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Research Article

On Different Classes of Algebraic Polynomials with Random Coefficients

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The expected number of real zeros of the polynomial of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ is a sequence of standard Gaussian random variables, is known. For n large it is shown that this expected number in $(-\infty, \infty)$ is asymptotic to $(2/\pi)\log n$. In this paper, we show that this asymptotic value increases significantly to $\sqrt{n+1}$ when we consider a polynomial in the form $a_0\binom{n}{0}^{1/2}x/\sqrt{1} + a_1\binom{n}{1}^{1/2}x^2/\sqrt{2} + a_2\binom{n}{2}^{1/2}x^3/\sqrt{3} + \cdots + a_n\binom{n}{n}^{1/2}x^{n+1}/\sqrt{n+1}$ instead. We give the motivation for our choice of polynomial and also obtain some other characteristics for the polynomial, such as the expected number of level crossings or maxima. We note, and present, a small modification to the definition of our polynomial which improves our result from the above asymptotic relation to the equality.

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1. Introduction

The classical random algebraic polynomial has previously been defined as

$$T(x) \equiv T_n(x,\omega) = \sum_{j=0}^{n} a_j(\omega)x^j, \tag{1.1}$$

where, for $(\Omega, \mathcal{A}, \Pr)$ a fixed probability space, $\{a_j(\omega)\}_{j=0}^{n}$ is a sequence of independent random variables defined on Ω. For n large, the expected number of real zeros of $T(x)$, in the interval $(-\infty, \infty)$, defined by $\mathrm{EN}_{0,T}(-\infty,\infty)$, is known to be asymptotic to $(2/\pi)\log n$. For this case the coefficients $a_j \equiv a_j(\omega)$ are assumed to be identical normal standard. This asymptotic value was first obtained by the pioneer work of Kac [1] and was recently significantly improved by Wilkins [2], who reduced the error term involved in this asymptotic formula to $O(1)$. Since then, many other mathematical properties of $T(x)$ have been studied and they are listed in [3] and more recently in [4].

The other class of random polynomials is introduced in an interesting article of Edelman and Kostlan [5] in which the jth coefficients of $T(x)$ in (1.1) have nonidentical variance $\binom{n}{j}$. It is interesting to note that in this case the expected number of zeros significantly increased to $\sqrt{n}$, showing that the curve representing this type of polynomial oscillates significantly more than the classical polynomial (1.1) with identical coefficients. As it is the characteristic of $\binom{n}{j}$, $j = 0, 1, 2, \ldots, n$ maximized at the middle term of $j = [n/2]$, it is natural to conjecture that for other classes of distributions with this property the polynomial will also oscillate significantly more. This conjecture is examined in [6, 7]. This interesting and unexpected property of the latter polynomial has its close relation to physics reported by Ramponi [8], which together with its mathematical interest motivated us to study the polynomial

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n} a_j \binom{n}{j}^{1/2} \frac{x^{j+1}}{\sqrt{j+1}}. \tag{1.2}$$

As we will see, because of the presence of the binomial elements in (1.2), we can progress further than the classical random polynomial defined in (1.1). However, even in this case the calculation yields an asymptotic result rather than equality. With a small change to the definition of the polynomial we show that the result improves. To this end we define

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n} a_j \binom{n}{j}^{1/2} \frac{x^{j+1}}{\sqrt{j+1}} + \frac{a^*}{\sqrt{n+1}}, \tag{1.3}$$

where a^* is mutually independent of and has the same distribution as $\{a_j\}_{j=0}^{n}$. We prove the following.

Theorem 1.1. *When the coefficients a_j of $P(x)$ are independent standard normal random variables, then the expected number of real roots is asymptotic to*

$$EN_{0,P}(-\infty, \infty) \sim \sqrt{n+1}. \tag{1.4}$$

Corollary 1.2. *With the same assumption as Theorem 1.1 for the coefficients a_j and a^* one has*

$$EN_{0,Q}(-\infty, \infty) = \sqrt{n+1}. \tag{1.5}$$

Also of interest is the expected number of times that a curve representing the polynomial cuts a level K. We assume K is any constant such that

$$\begin{gathered} \text{(i) } K^2 \leq \frac{e^n}{n^2}, \\ \text{(ii) } \frac{1}{n^2} = o\left(K^2\right), \\ \text{(iii) } K^2 = o\left(\frac{e^n}{n^2}\right). \end{gathered} \tag{1.6}$$

For example, any absolute constant $K \neq 0$ satisfies these conditions. Defining $\mathrm{EN}_{K,P}$ as the expected number of real roots of $P(x) = K$, we can generalize the above theorem to the following one.

Theorem 1.3. *When the coefficients a_j have the same distribution as in Theorem 1.1, and K obeys the above conditions (i)–(iii), the asymptotic estimate for the expected number of K-level crossings is*

$$EN_{K,Q}(-\infty,\infty) \sim EN_{K,P}(-\infty,\infty) \sim \sqrt{n+1}. \tag{1.7}$$

The other characteristic which also gives a good indication of the oscillatory behavior of a random polynomial is the expected number of maxima or minima. We denote this expected number by EN_P^M for polynomial $P(x)$ given in (1.2) and, since the event of tangency at the x-axis has probability zero, we note that this is asymptotically the same as the expected number of real zeros of $P'(x) = dP(x)/dx$. In the following theorem, we give the expected number of maxima of the polynomial.

Theorem 1.4. *With the above assumptions on the coefficients a_j, then the asymptotic estimate for the expected number of maxima of $P(x)$ is*

$$EN_P^M(-\infty,\infty) \sim \sqrt{n}. \tag{1.8}$$

Corollary 1.5. *With the above assumptions for the coefficients a_j and a^* one has*

$$EN_Q^M(-\infty,\infty) \sim \sqrt{n}. \tag{1.9}$$

2. Proof of Theorem 1.1

We use a well-known Kac-Rice formula, [1, 9], in which it is proved that

$$\mathrm{EN}_{0,P}(a,b) = \int_a^b \frac{\Delta}{\pi A^2}\,dx, \tag{2.1}$$

where $P'(x)$ represents the derivative with respect to x of $P(x)$. We denote

$$A^2 = \mathrm{var}\big(P(x)\big), \qquad B^2 = \mathrm{var}\big(P'(x)\big), \qquad C = \mathrm{cov}\big(P(x), P'(x)\big), \qquad \Delta^2 = A^2B^2 - C^2. \tag{2.2}$$

Now, with our assumptions on the distribution of the coefficients, it is easy to see that

$$A^2 = \sum_{j=0}^{n} \binom{n}{j} \frac{x^{2j+2}}{j+1} = \frac{\left(1+x^2\right)^{n+1}}{n+1} - \frac{1}{n+1}, \tag{2.3}$$

$$B^2 = \sum_{j=0}^{n} \binom{n}{j} (j+1)x^{2j} = \left(1+x^2\right)^{n-1}\left(1+x^2+nx^2\right), \tag{2.4}$$

$$C = \sum_{j=0}^{n} \binom{n}{j} x^{2j+1} = x\left(1+x^2\right)^{n}. \tag{2.5}$$

We note that, for all sufficiently large n and x bounded away from zero, from (2.3) we have

$$A^2 \sim \frac{\left(1+x^2\right)^{n+1}}{n+1}. \tag{2.6}$$

This together with (2.1), (2.4), and (2.5) yields

$$\mathrm{EN}_0(-\infty,\infty) \sim \frac{2}{\pi}\int_0^{\epsilon}\frac{\Delta}{A^2}dx + \frac{2}{\pi}\int_{\epsilon}^{\infty}\frac{\sqrt{n+1}}{1+x^2}\,dx, \tag{2.7}$$

where $\epsilon > 0$, $\epsilon \to 0$ as $n \to \infty$. The second integral can be expressed as

$$\frac{2\sqrt{n+1}}{\pi}\left\{\frac{\pi}{2} - \arctan\epsilon\right\} \longrightarrow \sqrt{n+1} \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

In the first integral, the expression (Δ/A^2) has a singularity at $x = 0$:

$$\frac{\Delta}{A^2} = \sqrt{\frac{(n+1)\left\{(1+x^2)^{2n} - (1+x^2)^{n-1}(1+x^2+nx^2)\right\}}{\left\{(1+x^2)^{n+1}-1\right\}^2}}. \tag{2.9}$$

Notice that the expression in (2.9) is bounded from above:

$$\frac{\Delta}{A^2} < \frac{\sqrt{(n+1)(1-D)}}{1+x^2}, \tag{2.10}$$

where

$$\begin{aligned} D &= \frac{1+nx^2(1+x^2)^{n-1} - (1+x^2)^n}{\left\{(1+x^2)^n - 1\right\}^2} \\ &= \frac{(n-1)(1+x^2)^{n-2} + (n-2)(1+x^2)^{n-3} + \cdots + 3(1+x^2)^2 + 2(1+x^2) + 1}{\left\{(1+x^2)^{n-1} + (1+x^2)^{n-2} + \cdots + (1+x^2) + 1\right\}^2}. \end{aligned} \tag{2.11}$$

When $x = 0$, we have

$$D = \frac{n^2 - n}{2n^2} \tag{2.12}$$

and therefore

$$\frac{\Delta}{A^2} < \frac{n+1}{\sqrt{2n}} \sim \sqrt{\frac{n+1}{2}}, \tag{2.13}$$

which means that the integrand in the first integral of (2.7) is bounded for every n. When $x > 0$, it can easily be seen that

$$1 > D > \frac{\sum_{j=0}^{n-2}(1+j)}{n^2(1+x^2)^{2n-2}} > 0, \tag{2.14}$$

and therefore

$$\frac{\Delta}{A^2} < \frac{\sqrt{n+1}}{1+x^2}. \tag{2.15}$$

Hence, the first integral that appears in (2.7) is bounded from above as follows:

$$\frac{2}{\pi}\int_0^{\epsilon}\frac{\Delta}{A^2}\,dx<\frac{2}{\pi}\int_0^{\epsilon}\frac{\sqrt{n+1}}{1+x^2}\,dx=\frac{2(\arctan\epsilon)\sqrt{n+1}}{\pi}=o(\sqrt{n+1}) \tag{2.16}$$

by the choice of ϵ. Altogether, the value of the first integral in (2.7) is of a smaller order of magnitude than the value of the second integral, and we have from (2.7)

$$\mathrm{EN}_0(-\infty,\infty)\sim\sqrt{n+1} \tag{2.17}$$

which completes the proof of Theorem 1.1.

In order to obtain the proof of Corollary 1.2, we note that the above calculations remain valid for B^2 and C. However, for A^2 we can obtain the exact value rather than the asymptotic value. To this end, we can easily see that

$$A_Q^2=\operatorname{var}\bigl(Q(x)\bigr)=\sum_{j=0}^{n}\binom{n}{j}\frac{x^{2j+2}}{j+1}+\frac{1}{n+1}=\frac{\bigl(1+x^2\bigr)^{n+1}}{n+1}. \tag{2.18}$$

Substituting this value instead of (2.3) together with (2.4) and (2.5) in the Kac-Rice formula (2.1), we get a much more straight forward expression than that in the above proof:

$$\mathrm{EN}_{0,Q}(-\infty,\infty)=\frac{1}{\pi}\int_0^{\infty}\frac{\sqrt{n+1}}{1+x^2}\,dx=\sqrt{n+1}. \tag{2.19}$$

This gives the proof of Corollary 1.2.

3. Level crossings

To find the expected number of K-level crossings, we use the following extension to the Kac-Rice formula as it was used in [10]. It is shown that in the case of normal standard distribution of the coefficients

$$\mathrm{EN}_K(a,b)=I_1(a,b)+I_2(a,b) \tag{3.1}$$

with

$$I_1(a,b)=\int_a^b\frac{\Delta}{\pi A^2}\exp\left(-\frac{B^2K^2}{2\Delta^2}\right)dx, \tag{3.2}$$

$$I_2(a,b)=\int_a^b\frac{\sqrt{2}KC}{\pi A^3}\exp\left(-\frac{K^2}{2A^2}\right)\operatorname{erf}\left(-\frac{KC}{\sqrt{2}A\Delta}\right)dx, \tag{3.3}$$

where, as usual, $\operatorname{erf}(x)=\int_0^x\exp(-t)dt\le\sqrt{\pi}/2$. Since changing x to $-x$ leaves the distribution of the coefficients unchanged, $\mathrm{EN}_K(-\infty,0)=\mathrm{EN}_K(0,\infty)$. Hence to what follows we are only concerned with $x\ge 0$. Using (2.3)–(2.5) and (3.2) we obtain

$$I_1(-\infty,\infty)=\frac{2\sqrt{n+1}}{\pi}\int_0^{\infty}\frac{1}{1+x^2}\exp\left(-\frac{K^2(n+1)\bigl(1+x^2+nx^2\bigr)}{2\bigl(1+x^2\bigr)^{n+1}}\right)dx. \tag{3.4}$$

Using substitution $x = \tan\theta$ in (3.4) we can see that

$$I_1(-\infty,\infty) = J_1\left(0,\frac{\pi}{2}\right) = \frac{2\sqrt{n+1}}{\pi}\int_0^{\pi/2} \exp\left(\frac{-K^2(n+1)}{2}(1+n\sin^2\theta)\cos^{2n}\theta\right)d\theta, \tag{3.5}$$

where the notation J_1 emphasizes integration in θ. In order to progress with the calculation of the integral appearing in (3.5), we first assume $\theta > \delta$, where $\delta = \arccos(1-1/(n\epsilon))$, where $\epsilon = 1/\{2\log(nK)\}$. This choice of ϵ is indeed possible by condition (i). Now since $\cos\theta < (1-1/(n\epsilon))$, we can show that

$$\cos^{2n}\theta < \left(1-\frac{1}{n\epsilon}\right)^{2n} = \left(\left(1-\frac{1}{n\epsilon}\right)^{-n\epsilon}\right)^{-2/\epsilon} \sim \exp\left(-\frac{2}{\epsilon}\right) \longrightarrow 0 \tag{3.6}$$

as $n\to\infty$. Now we are in a position to evaluate the dominated term which appears in the exponential term in (3.5). From (3.6), it is easy to see that for our choice of θ

$$K^2n^2\cos^{2n-2}\theta < K^2n^2\exp\left(-\frac{2}{\epsilon}\right) = K^2n^2\exp\left(-4\log(nK)\right) = (Kn)^{-2} \longrightarrow 0, \tag{3.7}$$

by condition (ii). Therefore, for all sufficiently large n, the argument of the exponential function in (3.5) is reduced to zero, and hence the integrand is *not* a function of θ and we can easily see by the bounded convergence theorem and condition (iii) that

$$J_1\left(\delta,\frac{\pi}{2}\right) \sim \sqrt{n+1}. \tag{3.8}$$

Since the argument of the exponential function appearing in (3.5) is always negative, it is straight forward for our choice of δ and ϵ to see that

$$J_1(0,\delta) < \frac{2\sqrt{n+1}}{\pi}\int_0^{\delta} d\theta = \frac{2}{\pi}\sqrt{n+1}\ \arccos\left(1-\frac{2\log(nK)}{n}\right) = o(\sqrt{n+1}), \tag{3.9}$$

by condition (iii). As $I_1(-\infty,\infty) = J_1(0,\delta) + J_1(\delta,\pi/2)$, by (3.8) and (3.9) we see that

$$I_1(-\infty,\infty) \sim \sqrt{n+1}. \tag{3.10}$$

Now we obtain an upper limit for I_2 defined in (3.3). To this end, we let $v = K/(\sqrt{2}A)$. Then we have

$$I_2(-\infty,\infty) \le \frac{|K|}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{C}{A^3}\exp\left(-\frac{K^2}{2A^2}\right)dx = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-v^2\right)dv \le \frac{2}{\sqrt{\pi}}. \tag{3.11}$$

This together with (3.10) proves that $\mathrm{EN}_{K,Q}(-\infty,\infty) \sim \sqrt{n+1}$. The theorem is proved for polynomial $Q(x)$ given in (1.3).

Let us now prove the theorem for polynomial $P(x)$ given in (1.2), that is

$$\mathrm{EN}_{K,P}(-\infty,\infty) \sim \sqrt{n+1}. \tag{3.12}$$

The proof in this case repeats the proof for $\mathrm{EN}_{K,Q}(-\infty,\infty)$ above, except that the equivalent of (3.4) will be an asymptotic rather than an exact equality, and the derivation of the equivalent of (3.9) is a little more involved, as shown below. Going back from the new variable θ to the original variable x gives

$$J_1(0,\delta) = \frac{2}{\pi}\int_0^{\tan\delta} \frac{\Delta}{A^2} \exp\left(-\frac{B^2K^2}{2\Delta^2}\right) dx < \frac{2}{\pi}\int_0^{\tan\delta} \frac{\Delta}{A^2}\, dx, \tag{3.13}$$

where Δ/A^2 is given by (2.9). Then by the same reasoning as in the proof of Theorem 1.1,

$$\begin{aligned} J_1(0,\delta) &< \frac{2\sqrt{n+1}}{\pi}\arctan(\tan\delta) = \frac{2\sqrt{n+1}}{\pi}\delta \\ &= \frac{2\sqrt{n+1}}{\pi}\arccos\left(1-\frac{2\log(nK)}{n}\right) = o(\sqrt{n+1}), \end{aligned} \tag{3.14}$$

by condition (iii). This completes the proof of Theorem 1.3.

4. Number of maxima

In finding the expected number of maxima of $P(x)$, we can find the expected number of zeros of its derivative $P'(x)$. To this end we first obtain the following characteristics needed in order to apply them into the Kac-Rice formula (2.1),

$$\begin{aligned} A_M^2 &= \operatorname{var}\left(P'(x)\right) = \sum_{j=0}^{n}\binom{n}{j}(j+1)x^{2j} \\ &= \left(1+x^2\right)^{n-1}\left(1+x^2+nx^2\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} B_M^2 &= \operatorname{var}\left((P''(x)\right) = \sum_{j=0}^{n}\binom{n}{j} j^2(j+1)x^{2j-2} \\ &= n\left(1+x^2\right)^{n-3}\left(2+4nx^2+nx^4+n^2x^4\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} C_M &= \operatorname{cov}\left(P'(x),P''(x)\right) = \sum_{j=0}^{n}\binom{n}{j} j(j+1)x^{2j-1} \\ &= nx\left(1+x^2\right)^{n-2}\left(2+x^2+nx^2\right). \end{aligned} \tag{4.3}$$

Hence from (4.1)–(4.3) we obtain

$$\Delta_M^2 = A_M^2B_M^2 - C_M^2 = n\left(1+x^2\right)^{2n-4}\left[2+nx^4+n^2x^4+2x^2+2nx^2\right]. \tag{4.4}$$

Now from (4.1) and (4.5) we have

$$\frac{\Delta_M}{A_M^2} = \frac{\sqrt{n\left(2+nx^4+n^2x^4+2x^2+4nx^2\right)}}{\left(1+x^2\right)\left(1+x^2+nx^2\right)}. \tag{4.5}$$

As the value of x increases, the dominating terms in (4.5) change. For accuracy therefore, the interval needs to be broken up. In this case, the interval $(0, \infty)$ was divided into two subintervals. First, choose $\epsilon < x < \infty$ such that $\epsilon = n^{-1/4}$, then

$$\frac{\Delta_M}{A_M^2} \sim \frac{\sqrt{n}}{1 + x^2}. \tag{4.6}$$

Substituting into the Kac-Rice formula (2.1) yields

$$\mathrm{EN}_P^M(\epsilon, \infty) \sim \frac{1}{\pi} \int_\epsilon^\infty \frac{\sqrt{n}}{1 + x^2} dx = \frac{\sqrt{n}}{2}. \tag{4.7}$$

Now we choose $0 < x < \epsilon$. Since for n sufficiently large the term $n^2 x^4$ is significantly larger than nx^4 and also since for this range of x we can see $2x^2 < 1$, we can obtain an upper limit for (4.5) as

$$\frac{\Delta_M}{A_M^2} < \frac{\sqrt{n(3 + 2n^2 x^4 + 4nx^2)}}{1 + nx^2} < \frac{\sqrt{n(3 + 6nx^2 + 3n^2 x^4)}}{1 + nx^2} = \sqrt{3n}. \tag{4.8}$$

Substituting this upper limit into Kac-Rice formula, we can see

$$\mathrm{EN}_P^M(0, \epsilon) = \int_0^\epsilon \frac{\Delta_M}{\pi A_M^2} dx < \sqrt{3n}\epsilon = o(n^{1/4}). \tag{4.9}$$

This together with (4.7) completes the proof of Theorem 1.4. To prove Corollary 1.5, it suffices to notice that since $Q'(x) = P'(x)$ and $Q''(x) = P''(x)$, all the arguments in the above proof apply to polynomial $Q(x)$, and we have therefore $\mathrm{EN}_P^M(a, b) = \mathrm{EN}_Q^M(a, b)$.

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