employed to determine an equivalent martingale measure. Simulation experiments are conducted to illustrate the practical implementation of the model and to highlight some features that can be obtained from our model.

1. Introduction

In recent years, participating life insurance products become more and more popular in major insurance and finance markets around the world. These products can be regarded as investment plans with associated life insurance benefits, a specified benchmark return, a guarantee of an annual minimum rate of return, and a specified rule of the distribution of annual excess investment return above the guaranteed return. To enter the contract, policyholders pay annual premiums to an insurer, who will then manage and invest the funds in a specified reference portfolio. One key feature of these investment plans is the sharing of profits from an investment portfolio between the policyholders and the insurer. The specified rule of surplus distribution commonly used by insurers is known as reversionary bonus, which is employed to credit interest at or above a specified amount of guaranteed rate to the policyholders every year. The policyholders can receive an additional bonus at the maturity of

the contract, namely, the terminal bonus, if the terminal surplus of the fund is positive at the maturity. If the insurer defaults at the maturity of the policy, the policyholders can only receive the outstanding assets. For more comprehensive discussion on various features of participating policies, refer to Grosen and Jørgensen [1]. Due to the internationally growing trend of adopting the market-based and fair valuation accountancy standards for the implementation of risk management practice for participating policies, it is practically important to develop appropriate, realistic, and objective models for valuing these policies.

Earlier works on exploring the use of the modern option pricing theory to value embedded options in with-profits life insurance policies go back to Brennan and Schwartz [2, 3] and Boyle and Schwartz [4]. Since then, there has been considerable interest on utilizing option pricing theory and its modern technologies to determine fair values of these policies. Grosen and Jørgensen [1] develop a flexible contingent claims model to incorporate the minimum rate guarantee, bonus distribution, and surrender risk. Prieul et al. [5] adopt a partial differential equation approach to value a participating policy and employ the method of similarity transformations of variables to reduce the dimension of the partial differential equation governing the value of the policy. Bacinello [6, 7] adopt binomial schemes for computing the numerical solutions to the fair valuation problems of participating policies with various contractual features. Bacinello [7] introduces a model for describing the feature of annual premiums. Grosen and Jørgensen [8] use a barrier option framework to study and document the effect of regulatory intervention rules on reducing the insolvency risk of the policies. Chu and Kwok [9] develop a flexible contingent claims model that describes rate guarantee, bonuses, and default risk. Siu [10] considers the pricing of a participating policy with surrender options when the market values of the reference portfolio are governed by a Markov-modulated geometric Brownian motion.

In this paper, we propose a model for valuing participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. We suppose that the jump component is specified by the class of Markov-modulated kernel-biased completely random measures. The class of kernel-biased completely random measures is a wide class of jump-type processes. It has a very nice representation, which is a generalized kernel-based mixture of Poisson random measures (or, in general, random measures). The main idea of the kernel-biased completely random measure is to provide various forms of distortion of jump sizes of a completely random measure using the kernel function. This provides a great deal of flexibility in modeling different types of finite and infinite jump activities compared with some existing models in the literature. We also provide additional flexibility to incorporate the impact of structural changes in macroeconomic conditions and business cycles on the valuation of participating policies by introducing an observable, continuous-time and finite-state Markov chain. Here the states of the Markov chain may be interpreted as proxies of some observable macroeconomic indicators, such as gross domestic product and retail price index. They might also be considered economic ratings of a region or sovereign ratings. The model we considered here is general enough to nest a number of important and popular models for asset price dynamics in finance, including the two important classes of models, namely, the jump-diffusion models and the Markovian regime-switching models. These models are justified empirically in the literature and are shown to be practically useful for pricing and hedging derivatives. Our model can also be related to other important and popular classes of financial models, namely, the VG model pioneered by Madan et al. [11] and the CGMY model pioneered by Carr et al. [12].

For valuing participating products under the generalized jump-diffusion model, we employ a well-known tool in actuarial science, namely, the Esscher transform, which provides a convenient and flexible way to determine an equivalent martingale measure under the incomplete market setting. We consider various special cases of the Markov-modulated kernel-biased completely random measure for the jump component, namely, the Markov-modulated generalized Gamma (MGG) process, the scale-distorted version of the MGG process, and the power-distorted version of the MGG process. The MGG process encompasses the Markov-modulated weighted Gamma (MWG) process and the Markov-modulated inverse Gaussian (MIG) process as special cases. We compare the fair values of the options embedded in the participating products implied by our generalized jump-diffusion models with those obtained from other existing models in the literature via simulation experiments and highlight some features of the qualitative behavior of the fair values that can be obtained from different parametric specifications of our model. The paper is outlined as follows.

Section 2 presents the generalized jump-diffusion model for the market value of the reference asset and the Esscher transform for valuation. We also provide some discussion for the hedging and risk management issues. In Section 3, we consider three important parametric cases of the Markov-modulated kernel-biased completely random measures, namely, the MGG, the scale-distorted and power-distorted versions of the MGG process. The simulation procedure and the simulation results of the fair values of the options embedded in the policy are presented and discussed in Section 4. The final section summarizes this paper. The proofs of the lemmas and propositions are presented in the appendix.

2. The valuation model

In this section, we consider a financial model consisting of a risk-free money market account and a reference risky asset or portfolio. We suppose that the market value of the reference asset is governed by a jump-diffusion model with the jump component being specified as a kernel-biased completely random measure with Markov-switching compensator. We assume that the market is frictionless and that the mortality risk and surrender option are absent. We further impose certain assumptions on the rule of bonus distribution in our valuation model. We aim at developing a fair valuation model for participating life insurance policies which can incorporate the impact of the switching behavior of the states of the economy on the market value of the reference asset and fair value of the policy. The market described by the model is incomplete in general (see [13–16]). Hence, there are infinitely many equivalent martingale measures and there is a range of no-arbitrage prices for a policy. Here, we determine an equivalent martingale measure by the Esscher transform. In the sequel, we introduce the set up of our model.

2.1. The price dynamics

In this subsection, we describe the price dynamics of the reference portfolio underlying the participating policy. Firstly, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the economy. We describe the states of the economy by a continuous-time Markov chain $\{X_t\}_{t \in \mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{S} := (s_1, s_2, \ldots, s_N)$. Without loss of generality, we can identify the state space of the process $\{X_t\}_{t \in \mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^N$.

Write Q for the generator or Q-matrix $[q_{ij}]_{i,\,j=1,2,\dots,N}$. Then, from Elliott et al. [17], we have the following semimartingale decomposition for the process $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t QX_s ds + M_t. \tag{2.1}$$

Here $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$.

Let $\{r(t,X_t)\}_{t\in\mathcal{T}}$ be the instantaneous market interest rate of a bank account or a money market account, which depends on the state of the economy. That is,

$$r(t,X_t) = \langle \mathbf{r}, X_t \rangle = \sum_{i=1}^{N} r_i \langle X_t, e_i \rangle, \quad t \in \mathcal{T}, \tag{2.2}$$

where $\mathbf{r} := (r_1, r_2, \dots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \dots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in the space $\mathcal{R}^N$.

For notational simplicity, we write r_t for $r(t,X_t)$. In this case, the dynamics of the price process $\{B_t\}_{t\in\mathcal{T}}$ for the bank account is described by

$$\begin{aligned} dB_t &= r(t,X_t)B_t dt, \\ B_0 &= 1. \end{aligned} \tag{2.3}$$

In the sequel, we first describe a Markov-switching kernel-biased completely random measure. James [18, 19] propose a kernel-biased representation of completely random measures, which provides a great deal of flexibility in modeling different types of finite and infinite jump activities by choosing different kernel functions. Here we employ the kernel-biased representation of completely random measures proposed by James [18, 19] and adapt this representation to the Markov-modulated case in which the compensator of the underlying random measure switches over time according to the state of $\{X_t\}_{t\in\mathcal{T}}$.

Let $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ denote a measurable space, where $\mathcal{B}(\mathcal{T})$ is the Borel σ-field generated by the open subsets of $\mathcal{T}$. Write $\mathcal{B}_0$ for the family of Borel sets $U \in \mathcal{R}^+$, whose closure $\overline{U}$ does not contain the point 0. Let $\mathcal{X}$ denote $\mathcal{T} \times \mathcal{R}^+$. The measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is then given by $(\mathcal{T} \times \mathcal{R}^+, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}_0)$.

For each $U \in \mathcal{B}_0$, let $N_{X_t}(\cdot, U)$ denote a Markov-switching Poisson random measure on the space $\mathcal{X}$. Write $N_{X_t}(dt, dz)$ for the differential form of the measure $N_{X_t}(t, U)$. Let $\rho_{X_t}(dz|t)$ denote a Markov-switching Lévy measure on the space $\mathcal{X}$ depending on t and the state X_t; η is a σ-finite (nonatomic) measure on $\mathcal{T}$. Note that if $X_t = e_i$ $(i = 1, 2, \dots, N)$, write $\rho_i := \rho_{e_i}(dz|t)$. To ensure the existence of the kernel-biased completely random measure to be defined in the sequel (see [18–20]), we assume that for an arbitrary strictly positive function on $\mathcal{R}^+$, h, ρ_i, and η are selected in such a way that for each bounded set $\mathcal{B}$ in $\mathcal{T}$,

$$\sum_{i=1}^{N} \int_{\mathcal{B}} \int_{\mathcal{R}^+} \min\left(h(z), 1\right) \rho_i(dz|t) \eta(dt) < \infty. \tag{2.4}$$

We assume that the Markov-switching intensity measure $\nu_{X_t}(dt, dz)$ for the Poisson random measure $N_{X_t}(dt, dz)$ is given by

$$\nu_{X_t}(dt, dz) := \rho_{X_t}(dz|t)\eta(dt) = \sum_{i=1}^{N} \left(\rho_i(dz|t)\langle X_t, e_i \rangle\right)\eta(dt). \tag{2.5}$$

By modifying the kernel-biased representation of James [18, 19], we define a Markov-modulated kernel-biased completely random measure $\mu_{X_t}(dt)$ on $\mathcal{T}$ as follows:

$$\mu_{X_t}(dt) := \int_{\mathcal{R}^+} h(z) N_{X_t}(dt, dz), \qquad (2.6)$$

which is a kernel-based mixture of the Markov-modulated Poisson random measure $N_{X_t}(dt, dz)$ over the state space of the jump size $\mathcal{R}^+$ with the mixing kernel function $h(z)$. See also Perman et al. [20] for discussion on representations of completely random measures. In general, we can replace the Poisson random measure with a random measure and choose some quite exotic functions for $h(z)$ to generate different types of finite and infinite jump activities.

Let m_{X_t} denote the mean measure of μ_{X_t}. That is,

$$m_{X_t}(dt) = \int_{\mathcal{R}^+} h(z) \nu_{X_t}(dt, dz) = \sum_{i=1}^{N} \left(\int_{\mathcal{R}^+} h(z) \rho_i(dz|t) \langle X_t, e_i \rangle \eta(dt) \right). \qquad (2.7)$$

Let $\{W_t\}_{t \in \mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the $\mathcal{P}$-augmentation of its natural filtration $\mathcal{F}^W := \{\mathcal{F}_t^W\}_{t \in \mathcal{T}}$. We suppose that W, X, and $\tilde{\mu}_{X_t}(dt)$ are independent. Let $\widetilde{N}_{X_t}(dt, dz)$ denote the compensated Poisson random measure defined by

$$\widetilde{N}_{X_t}(dt, dz) := N_{X_t}(dt, dz) - \rho_{X_t}(dz|t)\eta(dt). \qquad (2.8)$$

Let μ_t and σ_t denote the drift and volatility of the market value of the reference asset, respectively. We suppose that μ_t and σ_t are given by

$$\mu_t := \langle \boldsymbol{\mu}, X_t \rangle = \sum_{i=1}^{N} \mu_i \langle X_t, e_i \rangle,$$
$$\sigma_t := \langle \boldsymbol{\sigma}, X_t \rangle = \sum_{i=1}^{N} \sigma_i \langle X_t, e_i \rangle, \qquad (2.9)$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)$; $\mu_i \in \mathcal{R}$ and $\sigma_i > 0$, for each $i = 1, 2, \ldots, N$.

Then, we assume that the dynamic of the market value A of the reference portfolio is governed by the following general geometric jump-diffusion process with a Markov-switching kernel-biased completely random measure:

$$dA_t = A_{t-} \left[\mu_t dt - m_{X_t}(dt) + \sigma_t dW_t + \int_{\mathcal{R}^+} (e^{h(z)} - 1) N_{X_t}(dt, dz) \right]. \qquad (2.10)$$

By convention, we suppose that $A_0 = 1$, $\mathcal{P}$-a.s. Similar to the Merton jump-diffusion model, the drift term of A is given by the mean $\mu_t dt$ minus the Markov-switching compensator $m_{X_t}(dt)$ of $\mu_{X_t}(dt)$. We can then write the dynamic of A as follows:

$$dA_t = A_{t-} \left[\mu_t dt + \int_{\mathcal{R}^+} (e^{h(z)} - 1 - h(z)) \rho_{X_t}(dz|t)\eta(dt) + \sigma_t dW_t + \int_{\mathcal{R}^+} (e^{h(z)} - 1) \widetilde{N}_{X_t}(dt, dz) \right]. \qquad (2.11)$$

In general, one can consider the situation that the drift μ_t and the volatility σ_t depend not only on the current economic state X_t, but also other state variables or market information, such as the current value of the reference portfolio A_t, when dealing with a long-term maturity. This represents an interesting and practically relevant direction for further generalizing the model. To focus on modeling and examining the impact of transitions of economic states on the price dynamics of the reference portfolio and the fair value of the policy, we assume here that μ_t and σ_t depend on the current economic state X_t only.

Let $Y_t := \ln(A_t)$. Note that $Y_0 = 0$, $\mathcal{P}$-a.s., since $A_0 = 1$. Then, by Itô's formula,

$$dY_t = \left(\mu_t - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t + \int_{\mathcal{R}^+} h(z)\widetilde{N}_{X_t}(dt, dz). \qquad (2.12)$$

2.2. The crediting scheme

Now, we describe the scheme for evaluating the interest rate credited to the policy reserve. Let R_t denote the book value of the policy reserve and D_t the bonus reserve, at time $t \in \mathcal{T}$. Then, as in Chu and Kwok [9], we have the following accounting identity for A_t, R_t, and D_t:

$$A_t = R_t + D_t, \quad t \in \mathcal{T}, \qquad (2.13)$$

where $R_0 := \alpha_p A_0$, $\alpha_p \in (0, 1]$, and R_0 is interpreted as the single initial premium paid by the policyholder for acquiring the contract and α_p is the cost allocation parameter. In this case, the α_p-portion of the initial asset portfolio is financed by the policyholder.

Write $c_R(A, R)$ for the interest rate credited to the policy reserve. Then we have

$$dR_t = c_R(A, R)R_t dt. \qquad (2.14)$$

In practice, the specification of $c_R(A, R)$ depends on the rule of bonus distribution, which is decided by the management level of an insurance company. Typically, an insurer distributes to his/her policyholder a certain proportion, say δ, of the excess of the ratio of bonus reserve D_t to the policy reserve R_t over the target ratio β, which is a long-term constant target ratio specified by the management. The proportional constant δ is called the reversionary bonus distribution rate and it is assumed that $\delta \in (0, 1]$. For the crediting scheme of interest rate, it is also assumed that there is a specified guarantee rate r_g for the minimum interest rate credited to the policyholder's account. This means that the interest rate $c_R(A, R) \geq r_g$. Here, we adopt the interest rate crediting scheme used in Chu and Kwok [9] as follows:

$$c_R(A_t, R_t) = \max\left(r_g, \left(\ln\frac{A_t}{R_t} - \beta\right)\right), \qquad (2.15)$$

where the interest rate $c_R(A_t, R_t)$ credited to the policyholder's account depends on both the reversionary bonus β and the guaranteed rate r_g.

2.3. Pricing by the Esscher transform

In this subsection, we describe how to determine an equivalent martingale measure by the Esscher transform in the incomplete market specified by the generalized jump-diffusion

model. Firstly, we provide a short discussion on other existing approaches for option valuation in an incomplete market.

Different approaches have been proposed in the literature on how to pick an equivalent martingale pricing measure in an incomplete market. Föllmer and Sondermann [21], Föllmer and Schweizer [22] introduce the notion of a minimal martingale measure and select a unique equivalent martingale measure via risk-minimization. Duffie and Richardson [23] and Schweizer [24] propose the mean-variance criterion for determining an equivalent martingale measure. Davis [25] adopts the marginal rate of substitution, which is a sound equilibrium argument in economic theory, to pick a pricing measure by solving a utility maximization problem. The pioneering work by Gerber and Shiu [26] provides a pertinent solution to the option pricing problem in an incomplete market by the Esscher transform, a time-honored tool in actuarial science introduced by Esscher [27]. The Esscher transform provides market practitioners with a convenient and flexible way to value options. Here, we employ the regime-switching Esscher transform in the work of Elliott et al. [16] and present the idea of this transform in the sequel.

Firstly, we describe the information structure of the model. Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t \in \mathcal{T}}$ and $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration generated by X and Y, respectively. For each $i = 1, 2$ and $t \in \mathcal{T}$, write $\mathcal{G}_t$ for the σ-algebra $\mathcal{F}_T^X \vee \mathcal{F}_t^Y$. Let $BM(\mathcal{T})$ denote the collection of $\mathcal{B}(\mathcal{T})$-measurable and nonnegative functions with compact support on $\mathcal{T}$. Write $\mathcal{B}(\mathcal{T})$ for the Borel σ-field of $\mathcal{T}$. For each process $\theta \in BM(\mathcal{T})$, write $(\theta \cdot Y)_t$ for $\int_0^t \theta_u dY_u$, for each $t \in \mathcal{T}$. Let $\mathcal{M}_Y(\theta)_t := E_{\mathcal{P}}[e^{-(\theta \cdot Y)_t} \mid \mathcal{F}_T^X]$, where $E_{\mathcal{P}}$ represents expectation under $\mathcal{P}$.

Let $\{\Lambda_t\}_{t \in \mathcal{T}}$ denote a $\mathcal{G}$-adapted stochastic process defined as below:

$$\Lambda_t := \frac{e^{-(\theta \cdot Y)_t}}{\mathcal{M}_Y(\theta)_t}, \quad t \in \mathcal{T}. \tag{2.16}$$

Applying Itô's differentiation rule for jump-diffusion processes (see, e.g., [28, 29]), we have

$$
\begin{aligned}
e^{-(\theta \cdot Y)_t} = 1 &- \int_0^t e^{-(\theta \cdot Y)_s} \theta_s \left(\mu_s - \frac{1}{2}\sigma_s^2 \right) ds - \int_0^t e^{-(\theta \cdot Y)_s} \theta_s \sigma_s dW_s \\
&- \int_0^t \int_{\mathcal{R}^+} e^{-(\theta \cdot Y)_{s-}} \theta_{s-} h(z) \widetilde{N}_{X_s}(ds, dz) + \frac{1}{2}\int_0^t e^{-(\theta \cdot Y)_s} \theta_s^2 \sigma_s^2 ds \\
&+ \int_0^t \int_{\mathcal{R}^+} e^{-(\theta \cdot Y)_{s-}} \left(e^{-\theta_s h(z)} - 1 \right) \widetilde{N}_{X_s}(ds, dz) \\
&+ \int_0^t \int_{\mathcal{R}^+} e^{-(\theta \cdot Y)_{s-}} \left(e^{-\theta_s h(z)} - 1 + \theta_s h(z) \right) \rho_{X_s}(dz|s)\eta(ds).
\end{aligned}
\tag{2.17}
$$

Conditioning on $\mathcal{F}_T^X$ for both sides of (2.17),

$$
\begin{aligned}
E\big[e^{-(\theta \cdot Y)_t} \big| \mathcal{F}_T^X\big] = 1 &- \int_0^t E\big[e^{-(\theta \cdot Y)_s} \big| \mathcal{F}_T^X\big] \theta_s \left(\mu_s - \frac{1}{2}\sigma_s^2 \right) ds + \frac{1}{2}\int_0^t E\big[e^{-(\theta \cdot Y)_s} \big| \mathcal{F}_T^X\big] \theta_s^2 \sigma_s^2 ds \\
&+ \int_0^t \int_{\mathcal{R}^+} E\big[e^{-(\theta \cdot Y)_{s-}} \big| \mathcal{F}_T^X\big] \left(e^{-\theta_s h(z)} - 1 + \theta_s h(z) \right) \rho_{X_s}(dz|s)\eta(ds).
\end{aligned}
\tag{2.18}
$$

Hence,

$$
\mathcal{M}_Y(\theta)_t = \exp\left[-\int_0^t \theta_s\left(\mu_s - \frac{1}{2}\sigma_s^2\right)ds + \frac{1}{2}\int_0^t \theta_s^2\sigma_s^2 ds + \int_0^t\int_{\mathcal{R}^+}(e^{-\theta_s h(z)} - 1 + \theta_s h(z))\rho_{X_s}(dz|s)\eta(ds)\right].
$$

$$(2.19)$$

Therefore,

$$
\Lambda_t = \exp\left[-\int_0^t \theta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2\sigma_s^2 ds - \int_0^t\int_{\mathcal{R}^+}\theta_s h(z)\widetilde{N}_{X_s}(ds,dz)\right.
$$

$$
\left. - \int_0^t\int_{\mathcal{R}^+}(e^{-\theta_s h(z)} - 1 + \theta_s h(z))\rho_{X_s}(dz|s)\eta(ds)\right].
$$

$$(2.20)$$

Lemma 2.1. Λ *is a* $(\mathcal{G},\mathcal{P})$*-martingale.*

Then, the Esscher transform $Q\sim\mathcal{P}$ on $\mathcal{G}_t$ with respect to $\{\theta_t \mid t \in \mathcal{T}\}$ is defined as

$$
\left.\frac{dQ}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \Lambda_t, \quad t \in \mathcal{T}.
$$

$$(2.21)$$

Harrison and Kreps [30] and Harrison and Pliska [31, 32] establish the relationship between the absence of arbitrage opportunities and the existence of an equivalent martingale measure. This is called the fundamental theorem of asset pricing. Delbaen and Schachermayer [33] point out that the equivalent relationship does not hold "true" in general and show that the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. Write $\tilde{A}_t := \exp(-\int_0^t r_u du)A_t$. In our setting, the martingale condition is given by

$$
\tilde{A}_s = E_Q[\tilde{A}_t|\mathcal{G}_s], \quad \text{for any } t,s \in \mathcal{T} \text{ with } t \geq s,
$$

$$(2.22)$$

where E_Q represents expectation under Q.

Proposition 2.2. *Suppose there exists a function* $\eta'(\cdot) : (0,T)\to\mathcal{R}^+$ *such that* $\eta(dt) = \eta'(t)dt$. *Then, the martingale condition is satisfied if and only if* θ_t *satisfies*

$$
\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathcal{R}^+}\left[e^{-\theta_t h(z)}\left(e^{h(z)} - 1\right) - h(z)\right]\rho_{X_t}(dz|t)\eta'(t) = 0,
$$

$$(2.23)$$

for each $t \in \mathcal{T}$.

Proposition 2.3. *Let* $\widetilde{W} := \{\widetilde{W}_{t\in T}\}$ *denote a standard Brownian motion and let* $\widetilde{N}^Q_{X_t}(dt,dz)$ *denote a compensated Markov-modulated Poisson random measure with compensator* $\rho^Q_{X_t}(dz|t)\eta'(t)dt$ *under* Q, *where* $\rho^Q_{X_t}(dz|t) := e^{-\theta_t h(z)}\rho_{X_t}(dz|t)$. *Then, under* Q,

$$
dY_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_{\mathcal{R}^+}\left(1 - e^{h(z)} + h(z)\right)\rho^Q_{X_t}(dz|t)\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+}h(z)\widetilde{N}^Q_{X_t}(dt,dz),
$$

$$(2.24)$$

where X is governed by (2.1).

2.4. Fair valuation

Here, we present the procedure for the fair valuation based on an equivalent martingale measure chosen by the regime-switching Esscher transform in the last subsection.

Let $V(A_T, R_T, X_T)$ denote the terminal payoff of the participating policy on the policy's maturity date T, when the state of the economy X_T at time T is X. Then,

$$V(A_T, R_T, X_T) = \begin{cases} A_T, & \text{if } A_T < R_T, \\ R_T, & \text{if } R_T \leq A_T \leq \dfrac{R_T}{\alpha_p}, \\ R_T + \gamma P_{1T}, & \text{if } A_T > \dfrac{R_T}{\alpha_p}, \end{cases} \tag{2.25}$$

where γ is the terminal bonus distribution rate and $P_{1T} := \max(\alpha_p A_T - R_T, 0)$ is the terminal bonus option.

Let $P_{2T} := \max(R_T - A_T, 0)$, where P_{2T} represents the terminal default option on the policy's maturity date T. Then, the terminal payoff $V(A_T, R_T, X_T)$ can be written in the following form:

$$V(A_T, R_T, X_T) = R_T + \gamma P_{1T} - P_{2T}. \tag{2.26}$$

Note that the bonus option can be viewed as a standard European call option that grants the policyholder the right to pay the policy value as a strike price to receive α_p-portion of the asset portfolio. Instead of evaluating the fair value of the terminal payoff of the policy, we consider the fair valuation for each of the components of the terminal payoff of the policy, namely, the guaranteed benefit R_T, the terminal bonus option P_{1T}, and the terminal default option P_{2T}. Given knowledge of $\mathcal{G}_t$, the conditional fair values of the guaranteed benefit, the terminal bonus option, and the terminal default option at time t are, respectively,

$$G(t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) R_T \,\Big|\, \mathcal{G}_t\right],$$

$$P_1(t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) P_{1T} \,\Big|\, \mathcal{G}_t\right], \tag{2.27}$$

$$P_2(t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) P_{2T} \,\Big|\, \mathcal{G}_t\right].$$

Note that the discount factor here is stochastic and switches over time according to the states of the Markov chain.

As in Buffington and Elliott [14, 15], given that $A_t = A$, $R_t = R$ and $X_t = X$, the fair values of the guaranteed benefit, the terminal bonus option, and the terminal default option at time t are, respectively,

$$G(A_t, R_t, X_t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) R_T \mid A_t = A, \ R_t = R, \ X_t = X\right], \tag{2.28}$$

$$P_1(A_t, R_t, X_t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) P_{1T} \mid A_t = A, \ R_t = R, \ X_t = X\right], \tag{2.29}$$

$$P_2(A_t, R_t, X_t) = E_Q\left[\exp\left(-\int_t^T r_s ds\right) P_{2T} \mid A_t = A, \ R_t = R, \ X_t = X\right]. \tag{2.30}$$

2.5. Hedging and risk management

Besides fair valuation of the options embedded in the participating policy, it is interesting to investigate how the risks inherent in these options can be hedged once the policy has been sold from a risk management perspective. The main focus of the current paper is the fair valuation issue of the policy. In practice, the hedging and risk management issues of the policy are also important. So, we provide some discussion for the hedging and risk management issues of the policy here. The hedging and risk management issues of the policy are certainly interesting and important topics for future research.

There are different ways to hedge the risks inherent in the options embedded in the policy. Hedging via the Greeks and the risk-minimizing hedging represent two popular approaches to hedging these risks. However, due to the fact that the market model considered here is incomplete, perfect hedging cannot be achieved. Here we discuss the use of the Greeks to hedge the risks inherent in the options, namely, the guaranteed benefit, the terminal bonus option, and the default option, embedded in the policy. Note that hedging using the Greeks is only an approximating hedging strategy and that it cannot provide a perfect hedging result due to the market incompleteness. There are different approaches to compute the Greeks based on the Monte Carlo simulation of the price paths. The basic method is the Monte Carlo finite-difference approach. The key idea of this method is to compute the finite difference approximation of the differentials using the Monte Carlo simulation. For illustration, we consider the use of this method to compute the Delta. Suppose $\widehat{V}(A)$ and $\widehat{V}(A + \epsilon)$ denote Monte Carlo estimators of "true" prices $V(A)$ and $V(A + \epsilon)$, respectively, where A represents the initial value of the reference portfolio and ϵ is a (small) positive constant. Then, the Delta $\Delta(A)$ of an option evaluated at the initial value A can be estimated by the finite-difference estimator as follows:

$$\widehat{\Delta}(A) = \frac{\widehat{V}(A + \epsilon) - \widehat{V}(A)}{\epsilon}. \tag{2.31}$$

Glynn [34] shows that if the simulations of the two estimators $\widehat{V}(A)$ and $\widehat{V}(A + \epsilon)$ are drawn independently, the best possible convergence rate is $n^{-1/4}$, where n is the number of simulation runs. The convergence rate can be improved using the central difference $(\widehat{V}(A + \epsilon) - \widehat{V}(A - \epsilon))/2\epsilon$. In this case, the best possible convergence rate is $n^{-1/3}$. The convergence rate can further be improved using common random numbers for both Monte Carlo estimators. The optimal convergence rate one can achieve in this case is $n^{-1/2}$, which is the same as the best possible convergence rate of a crude Monte Carlo method.

Other approaches that enhance the efficiency of the computation of the Greeks based on the Monte Carlo simulation include the simple differentiation approach proposed by Broadie and Glasserman [35] and the Malliavin calculus approach discussed by Fournié et al. [36, 37]. Chen and Glasserman [38] nvestigate the connection between the Malliavin calculus approach and the traditional approach based on the pathwise method and likelihood method. Recently, the Malliavin calculus approach for the Monte Carlo computation of the Greeks for jump-diffusion models and Lévy processes has been developed by several authors, including León et al. [39], El-Khatib and Privault [40], Davis and Johansson [41], and others. It is interesting to explore how the Malliavin calculus approach for jump-diffusion processes and Lévy processes can be extended to deal with the hedging of the risks inherent in the options embedded in the participating policy under the Markovian regime-switching jump-diffusion model considered

here. Besides using the Malliavin calculus approach, one may also consider the use of an extended Clark-Haussman-Ocone formula using the white noise analysis in the work of Aase et al. [42] to hedge the risks of the options embedded in the policy under the Markovian regime-switching jump-diffusion model. This also represents an interesting topic for further research.

3. Various parametric specifications to the jump component

In the previous section, we have defined a general jump-diffusion process with the jump component specified by a kernel-biased Markov-modulated completely random measure. Here, we consider some parametric cases of the general jump process by specifying some particular forms of the kernel function and the Markovian regime-switching intensity measure. These parametric cases include the MGG process, the scale-distorted and power-distorted versions of the MGG process, and their special cases. We also derive the risk-neutral dynamics for the logarithmic return process $\{Y_t\}_{t\in\tau}$ under Q for various parametric specifications which will be used for computing the fair values of the policies in Section 4. It is interesting to note that the kernel-biased completely random measure has some connections to some important Lévy processes in the literature including the VG process by Madan et al. [11] and the CGMY model of Carr et al. [12]. We also discuss these connections in this section.

3.1. Markov-modulated generalized Gamma (MGG) process

The generalized Gamma (GG) process is a wide class of jump-type processes, which consists of the weighted Gamma (WG) process and the inverse Gaussian (IG) process as special cases. The GG process is a special case of the kernel-biased completely random measure and can be obtained by setting the kernel function $h(z) = z$ and choosing a particular parametric form of the compensator measure. To provide more flexibility in describing the impact of the states of an economy on the jump component, we consider a Markov-modulated GG process, called the MGG process, whose compensator switches over time according to the states of the economy. We first describe the MGG process in the sequel.

Let $\alpha \geq 0$ denote a constant shape parameter of the MGG process. We suppose that the scale parameter of the MGG process $b(t) := b(t, X_t)$ switches over time according to the states of the Markov chain X and is given by

$$b(t) := \langle \mathbf{b}, X_t \rangle = \sum_{i=1}^{N} b_i \langle X_t, e_i \rangle, \tag{3.1}$$

where $\mathbf{b} := (b_1, b_2, \ldots, b_N) \in \mathcal{R}^N$ and $b_i \geq 0$, for each $i = 1, 2, \ldots, N$.

Then, the Markov-switching intensity process of the MGG process is

$$\rho_{X_t}(dz|t)\eta'(t)dt = \frac{1}{\Gamma(1-\alpha)}e^{-\langle \mathbf{b}, X_t \rangle z}z^{-\alpha-1}dz\eta'(t)dt = \sum_{i=1}^{N}\frac{1}{\Gamma(1-\alpha)}e^{-b_i z}z^{-\alpha-1}\langle X_t, e_i \rangle dz\eta'(t)dt. \tag{3.2}$$

In this case, the martingale condition becomes

$$\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathcal{R}^+}\left[e^{-\theta_t z}(e^z - 1) - z\right]\rho_{X_t}(dz|t)\eta'(t) = 0, \tag{3.3}$$

where $\rho_{X_t}(dz \mid t)\eta'(t)$ is given by (3.2).

Write $\rho^Q_{X_t}(dz|t) := e^{-\theta_t z}\rho_{X_t}(dz|t)$, where θ_t satisfies (3.3). Let $N^Q_{X_t}(dt, dz)$ denote a Poisson random measure with Markov-switching compensator $\rho^Q_{X_t}(dz|t)\eta'(t)dt$ under Q. Then, under Q, the dynamic of Y is

$$dY_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_{\mathcal{R}^+}(1 - e^{h(z)})\rho^Q_{X_t}(dz|t)\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+}h(z)N^Q_{X_t}(dt, dz). \qquad (3.4)$$

When $\alpha = 0$, the MGG process reduces to a Markov-modulated WG (MWG) process. That is, the Markov-switching intensity of the MWG process is

$$\rho_{X_t}(dz|t)\eta'(t)dt = \frac{e^{-\langle \mathbf{b}, X_t \rangle z}}{z}dz\eta'(t)dt = \sum_{i=1}^{N}\frac{e^{-b_i z}}{z}\langle X_t, e_i \rangle dz\eta'(t)dt. \qquad (3.5)$$

In this case, the martingale condition becomes

$$\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathcal{R}^+}\left[e^{-\theta_t z}(e^z - 1) - z\right]\left(\frac{e^{-\langle \mathbf{b}, X_t \rangle z}}{z}\right)dz\eta'(t) = 0. \qquad (3.6)$$

Under Q,

$$dY_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_{\mathcal{R}^+}(1 - e^{h(z)})\left(\frac{e^{-(\theta_t + \langle \mathbf{b}, X_t \rangle)z}}{z}\right)dz\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+}h(z)N^Q_{X_t}(dt, dz), \qquad (3.7)$$

where $N^Q_{X_t}(dt, dz)$ is a Poisson random measure with Markov-switching compensator,

$$\rho^Q_{X_t}(dz|t)\eta'(t)dt = \left(\frac{e^{-(\theta_t + \langle \mathbf{b}, X_t \rangle)z}}{z}\right)dz\eta'(t)dt, \qquad (3.8)$$

and θ_t satisfies (3.6).

When $\alpha = 1/2$, the MGG becomes a Markov-modulated IG (MIG) process. In this case, the martingale condition becomes

$$\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathcal{R}^+}\left[e^{-\theta_t z}(e^z - 1) - z\right]\left(\frac{1}{\Gamma(1/2)\sqrt{z}}e^{-\langle \mathbf{b}, X_t \rangle z}dz\right)dz\eta'(t) = 0. \qquad (3.9)$$

Under Q, the dynamic of Y is

$$dY_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_{\mathcal{R}^+}(1 - e^{h(z)})\left(\frac{1}{\Gamma(1/2)\sqrt{z}}e^{-(\theta_t + \langle \mathbf{b}, X_t \rangle)z}\right)dz\eta'(t)dt$$
$$+ \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+}h(z)N^Q_{X_t}(dt, dz), \qquad (3.10)$$

where the intensity process for $N^Q_{X_t}(dt, dz)$ is

$$\rho^Q_{X_t}(dz|t)\eta'(t)dt = \left(\frac{1}{\Gamma(1/2)\sqrt{z}}e^{-(\theta_t + \langle \mathbf{b}, X_t \rangle)z}\right)dz\eta'(t)dt, \qquad (3.11)$$

and θ_t satisfies (3.9).

3.2. The scale and power distortions of the MGG process

In this section, we consider the scale-distorted and power-distorted versions of the MGG process. The scale-distorted and power-distorted versions of the MGG process provide additional flexibility for describing various jump-type behaviors. They can describe the overstate and understate of jump amplitudes due to overreaction and underreaction of market participants to extraordinary events, respectively. For the scale-distorted version of the MGG process, the kernel function $h(z) = cz$, where c is a positive constant. When $c > 1$, jump sizes are overstated. When $0 < c < 1$, jump sizes are understated. For the power-distorted version of the MGG process, the kernel function $h(z) = z^q$, where $q > 0$. When $q > 1$, small jump sizes (i.e., $0 < z < 1$) are understated and large jump sizes (i.e., $z > 1$) are overstated. When $0 < q < 1$, small jump sizes are overstated and large jump sizes are understated. The scale-distorted and power-distorted versions of the MGG process with different scale and power-distorted parameters can generate different types of behaviors of market participants when they react to extraordinary events. They can shed lights on understanding the impact of these market participants' behaviors on the price dynamic of the reference asset from a behavioral finance perspective. In general, one can also assume that the scale c for the scale-distorted version of the MGG process and the power q for the power-distorted version of the MGG process also switch over time according to the state of the Markov chain. In this case, the overstate and understate of the jump amplitudes also depend on the state of the economy. Here, we consider the case that both c and q are constants for illustration.

For both the scale-distorted and power-distorted versions of the MGG process, the Markov-switching intensity processes are the same as that of the MGG process. For the scale-distorted version of the MGG process, the martingale condition is given by

$$\mu_t - r_t - \theta_t \sigma_t^2 + \int_{\mathcal{R}^+} \left[e^{-\theta_t cz} (e^{cz} - 1) - cz \right] \rho_{X_t}(dz|t)\eta'(t) = 0, \qquad (3.12)$$

where $\rho_{X_t}(dz|t)\eta'(t)$ is given by (3.2).

Under Q, the dynamic of Y is

$$dY_t = \left(r_t - \frac{1}{2}\sigma_t^2 \right) dt + \int_{\mathcal{R}^+} (1 - e^{cz}) \rho_{X_t}^Q(dz|t)\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+} cz N_{X_t}^Q(dt, dz), \qquad (3.13)$$

where $\rho_{X_t}^Q(dz|t) = e^{-\theta_t z} \rho_{X_t}(dz|t)\eta(dt)$ and $\rho_{X_t}(dz|t)\eta'(t)dt$ is given by (3.2).

For the power-distorted version of the MGG process, the martingale condition is

$$\mu_t - r_t - \theta_t \sigma_t^2 + \int_{\mathcal{R}^+} \left[e^{-\theta_t z^q} (e^{z^q} - 1) - z^q \right] \rho_{X_t}(dz|t)\eta'(t) = 0, \qquad (3.14)$$

where $\rho_{X_t}(dz|t)\eta'(t)$ is specified by (3.2).

Under Q, the dynamic of Y is

$$dY_t = \left(r_t - \frac{1}{2}\sigma_t^2 \right) dt + \int_{\mathcal{R}^+} (1 - e^{z^q}) \rho_{X_t}^Q(dz|t)\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+} z^q N_{X_t}^Q(dt, dz), \qquad (3.15)$$

where $\rho_{X_t}^Q(dz|t) = e^{-\theta_t z} \rho_{X_t}(dz|t)$ and $\rho_{X_t}(dz|t)\eta'(t)$ is given by (3.2).

When $\alpha = 0$, the scale-distorted version of the MGG process becomes the scale-distorted version of the MWG process and the power-distorted version of the MGG process reduces to the power-distorted version of the MWG process. When $\alpha = 1/2$, the scale-distorted and power-distorted versions of the MGG process become the scale-distorted and power-distorted versions of the MIG process, respectively.

3.3. Connections to the VG and CGMY processes

Now, we outline some connections of a modified version of the Markov-modulated kernel-biased completely random measure to the VG and CGMY processes. We first provide some discussions on the VG model. The VG process can be represented in a number of equivalent ways, namely, the representation based on the time-changed Brownian motion, the difference between two gamma processes, the Lévy measure representation, the predictable compensator representation, where the predictable compensator representation is closely related to the Lévy measure representation in a fundamental way. Madan et al. [11] and Elliott and Royal [43] provide detailed discussion for different representations of the VG process. It has been shown by Elliott and Royal [43] that the predictable compensator of the VG process is the same as the Lévy measure. The Lévy measure of the VG process is given by (see [11])

$$\nu^{\mathrm{VG}}(dz, dt) = k^{\mathrm{VG}}(z)dz\,dt = \left(\frac{C\exp(-Mz)}{z}I_{\{z>0\}} - \frac{C\exp(Gz)}{z}I_{\{z<0\}} \right)dz\,dt, \qquad (3.16)$$

where $C, M, G \in \mathcal{R}^+$ are parameters of the VG process.

We consider a Markov-modulated version of the VG process with the following Markov-switching compensator:

$$
\begin{aligned}
\nu^{\mathrm{VG}}_{X_t}(dz, dt) &= k^{\mathrm{VG}}_{X_t}(z)dz\,dt \\
&= \left(\frac{C\exp\left(-\langle \mathbf{M}, X_t \rangle z\right)}{z}I_{\{z>0\}} - \frac{C\exp\left(\langle \mathbf{G}, X_t \rangle z\right)}{z}I_{\{z<0\}} \right)dz\,dt,
\end{aligned}
\qquad (3.17)
$$

where $\mathbf{M} := (M_1, M_2, \ldots, M_N) \in \mathcal{R}^N$ and $\mathbf{G} := (G_1, G_2, \ldots, G_N) \in \mathcal{R}^N$ with $M_i > 0$ and $G_i > 0$, for each $i = 1, 2, \ldots, N$.

The Markov-modulated VG process can be related to a modified version of the Markov-switching kernel-biased completely random measure by suitable matching of the model parameters. Consider two Markov-switching Poisson random measures $N^k_{X_t}(dz, dt)$, $k = 1, 2$, with the following intensity processes:

$$\rho^k_{X_t}(dz|t)\eta'(t)dt = \frac{e^{(-1)^k\langle \mathbf{b}^k, X_t \rangle z}}{z}dz\eta'(t)dt, \qquad (3.18)$$

where $\mathbf{b}^k := (b^k_1, b^k_2, \ldots, b^k_N) \in \mathcal{R}$ with $b^k_i > 0$, for each $i = 1, 2, \ldots, N$.

Write $N_{X_t}(dz, dt) := N^1_{X_t}(dz, dt)I_{\{z>0\}} + N^2_{X_t}(dz, dt)I_{\{z<0\}}$. Then, the intensity process of $N_{X_t}(dz, dt)$ is

$$\rho_{X_t}(dz|t)\eta'(t)dt = \left(\frac{e^{-\langle \mathbf{b}^1, X_t \rangle z}}{z}I_{\{z>0\}} - \frac{e^{\langle \mathbf{b}^2, X_t \rangle z}}{z}I_{\{z<0\}} \right)dz\eta'(t)dt. \qquad (3.19)$$

Suppose, for each $k = 1, 2$, that $h_k(\cdot) : \mathcal{R}^+ \to \mathcal{R}^+$ satisfies the following condition:

$$\sum_{i=1}^{N} \int_{\mathcal{B}} \int_{\mathcal{R}^+} \min\left(h_k(z), 1\right) \rho_i^k(dz|t) \eta(dt) < \infty. \tag{3.20}$$

Let $h(\cdot) : \mathcal{R} \to \mathcal{R}$ be a real-valued function defined as follows:

$$h(z) = h_1(z) I_{\{z > 0\}} - h_2(-z) I_{\{z < 0\}}. \tag{3.21}$$

Then, define a process $\mu(t)$ as follows:

$$\mu(t) := \int_0^t \int_{\mathcal{R}} h(z) N_{X_u}(dz, du). \tag{3.22}$$

Let $\eta(dt) = dt$ (i.e., $\eta(\cdot)$ is a uniform density). In this case, $\eta'(t) = 1$. We further assume that $b_i^1 = M_i$; $b_i^2 = G_i$; $C = 1$. Then, the Markov-modulated VG process with Markov-switching compensator $\nu_{X_t}^{\text{VG}}(dz, dt)$ coincides with the process $\mu(t)$, which is a modified version of the Markov-switching kernel-biased completely random measure.

In the sequel, we consider the CGMY process. From Carr et al. [12], the Lévy measure of the CGMY process is

$$\nu^{\text{CGMY}}(dz, dt) = k^{\text{CGMY}}(z) dz\, dt$$

$$= \left(C \frac{\exp\left(-G|z|\right)}{|z|^{1+Y}} I_{\{z < 0\}} + C \frac{\exp\left(-M|z|\right)}{|z|^{1+Y}} I_{\{z > 0\}} \right) dz\, dt, \tag{3.23}$$

where $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$.

We consider a Markov-modulated version of the CGMY process with the following Markov-switching compensator:

$$\nu_{X_t}^{\text{CGMY}}(dz, dt) = k_{X_t}^{\text{CGMY}}(z) dz\, dt$$

$$= \left(C \frac{\exp\left(-\langle \mathbf{G}, X_t \rangle |z|\right)}{|z|^{1+Y}} I_{\{z < 0\}} + C \frac{\exp\left(-\langle \mathbf{M}, X_t \rangle |z|\right)}{|z|^{1+Y}} I_{\{z > 0\}} \right) dz\, dt, \tag{3.24}$$

where $\mathbf{M} := (M_1, M_2, \ldots, M_N) \in \mathcal{R}^N$ and $\mathbf{G} := (G_1, G_2, \ldots, G_N) \in \mathcal{R}^N$ with $M_i > 0$ and $G_i > 0$, for each $i = 1, 2, \ldots, N$.

For each $k = 1, 2$, let $\overline{N}_{X_t}^k(dz, dt)$ denote a Markov-switching Poisson random measure with the following intensity process:

$$\rho_{X_t}^k(dz|t) \eta'(t) dt = \frac{1}{\Gamma(1-\alpha)} e^{-\langle \mathbf{b}^k, X_t \rangle z} z^{-\alpha-1} dz \eta'(t) dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{e^{-\langle \mathbf{b}^k, X_t \rangle |z|}}{|z|^{1+\alpha}} dz \eta'(t) dt. \tag{3.25}$$

Let $\overline{N}_{X_t}(dz, dt) := \overline{N}_{X_t}^1(dz, dt) I_{\{z > 0\}} + \overline{N}_{X_t}^2(dz, dt) I_{\{z < 0\}}$. Then, the Markov-switching intensity process for $\overline{N}_{X_t}(dz, dt)$ is

$$\rho_{X_t}(dz|t) \eta'(t) dt = \left(\frac{1}{\Gamma(1-\alpha)} \frac{e^{-\langle \mathbf{b}^1, X_t \rangle |z|}}{|z|^{1+\alpha}} I_{\{z > 0\}} + \frac{1}{\Gamma(1-\alpha)} \frac{e^{-\langle \mathbf{b}^2, X_t \rangle |z|}}{|z|^{1+\alpha}} I_{\{z < 0\}} \right) dz \eta'(t) dt. \tag{3.26}$$

In this case, the kernel function h is given by (3.21). Define a process $\overline{\mu}(t)$ as below:

$$\overline{\mu}(t) := \int_0^t \int_{\mathcal{R}} h(z) \overline{N}_{X_u}(dz, du). \tag{3.27}$$

Let $\eta(dt) = dt$; $b_i^1 = M_i$; $b_i^2 = G_i$; $\alpha = Y$; $C = 1/\Gamma(1 - \alpha)$. Then, the Markov-modulated CGMY process coincides with the process $\overline{\mu}(t)$, which is a modified version of the Markov-switching kernel-biased completely random measure.

4. Simulation experiments and comparisons

In this section, we conduct simulation experiments to compare the fair values of the guaranteed benefit, the terminal bonus option, and the default option embedded in the participating policy implied by various parametric specifications of our generalized jump-type model described in Section 3 with those obtained from other existing models in the literature, such as the Merton jump-diffusion model, the VG process, and the geometric Brownian motion (GBM). We also document the impact of the regime-switching effect in the price dynamic of the reference portfolio on the fair values of the embedded options. We highlight some features of the qualitative behavior of the fair values of the embedded options that can be obtained from different parametric specifications of our model. Besides investigating the implications for the fair values of the embedded options, we also compare the default probabilities of the embedded options implied by various specifications of our jump-diffusion process with those implied by other models.

For simulating various parametric cases of our generalized jump-type process, we adopt the Poisson weighted algorithm by Lee and Kim [44] to simulate completely random measures with Markov-switching compensator. The Poisson weighted algorithm is applicable for a wide class of completely random measures, which are very difficult, if not impossible, to simulate directly in practice. The main idea of the Poisson weighted algorithm is that instead of generating jump sizes of a completely random measure directly from a nonstandard density function, one can first generate jump sizes from a proposed density function, which is a standard density function, like a gamma density, and then adjust the simulated jump sizes by the corresponding Poisson weights. The Poisson weights are simulated from a Poisson distribution with intensity parameter given by the odd ratio of the compensator of the completely random measure and the compensator corresponding to the proposed density.

In the sequel, we describe the modified Poisson weighted algorithm. For generality, we consider the full jump-diffusion model in Section 2, which is a Markov-modulated kernel-biased completely random measure. Suppose we wish to sample from the following process under the risk-neutral probability measure Q:

$$\mu^Q(t) = \int_0^t \int_{\mathcal{R}^+} h(z) N_{X_u}^Q(dz, du), \tag{4.1}$$

where the Markov-switching compensator for the Poisson random measure $N_{X_t}^Q(dz, dt)$ under Q is

$$v_{X_t}^Q(dz, dt) = \rho_{X_t}^Q(dz|t) \eta'(t) dt = \sum_{i=1}^N \left(\rho_i^Q(dz|t) \langle X_t, e_i \rangle \right) \eta'(t) dt. \tag{4.2}$$

Suppose we divide the time horizon $[0, T]$ of T years into nT subintervals with equal length of $\Delta := 1/n$, where T is a positive integer. That is, one year is divided into n subintervals. If $n = 252$, each subinterval represents one trading day. Write $[t_j, t_{j+1}]$ for the $(j + 1)$st subinterval, for each $j = 0, 1, \ldots, n - 1$. Let M denote the number of jumps of the completely random measure over a one-year time horizon. Here, M controls the degree of accuracy of the approximation by the Poisson weighted algorithm and can be set as different values according to the desirable degree of accuracy. The larger M is, the more accurate the approximation is. Here, we set $M = 100$ and $n = 252$. In general, they can be set as any positive numbers. We take $\eta'(t)$ to be a proper density (i.e., $\eta(T) := \int_0^T \eta'(t)dt < \infty$). For illustration, we assume here that $\eta'(t)$ is an unnormalized density function of a uniform density on $[0, T]$. Then, the Poisson weighted algorithm is described as follows.

(1) Simulate the Markov chain process $\{X_j \mid j = 1, 2, \ldots, nT\}$.

(2) Generate i.i.d. positive random variables $T_1, \ldots, T_{MT}$, which represent jump times, from the normalized density function $\eta'(t)/\eta(T)$, where $\eta(T) := \int_0^T \eta'(t)dt$.

(3) Generate the jump size W_i from the proposed density function g_{T_i}, where g_{T_i} is defined as a conditional density function knowledge of T_i, for each $i = 1, 2 \ldots, MT$. Here, g_{T_i} is assumed to be a gamma density function.

(4) For each fixed $i = 1, 2, \ldots, MT$, if $T_i \in [t_j, t_{j+1})$, calculate $\lambda_i = \eta(T)\rho_{X_j}^Q(W_i|T_i)/ MT g_{T_i}(W_i)$, for each $i = 1, \ldots, M$.

(5) Generate the Poisson weights Z_i from a Poisson distribution with intensity parameter λ_i.

(6) $\mu(t) = \sum_{i=1}^{MT} h(Z_i W_i) I_{(T_i \leq t)}$, where $h(\cdot)$ is the kernel function of the kernel-biased completely random measure.

The process generated by the Poisson weighting algorithm converges in distribution to the completely random measure $\mu(t)$ on the space $D[0, T]$ of real-valued functions defined on the compact domain $[0, T]$ with the Skorohod topology as $M \to \infty$. For the proof of the convergence, interested readers may refer to Lee and Kim [44].

In our simulation experiments, we compute the fair values for the embedded options, including the guaranteed benefit $G(0)$, the terminal bonus option $P_1(0)$, and the default option $P_2(0)$, all with maturity T equal to 20 years. Each fair value is computed by using 10 000 simulation paths over 20 years. All computations here were done by C++ codes. Note that $n = 252$ (i.e., $\Delta = 1/252$). In other words, we simulate daily observations for the dynamic of the reference asset A. We consider a two-state Markov chain model X with $N = 2$, where "$X_t = 1$" represents "Good" economy while "$X_t = 2$" represents "Bad" economy. We take the transition probability matrix for X as follows:

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}. \tag{4.3}$$

We generate 10 000 simulation paths for X over 20 years (i.e., $\{X_j\}_{j=1,2,\ldots,5040}$) and suppose that $X_0 = 1$. For each simulation path of X, we generate the log return series $\{Y_j\}_{j=1,2,\ldots,5040}$ for the

full model in Section 2 from the following discretized version of the risk-neutral log return process based on the forward Euler discretization scheme:

$$Y_{j+1} = Y_j + \left(r_{X_j} - \frac{1}{2}\sigma_{X_j}^2\right)\Delta + \int_{\mathcal{R}^+}(1 - e^{h(z)} + h(z))\rho_{X_j}^Q(dz|t_j)\Delta + \sigma_{X_j} \times \epsilon \times \sqrt{\Delta} + \widetilde{\mu}_{X_j}^Q(t_{j+1}) - \widetilde{\mu}_{X_j}^Q(t_j),$$

(4.4)

where $\epsilon \sim N(0,1)$ and

$$\widetilde{\mu}_{X_j}^Q(t) = \mu_{X_j}^Q(t) - \int_0^t\int_{\mathcal{R}^+}\rho_{X_j}^Q(dz|u)du.$$

(4.5)

We take the initial value Y_0 to be zero.

The forward Euler discretization scheme is a popular method to approximate the paths of a continuous-time process when performing Monte Carlo simulation and/or estimation of the process. It provides a natural, intuitive, and convenient way to discretize a continuous-time process. It has widely been adopted in approximating the paths of continuous-time asset price dynamics when performing Monte Carlo simulations in financial engineering (see, e.g., [45]). Under some conditions, it can be shown that the Euler approximation converges weakly to the target continuous-time process when the number of discretization intervals tends to infinity. Kloeden and Platen [46] provided an excellent discussion for the convergence of the forward Euler discretization scheme. They also presented numerous higher-order discretization schemes, which are more efficient than the Euler scheme, and discussed their convergence. These higher-order schemes include the Milstein scheme and the Platen-Wagner scheme. To illustrate the practical implementation of our model, we decide to use the forward Euler discretization scheme for being computationally convenient.

Let $\Delta Y_{j+1} := Y_{j+1} - Y_j$, for each $j = 0, 1, \ldots, 5039$; then,

$$\Delta Y_{j+1} := \left(r_{X_j} - \frac{1}{2}\sigma_{X_j}^2\right)\Delta + \int_{\mathcal{R}^+}(1 - e^{h(z)} + h(z))\rho_{X_j}^Q(dz|t_j)\Delta + \sigma_{X_j} \times \epsilon \times \sqrt{\Delta} + \widetilde{\mu}_{X_j}^Q(t_{j+1}) - \widetilde{\mu}_{X_j}^Q(t_j).$$

(4.6)

Given each simulated path of $\{\Delta Y_j\}_{j=1,2,\ldots,5039}$, $\{A_j\}_{j=1,2,\ldots,5040}$ is calculated as

$$A_j = A_0 e^{\sum_{k=1}^j \Delta Y_k},$$

(4.7)

where it is assumed that $A_0 = 100$.

We calculate the value of the policy reserve $\{R_t\}_{t=1,2,\ldots,20}$ annually over 20 years by applying the following forward Euler discretization scheme iteratively:

$$R_{t+1} = R_t \exp\left[\sum_{j=1}^{252}\max\left(r_g, \ln\left(\frac{A_{252t+j}}{R_t}\right) - \beta\right)\Delta\right],$$

(4.8)

where t represents the tth year and the initial value $R_0 = \alpha_p A_0$.

We assume some specimen values for the parameters for the participating policy:

$$r_g = 0.04; \qquad \beta = 0.5; \qquad \gamma = 0.7; \qquad \alpha_p = 0.6; \qquad A_0 = 100; \qquad T = 20 \text{ years}. \qquad (4.9)$$

We consider some specimen values for the model parameters:

$$\mu_1 = 0.10; \qquad \mu_2 = 0.05; \qquad r_1 = 0.035; \qquad r_2 = 0.015; \qquad \sigma_1 = 0.2; \qquad \sigma_2 = 0.4;$$
$$b_1 = 200.0, \qquad b_2 = 500.0. \tag{4.10}$$

We choose these specimen values to illustrate the practical implementation of our model. These values are in the reasonable ranges of magnitudes from a practical perspective and are consistent with the magnitudes of model parameters estimated in some empirical literature on jump-diffusion models. The estimation issue of our model, in particular, the Bayesian nonparametric estimation, is an interesting topic for future research. We plan to pursue this direction in our future research.

4.1. The MGG processes

In this subsection, we consider the MGG process, the GG process, the Merton jump-diffusion model, the VG process, and the GBM. We suppose that the shape parameter α for the (M)GG processes from 0.0 to 0.9, with an increment of 0.1. When $\alpha = 0.0$, the (M)GG process becomes the (M)WG process. When $\alpha = 0.5$, the (M)GG process becomes the (M)IG process. Other values of α generate different parametric forms of the (M)GG processes. We assume that the parameter values of the no-regime-switching versions of these processes match with those in the corresponding regime-switching processes when the economy is in "State 1." For the Merton jump diffusion model, we consider the following parameter values:

$$\mu = 0.1; \qquad r = 0.035; \qquad \sigma = 0.2; \qquad \mu_X = -0.05; \qquad \sigma_X = 0.07; \qquad \lambda = 0.6, \tag{4.11}$$

where the jump size X of the compound Poisson process follows a normal distribution with mean μ_X and variance σ_X^2; λ is the intensity parameter of the Poisson process.

We consider a VG process for the log return process $\{Y_t\}_{t \in \tau}$ with the following time-changed or subordinated Brownian motion representation:

$$Y_t = Z_{L(t,v)}, \quad t \in \tau, \tag{4.12}$$

where $Z_t = \theta t + \sigma_{\mathrm{VG}} W_t$; W_t is a standard Brownian motion; the subordinated process $L(t, v)$ is a Gamma process with unit mean rate and variance v.

We assume the following specimen values for the parameters of the VG process:

$$\theta = 0.0; \qquad v = 0.01; \qquad \sigma_{\mathrm{VG}} = 0.2. \tag{4.13}$$

The values of the parameters μ, r, and σ in this case are the same as those in the case of the Merton jump-diffusion model.

The fair values for the guaranteed benefit, the terminal bonus option, and the default option under the Merton jump-diffusion model and the VG model are evaluated under the Esscher transform. These fair values are computed using Monte Carlo simulations. In all figures, "with Markov switching" refers to the models with both the jump component and the model parameters being modulated by the two-state Markov chain; "without Markov switching" refers to the models with the jump component and constant model parameters; "no jump with Markov switching" refers to the Markovian regime-switching geometric Brownian
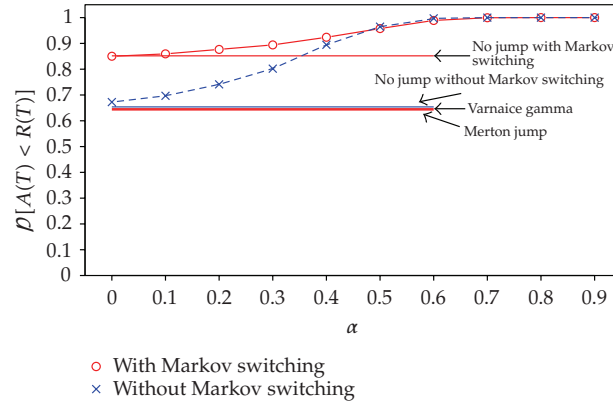
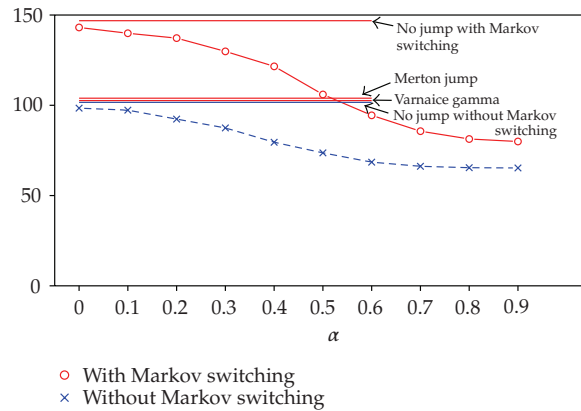**Figure 1:** Probabilities that the terminal asset value $A(T)$ is less than the terminal policy reserve $R(T)$ under $\mathcal{P}$.

Figure 2: Fair value of the guaranteed benefit.

motion; "no jump without Markov switching" refers to the geometric Brownian motion; "Merton jump" and "Variance Gamma" refer to the Merton jump-diffusion model and the variance Gamma process with constant model parameters, respectively.

Figures 1–4 display the numerical results for the probabilities that $A_T < R_T$ under $\mathcal{P}$ (i.e., the ruin probabilities) and the fair values for each of the embedded options. The numerical results for various parametric cases, namely, the MGG and GG processes with different values of the shape parameter α, the Merton jump-diffusion model, the VG process, and the GBM, are displayed here.

From Figure 1, we see that the probabilities that $A(T) < R(T)$ increase significantly as α does in both the regime-switching and no-regime-switching cases. When $\alpha < 0.5$, the probabilities that $A(T) < R(T)$ in the regime-switching case are significantly larger than their corresponding values in the no-regime-switching case. From Figures 2 to 4, the impact of α on the fair values of three embedded options is significant. We can also see that the effect of switching regimes on the fair values is also significant.
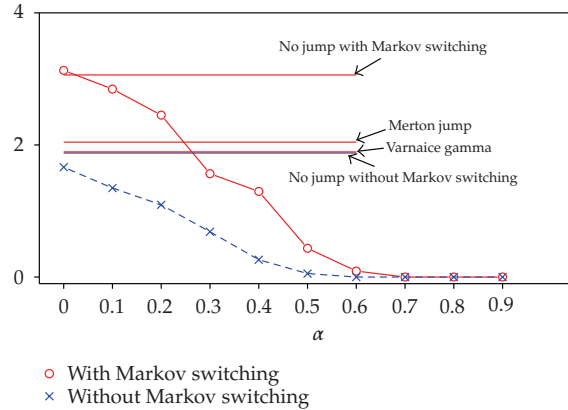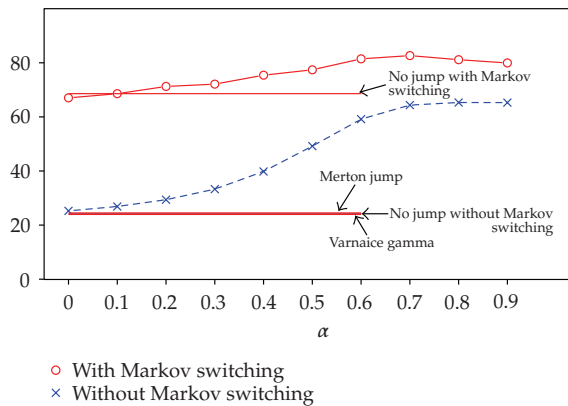
Figure 3: Fair value of the terminal bonus option.

Figure 4: Fair value of the default option.

4.2. The Scale-distorted version of the MGG and GG processes

We consider the scale-distorted version of the MGG and GG processes, the Merton jump-diffusion model, the VG process, and the GBM. We focus on investigating the impact of different values of the scale distortion parameter c on the underlying price behaviors, the payoff structures, and the fair values of the participating policy. In particular, we suppose that c takes values 0.5, 1.0 (i.e., no scale distortion), 2.0, and 3.0. Throughout this subsection, we suppose that the shape parameter $\alpha = 0.5$ for the scale-distorted version of the MGG and GG processes. The parameter values for the Merton jump-diffusion model and the VG model are given by those in Section 4.1.

Figures 5–8 display the numerical results for the ruin probabilities and the fair values of the three embedded options for various parametric cases, namely, the scale-distorted version of the MGG and GG processes with different values of the scale distortion parameter c, the Merton jump-diffusion model, the VG process, and the GBM.
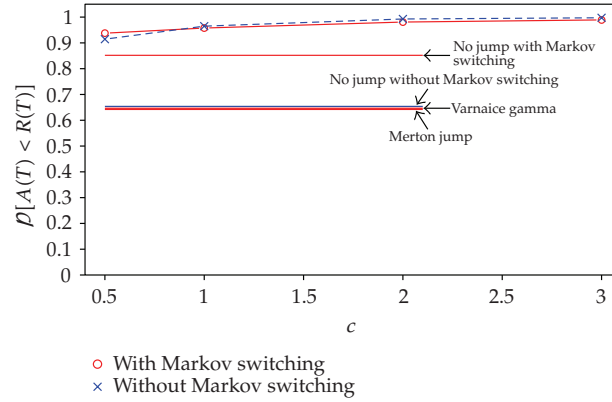
Figure 5: Probabilities that the terminal asset value $A(T)$ is less than the terminal policy reserve $R(T)$ under $\mathcal{P}$.
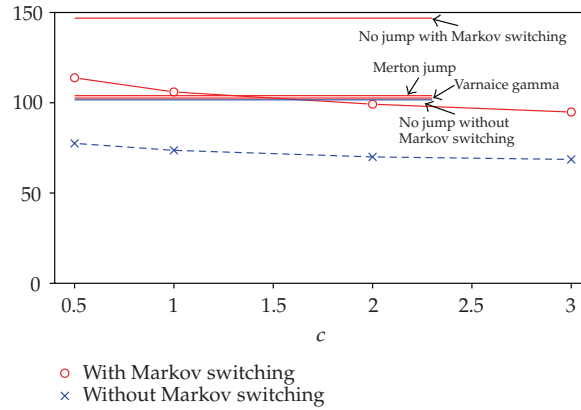

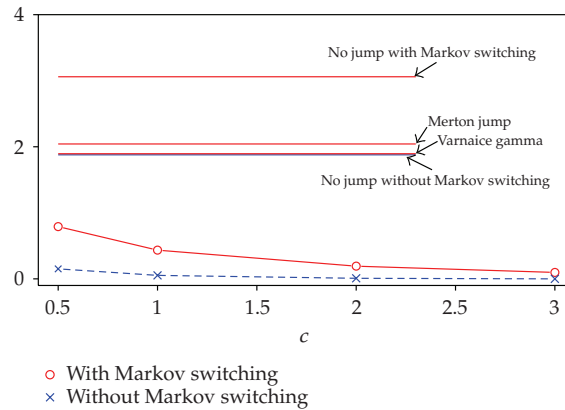**Figure 6:** Fair value of the guaranteed benefit.

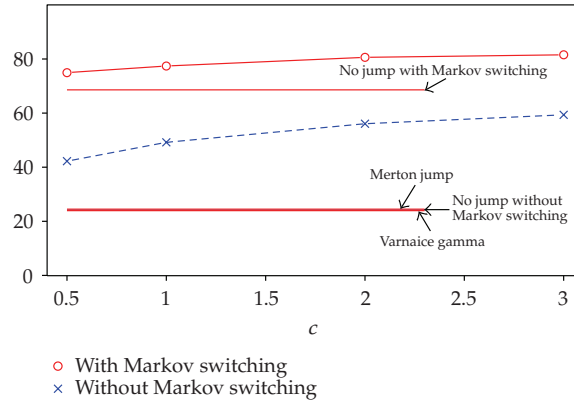**Figure 7:** Fair value of the terminal bonus option.
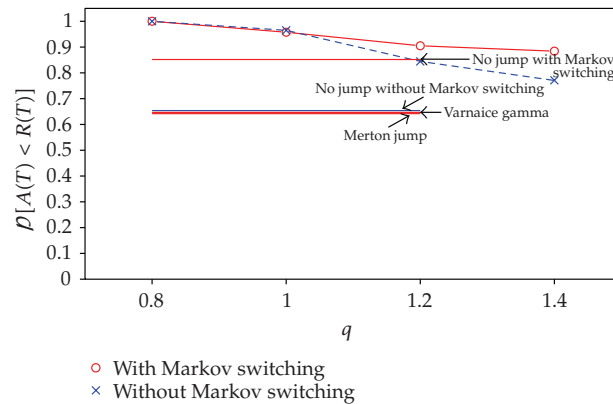
Figure 8: Fair value of the default option.

Figure 9: Probabilities that the terminal asset value $A(T)$ is less than the terminal policy reserve $R(T)$ under $\mathcal{P}$.

From Figures 5–8, the impact of the scale distortion parameter c on the the ruin probabilities and fair values of the embedded options is much less significant compared with that of the shape parameter α. From Figures 6–8, the effect of switching regimes on the fair values of the embedded options is still significant.

4.3. The power-distorted version of the MGG and GG processes

Now, we consider the power-distorted version of the MGG and GG processes, the Merton jump-diffusion model, the VG process, and the GBM. We suppose that the power distortion parameter q takes values 0.6, 0.8, 1.0 (i.e., no scale distortion), 1.2, and 1.4. Throughout this subsection, we suppose that the shape parameter $\alpha = 0.5$ for the power-distorted version of the MGG and GG processes. The Merton jump diffusion model and the VG process for fair valuation are the same as before.

Figures 9–12 display the numerical results for the ruin probabilities and the fair values of the three embedded options for various parametric cases, namely, the power-distorted version of the MGG and GG processes with different values of the power distortion parameter q, the Merton jump-diffusion model, the VG process, and the GBM.
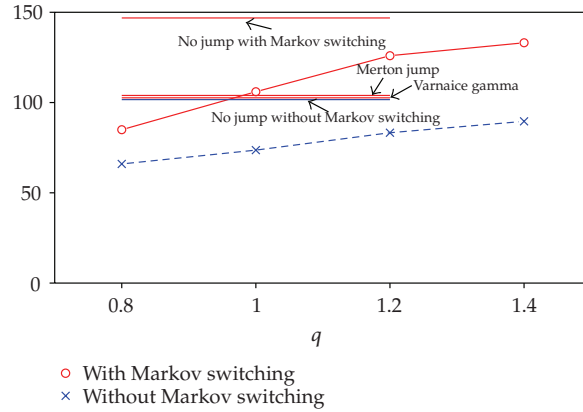
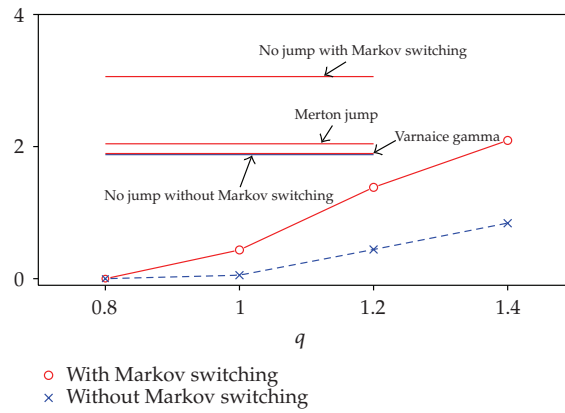**Figure 10:** Fair value of the guaranteed benefit.

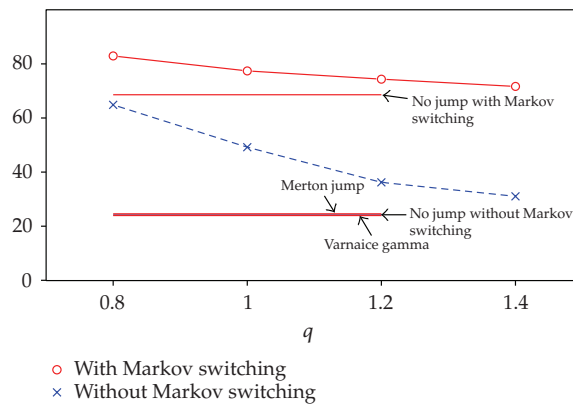**Figure 11:** Fair value of the terminal bonus option.

Figure 12: Fair value of the default option.

From Figures 9–12, the impact of the power distortion parameter q on the ruin probabilities and fair values of the embedded options is less significant than that of the shape parameter α, but more significant than that of the scale distortion parameter c. The effect of switching regimes on the fair values of the embedded options is still significant in this case (see Figures 10–12).

The market consistent valuation and the risk management of modern insurance products are important issues as highlighted by Solvency II and the International Accounting Standards Board (IASB). Solvency II refers to an amended set of regulatory requirements for insurance companies operating in the European Union region. Like the well-known regulatory requirements for banks and financial institutions, namely, Basel II, Solvency II operates as a three-pillar system, which consists of the quantitative evaluation of risk capitals, the review of the models by supervisors, and the disclosure of the risk information. So, Solvency II is also referred to as Basel II for insurance companies. Under Solvency II, insurance companies need to build their internal models for market consistent valuation and risk management and these models are then sent to supervisors for assessment and review. So, an important question for the insurance companies, perhaps the supervisors as well, is how to develop or build appropriate models for fair valuation and risk management of the insurance policies. An appropriate stochastic model for modeling the asset price dynamics plays a key role to answer this important question.

The numerical results have some implications for the market consistent valuation and the risk management practice of participating life insurance policies. They shed some lights on the importance of the correct specification of stochastic models for the long-term price movements of the reference portfolio. In particular, our study reveals that the specification of the parametric distribution of the jump component (i.e., the specification of the parameter α) and the incorporation of the regime-switching effect in the stochastic models for the long-term movements of the reference portfolio play a significant role in the market consistent valuation and the risk management via hedging of participating policies.

Besides providing some implications for the stochastic modeling of the asset price dynamics, the results of our studies also have some implications for the design of the products, for example, the choices of the target ratio β and the guaranteed rate r_g. From Figure 1, we see that the ruin probabilities increase substantially as α increases. Also, the ruin probabilities implied by the model without switching regimes are substantially lower than the corresponding values arising from the model with switching regimes. In other words, the mis-specification of the parametric distribution of the jump component in the stochastic model of the reference portfolio and/or neglecting the regime-switching effect may lead to underestimation of the ruin probabilities. So, the model risk is quite substantially here. One possible remedy for the model risk is to set the target ratio β and the guaranteed rate r_g in a more prudent way. For example, β can be set higher and r_g can be set lower, so that the interest rate credited in the scheme becomes lower and a lower terminal policy reserve may result. However, these two rates cannot be set as lower (higher) as we wish. It is important to consider the marketing/sale issue and the competition of other insurance companies which offer similar products when setting these rates.

5. Summary

We considered the pricing of participating life insurance policies when the market value of the reference asset is governed by a generalized jump-diffusion model with a Markov-switching

compensator, where the jump component is specified by a Markov-modulated kernel-biased completely random measure. The Esscher transform was adopted to determine an equivalent martingale measure under the incomplete market setting. Various parametric cases of the Markov-modulated kernel-biased completely random measure were considered. We conducted simulation experiments using the Poisson weighted algorithm to compare the fair values of participating products implied by our model with those obtained from other existing models in the literature and to highlight some features that can be obtained from our model. The simulation results reveal that the impacts of various specifications of jump component and the switching regimes on the fair values of the embedded options in participating products are significant. The results of our studies highlight the importance of the specification of the parametric distribution of the jump component and the incorporation of the regime-switching effect in the stochastic model for the reference portfolio underlying a participating policy. They also shed some lights on the design of the contractual structure of the policy.

Appendix

Proofs

Proof of Lemma 2.1. First, for any $t, s \in \mathcal{T}$ with $t \geq s$,

$$
\begin{aligned}
E\left[\frac{\Lambda_t}{\Lambda_s}\bigg|\mathcal{G}_s\right] = E\bigg\{ &\exp\bigg[-\int_s^t \theta_u \sigma_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 \sigma_u^2 du - \int_s^t \int_{\mathcal{R}^+} \theta_u h(z)\widetilde{N}_{X_u}(du, dz) \\
&-\int_s^t \int_{\mathcal{R}^+} (e^{-\theta_u h(z)} - 1 + \theta_u h(z))\rho_{X_u}(dz|u)\eta(du)\bigg]\bigg|\mathcal{G}_s\bigg\}.
\end{aligned}
\tag{A.1}
$$

Note that

$$
E\left[\exp\left(-\int_s^t \theta_u \sigma_u dW_u \right)\bigg|\mathcal{G}_s\right] = \exp\left(-\frac{1}{2}\int_s^t \theta_u^2 \sigma_u^2 du \right),
\tag{A.2}
$$

and, by James [18, 19],

$$
E\left[\exp\left(-\int_s^t \int_{\mathcal{R}^+} \theta_u h(z)\widetilde{N}_{X_u}(du, dz) \right)\bigg|\mathcal{G}_s\right] = \exp\left[\int_s^t \int_{\mathcal{R}^+} (e^{-\theta_u h(z)} - 1 + \theta_u h(z))\rho_{X_u}(dz|u)\eta(du)\right].
\tag{A.3}
$$

Hence,

$$
E\left[\frac{\Lambda_t}{\Lambda_s}\bigg|\mathcal{G}_s\right] = 1, \quad \mathcal{P}\text{-a.s.}
\tag{A.4}
$$

$\square$

Proof of Proposition 2.2. First, by Bayes' rule,

$$
\begin{aligned}
E_Q\left[\exp\left(-\int_0^t r_u du\right)A_t\Big|\mathcal{G}_0\right] &= \exp\left(-\int_0^t r_u du\right)E_P\left[\Lambda_t\exp\left(\int_0^t dY_u\right)|\mathcal{G}_0\right] \\
&= \exp\left(-\int_0^t r_u du\right)\frac{E_P\left[\exp\left(-\int_0^t(\theta_u-1)dY_u\right)|\mathcal{G}_0\right]}{\mathcal{M}_Y(\theta)_t} \\
&= \exp\left(-\int_0^t r_u du\right)\frac{\mathcal{M}_Y(\theta-1)_t}{\mathcal{M}_Y(\theta)_t} \\
&= \exp\left[\int_0^t\left(\mu_s - r_s - \frac{1}{2}\sigma_s^2\right)ds + \frac{1}{2}\int_0^t(1-2\theta_s)\sigma_s^2 ds\right. \\
&\qquad \left. + \int_0^t\int_{\mathcal{R}^+}\left(e^{-(\theta_s-1)h(z)} - e^{-\theta_s h(z)} - h(z)\right)\rho_{X_s}(dz|s)\eta'(s)ds\right].
\end{aligned}
$$

$$(A.5)$$

Then, by setting $s = 0$, the martingale condition implies that

$$1 = E_Q[\tilde{A}_t|\mathcal{G}_0]. \tag{A.6}$$

This implies that

$$\int_0^t\left(\mu_s - r_s - \frac{1}{2}\sigma_s^2\right)ds + \frac{1}{2}\int_0^t(1-2\theta_s)\sigma_s^2 ds + \int_0^t\int_{\mathcal{R}^+}\left(e^{-(\theta_s-1)h(z)} - e^{-\theta_s h(z)} - h(z)\right)\rho_{X_s}(dz|s)\eta'(s)ds = 0, \tag{A.7}$$

for each $t \in \mathcal{T}$. Hence (2.28) is proved. Given (2.28), the martingale condition is satisfied. $\square$

Proof of Proposition 2.3. Let $Z_u \in BM(\mathcal{T})$. Then, by Bayes' rule,

$$
\begin{aligned}
\mathcal{M}_Y^\theta(Z)_t &:= E_Q\left[e^{-(Z\cdot Y)_t} \mid \mathcal{G}_0\right] = E\left[\Lambda_t e^{-(Z\cdot Y)_t} \mid \mathcal{G}_0\right] \\
&= \exp\left\{-\int_0^t Z_s\left(\mu_s - \theta_s\sigma_s^2 - \frac{1}{2}\sigma_s^2\right)ds + \frac{1}{2}\int_0^t Z_s^2\sigma_s^2 ds\right. \\
&\qquad \left. + \int_0^t\int_{\mathcal{R}^+}\left[e^{-\theta_s h(z)}\left(e^{-Z_s h(z)} - 1\right) - Z_s h(z)\right]\rho_{X_s}(dz|s)\eta'(s)ds\right\}.
\end{aligned}
$$

$$(A.8)$$

From the martingale condition,

$$\mu_t - \theta_t\sigma_t^2 = r_t - \int_{\mathcal{R}^+}\left[e^{-\theta_t h(z)}\left(e^{h(z)} - 1\right) - h(z)\right]\rho_{X_t}(dz|t)\eta'(t)dt. \tag{A.9}$$

Hence,

$$
\begin{aligned}
\mathcal{M}_Y^\theta(Z)_t &= \exp\left\{ -\int_0^t Z_s\left(r_s - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t\int_{\mathcal{R}^+} Z_s\left(e^{h(z)} - 1 - h(z)\right)e^{-\theta_s h(z)}\rho_{X_s}(dz|s)\eta'(s)ds \right. \\
&\quad \left. + \frac{1}{2}\int_0^t Z_s^2\sigma_s^2 ds + \int_0^t\int_{\mathcal{R}^+}\left(e^{-Z_s h(z)} - 1 + Z_s h(z)\right)e^{-\theta_s h(z)}\rho_{X_s}(dz|s)\eta'(s)ds \right\} \\
&= \exp\left\{ -\int_0^t Z_s\left(r_s - \frac{1}{2}\sigma_s^2\right)ds - \int_0^t\int_{\mathcal{R}^+} Z_s\left(1 - e^{h(z)} + h(z)\right)\rho_{X_s}^Q(dz|s)\eta'(s)ds \right. \\
&\quad \left. + \frac{1}{2}\int_0^t Z_s^2\sigma_s^2 ds + \int_0^t\int_{\mathcal{R}^+}\left(e^{-Z_s h(z)} - 1 + Z_s h(z)\right)\rho_{X_s}^Q(dz|s)\eta'(s)ds \right\}.
\end{aligned}
\tag{A.10}
$$

Then, under Q,

$$
dY_t = \left(r_t - \frac{1}{2}\sigma_t^2\right)dt + \int_{\mathcal{R}^+}\left(1 - e^{h(z)} + h(z)\right)\rho_{X_t}^Q(dz|t)\eta'(t)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+} h(z)\widetilde{N}_{X_t}^Q(dt, dz).
\tag{A.11}
$$

Since X is independent with W and N, the probability law of X remains unchanged under the change of measures from $\mathcal{P}$ to Q. □

Acknowledgments

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