RANDOM FIXED POINT THEOREMS FOR MULTIVALUED NONEXPANSIVE NON-SELF-RANDOM OPERATORS

S. PLUBTIENG AND P. KUMAM

Received 8 March 2005; Revised 9 June 2005; Accepted 4 August 2005

Let (Ω, Σ) be a measurable space, with Σ a sigma-algebra of subset of Ω, and let C be a nonempty bounded closed convex separable subset of a Banach space X, whose characteristic of noncompact convexity is less than 1, $KC(X)$ the family of all compact convex subsets of X. We prove that a multivalued nonexpansive non-self-random operator $T : \Omega \times C \to KC(X)$, 1-$\chi$-contractive mapping, satisfying an inwardness condition has a random fixed point.

1. Introduction

In recent years there have appeared various random fixed point theorems for single-valued and set-valued random operators; see for example, Itoh [7], Ramírez [9], Tan and Yuan [10], Xu [12, 13] Yuan and Yu [15], and references therein.

Ramírez [9] proved the existence of random fixed point theorems for a random nonexpansive operator in the framework of Banach spaces with a characteristic of noncompact convexity $\varepsilon_\alpha(X)$ is less than 1. On the other hand, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a set-valued nonexpansive self-mapping and 1-χ-contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. Domínguez Benavides and Ramírez [5] proved a fixed point theorem for a multivalued nonexpansive non-self-mapping and 1-χ-contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the Kuratowski measure of noncompactness $\varepsilon_\alpha(X)$ is less than 1.

The purpose of the present paper is to prove a random fixed point theorem for multivalued nonexpansive non-self-random operators which is 1-χ-contractive mapping, in the framework of Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1 and satisfying an inwardness condition. Our result can also be seen as an extension of [5, Theorem 3.4].

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 43796, Pages 1–9
DOI 10.1155/JAMSA/2006/43796

2. Preliminaries and notations

We begin with establishing some preliminaries. By (Ω,Σ) we denote a measurable space with Σ a sigma-algebra of subset of Ω. Let (X,d) be a metric space. We denote by $CL(X)$ (resp., $\mathrm{CB}(X)$, $KC(X)$) the family of all nonempty closed (resp., closed bounded, compact convex) subset of X, and by H the Hausdorff metric on $\mathrm{CB}(X)$ induced by d, that is,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \tag{2.1}$$

for $A,B \in \mathrm{CB}(X)$, where $d(x,E) = \inf\{d(x,y) \mid y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a multivalued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf. [1, 2] for details). There is another different kind of continuity for multivalued operator: $T : C \to \mathrm{CB}(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [2]) that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

If C is a closed convex subset of Banach spaces X, then a multivalued mapping $T : C \to \mathrm{CB}(X)$ is said to be a *contraction* if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \leq k\|x-y\|, \quad x,y \in C, \tag{2.2}$$

and T is said to be *nonexpansive* if

$$H(Tx,Ty) \leq \|x-y\|, \quad x,y \in C. \tag{2.3}$$

A multivalued operator $T : \Omega \to 2^X$ is called (Σ)-measurable if, for any open subset B of X,

$$T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\} \tag{2.4}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable multivalued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \to 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(\cdot,x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega,\cdot)$, that is,

$$F(\omega) := \{x \in C : x \in T(\omega,x)\}. \tag{2.5}$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega,\cdot) : C \to 2^X$, $F(\omega)$ may be empty. A measurable operator $x : \Omega \to C$ is said to be a *random fixed point of an operator* $T : \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega,x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\omega,\cdot) \to 2^X$ is continuous.

A random operator $T:\Omega\times C\to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega\in\Omega$, the map $T:(\omega,\cdot)\to C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1 (Wagner, cf. [11]). *Let (X,d) be a complete separable metric space and $F:\Omega\to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2 (Itoh, cf. [7]). *Suppose $\{T_n\}$ is a sequence of measurable multivalued operator from Ω to $\mathrm{CB}(X)$ and $T:\Omega\to\mathrm{CB}(X)$ is an operator. If, for each $\omega\in\Omega$, $H(T_n(\omega),T(\omega))\to 0$, then T is measurable.*

Lemma 2.3 (Tan and Yuan, cf. [10]). *Let X be a separable metric space and Y a metric space. If $f:\Omega\times X\to Y$ is measurable in $\omega\in\Omega$ and continuous in $x\in X$, and if $x:\Omega\to X$ is measurable, then $f(\cdot,x(\cdot)):\Omega\to Y$ is measurable.*

As an easy application of Itoh [7, Proposition 3], we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space X, $T:\Omega\times C\to C$ a random continuous operator, and $F:\Omega\to 2^C$ a measurable closed-valued operator. Then for any $s>0$, the operator $G:\Omega\to 2^C$ given by*

$$G(\omega)=\{x\in F(\omega):\|x-T(\omega,x)\|<s\},\quad \omega\in\Omega, \tag{2.6}$$

is measurable and so is the operator $\mathrm{cl}\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5 (Domínguez Benavidel and Lopez Acedo, cf. [3]). *Suppose C is a weakly closed nonempty separable subset of a Banach space X, $F:\Omega\to 2^X$ measurable with weakly compact values, $f:\Omega\times C\to\mathbb{R}$ measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r:\Omega\to\mathbb{R}$ defined by*

$$r(\omega):=\inf_{x\in F(\omega)} f(\omega,x) \tag{2.7}$$

and the marginal map $R:\Omega\to X$ defined by

$$R(\omega):=\{x\in F(\omega):f(\omega,x)=r(\omega)\} \tag{2.8}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are, respectively, defined as the numbers

$$\begin{aligned}\alpha(B)&=\inf\{r>0:B\text{ can be covered by finitely many sets of diameter}\le r\},\\ \chi(B)&=\inf\{r>0:B\text{ can be covered by finitely many ball of radius}\le r\}.\end{aligned} \tag{2.9}$$

The separation measure of noncompacness of a nonempty bounded subset B of X is defined by

$$\beta(B)=\sup\{\varepsilon:\text{there exists a sequence }\{x_n\}\text{ in }B\text{ such that }\operatorname{sep}(\{x_n\})\ge\varepsilon\}. \tag{2.10}$$

Then a multivalued mapping $T : C \to 2^X$ is called *γ-condensing (resp., 1-γ-contractive)* where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset B of C with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp., } \gamma(T(B)) \le \gamma(B)). \tag{2.11}$$

Here $T(B) = \bigcup_{x\in B} Tx$. The random operator $T : \Omega \times C \to 2^X$ is said to be *1-γ-contractive* if, for each $\omega \in \Omega$, the map $T : (\omega, \cdot) \to 2^X$ is 1-γ-contractive.

Definition 2.1. Let X be a Banach space and $\phi = \alpha$, β, or χ. The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - d(0,A) : A \subset B_X \text{ is convex, } \phi(A) \ge \varepsilon\}, \tag{2.12}$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \ge 0 : \Delta_{X,\phi}(\varepsilon) = 0\}. \tag{2.13}$$

The following relationshops among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\varepsilon) \le \Delta_{X,\beta}(\varepsilon) \le \Delta_{X,\chi}(\varepsilon), \tag{2.14}$$

and consequently

$$\varepsilon_\alpha(X) \ge \varepsilon_\beta(X) \ge \varepsilon_\chi(X). \tag{2.15}$$

When X is a reflexive Banach space, we have some alternative expressions for the moduli of noncompact convexity associated β and χ:

$$\begin{aligned} \Delta_{X,\beta}(\varepsilon) &= \inf\{1 - \|x\| : \{x_n\} \subset B_X,\ x = w - \lim x_n,\ \operatorname{sep}(\{x_n\}) \ge \varepsilon\}, \\ \Delta_{X,\chi}(\varepsilon) &= \inf\{1 - \|x\| : \{x_n\} \subset B_X,\ x = w - \lim x_n,\ \chi(\{x_n\}) \ge \varepsilon\}. \end{aligned} \tag{2.16}$$

In order to study the fixed point theory for non-self-mappings, we must introduce some terminology for boundary condition. The inward set of C at $x \in C$ is defined by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \ge 0,\ y \in C\}. \tag{2.17}$$

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does. A multivalued mapping $T : C \to 2^X\{\emptyset\}$ is said to be *inward* on C if

$$Tx \subset I_C(x) \quad \forall x \in C. \tag{2.18}$$

Let $\bar{I}_C(x) := x + \{\lambda(z - x) : z \in C,\ \lambda \ge 1\}$. Note that for a convex C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be *weakly inward* on C if

$$Tx \subset \bar{I}_C(x) \quad \forall x \in C. \tag{2.19}$$

Let C be a nonempty bounded closed subset of Banach spaces X, and $\{x_n\}$ bounded sequence in X; we use $r(C,\{x_n\})$ and $A(C,\{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, that is,

$$\begin{aligned} r(C,\{x_n\}) &= \inf\left\{\limsup_n \|x_n - x\| : x \in C\right\}, \\ A(C,\{x_n\}) &= \left\{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\right\}. \end{aligned} \tag{2.20}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x-y\| : y \in D\} : x \in C\}. \tag{2.21}$$

Obviously, the convexity of C implies that $A(C,\{x_n\})$ is convex. Notice that $A(C,\{x_n\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X.

Let $\{x_n\}$ and C be nonempty bounded closed subsets of Banach spaces X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called *asymptotically uniform* with respect to C if $A(C,\{x_n\}) = A(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

LEMMA 2.6 (Goebel [6] and Lim [8]). *Let $\{x_n\}$ and C be as above. Then we have the following:*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular with respect to C;*
(ii) *if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C.*

Moreover, we also need the following lemma.

LEMMA 2.7 (Domínguez Benavides and Ramírez, cf. [4, Theorem 3.4]). *Let C be a closed convex subset of reflexive Banach spaces X, and let x_n be a bounded sequence in C which is regular with respect to C. Then*

$$r_C(A(C,x_n)) \le (1 - \Delta_{X,\beta}(1^-))r(C,\{x_n\}). \tag{2.22}$$

Moreover, if X satisfies the nonstrict Opial condition, then

$$r_C(A(C,x_n)) \le (1 - \Delta_{X,\chi}(1^-))r(C,\{x_n\}). \tag{2.23}$$

LEMMA 2.8 (Domínguez Benavides and Ramírez, cf. [5, Theorem 3.2]). *Let C be a closed convex subset of a reflexive Banach space X, and let $\{x_\beta : \beta \in D\}$ be a bounded ultranet. Then*

$$r_C(A(C,x_\beta)) \le (1 - \Delta_{X,\alpha}(1^-))r(C,\{x_\beta\}). \tag{2.24}$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.9 (Deimling, cf. [2]). *Let X be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous γ-condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ for all $x \in C$, then F has a fixed point. (Here $I_D(x)$ is called the inward set at x defined by $I_D(x) := \{x + \lambda(y - x) : \lambda \geq 0,\ y \in D\}$.)*

3. The result

In order to prove our first result, we need the following lemma which is proved along the proof of Kirk-Massa theorem as it appears in [14].

Lemma 3.1. *Let C be a nonempty closed bounded convex separable subset of a Banach space X. $T : C \to KC(X)$ is nonexpansive such that $T(C)$ is a bounded set which satisfies $Tx \subset I_C(x)$, $\forall x \in C$, $\{x_n\}$ is a sequence in C such that $\lim_n d(x_n, Tx_n) = 0$. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that $Tx \cap I_A(x) \neq \emptyset, \forall x \in A := A(C, \{z_n\})$.*

Lemma 3.1 is part (more or less) of the proof of [5, Theorem 3.4].

The next result states the main result of this work.

Theorem 3.2. *Let C be a nonempty closed bounded convex separable subset of Banach spaces X such that $\epsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(X)$ a multivalued nonexpansive random operator and 1-χ-contractive mapping, such that for each $\omega \in \Omega$, $T(\omega, C)$ is a bounded set, which satisfies the inwardness condition, that is, for each $\omega \in \Omega, T(\omega, x) \subset I_C(x), \forall x \in C$.*

Then T has a random fixed point.

Proof. Fix $x_0 \in C$, and consider the measurable function $x_0(\omega) \equiv x_0$. For each $n \geq 1$, define $T_n(\omega, \cdot) : C \to KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n} x_0(\omega) + \left(\frac{n-1}{n}\right) T(\omega, x), \quad \forall x \in C. \tag{3.1}$$

Then $T_n(\omega, \cdot)$ is a multivalued contraction and $T_n(\omega, x) \subset I_C(x), \forall x \in C$. Hence each T_n has a fixed point $z_n(\omega) \in C$. It is easily seen that $d(z_n(\omega), T(\omega, z_n(\omega))) \leq (1/n) \operatorname{diam} C \to 0$ as $n \to \infty$. Thus the set

$$F_n(\omega) = \left\{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} \operatorname{diam} C\right\} \tag{3.2}$$

is nonempty closed and convex. Furthermore, by Lemma 2.4, each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ such that

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} \operatorname{diam} C \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.3}$$

Define a function $f : \Omega \times C \to \mathbb{R}^+ := [0, \infty)$ by

$$f(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \quad x \in C. \tag{3.4}$$

By Lemma 2.3, it is easily seen that $f(\cdot, x)$ is measurable and $f(\omega, \cdot)$ is continuous and convex, therefore it is a weakly lower semicontinuous function. Note that; condition

$\varepsilon_\beta(X) < 1$ implies reflexivity (see [1]) and so C is a weakly compact. Hence, by Lemma 2.5, the marginal functions

$$r(\omega) := \inf_{x \in C} f(\omega, x),$$
$$A(\omega) := \{x \in C : f(\omega, x) = r(\omega)\} \tag{3.5}$$

are measurable. It is clearly that $A(\omega)$ is a weakly compact convex subset of C. For any $\omega \in \Omega$, we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect to C. Note that $A(\omega) = A(C, \{x_n(\omega)\})$, and $r(\omega) = r(C, \{x_n(\omega)\})$. We can apply inequality (2.22) in Lemma 2.7 to obtain

$$r_C(A(\omega)) \leq \lambda r(C, \{x_n(\omega)\}), \tag{3.6}$$

where $\lambda = 1 - \Delta_{X,\beta}(1^-) < 1$, since $\varepsilon_\beta(X) < 1$.

For each $\omega \in \Omega$ and $n \geq 1$, we define the multivalued contraction $T_n^1(\omega, \cdot) : A(\omega) \to KC(X)$ by

$$T_n^1(\omega, x) = \frac{1}{n} x_1(\omega) + \left(\frac{n-1}{n}\right) T(\omega, x), \tag{3.7}$$

for each $x \in C$. By Lemma 3.1, we note that $T(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \forall x \in A(\omega)$. Since $I_{A(\omega)}(x)$ is convex, it follows that $T_n^1(\omega, \cdot)$ satisfies the boundary condition, that is,

$$T_n^1(\omega, x) \cap I_{A(\omega)}(x) \neq \emptyset, \quad \forall x \in A(\omega). \tag{3.8}$$

Since $T_n^1(\omega, \cdot)$ is 1-χ-contractive mapping, it follows by [4, page 382] that $T_n^1(\omega, \cdot)$ is χ-condensing. Hence, by Lemma 2.9, $T_n^1(\omega, \cdot)$ has a fixed point $z_n^1(\omega) \in A(\omega)$, that is, $F(\omega) \cap A(\omega) \neq \emptyset$. Also it is easily seen that

$$\operatorname{dist}\left(z_n^1(\omega), T(\omega, z_n^1(\omega))\right) \leq \frac{1}{n} \operatorname{diam} C \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Thus $F_n^1(\omega) := \{x \in A(\omega) : d(x, T(\omega, x)) \leq (1/n) \operatorname{diam} C\}$ is nonempty closed and convex for each $n \geq 1$. By Lemma 2.4, each F_n^1 is measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1. Thus we have $x_n^1(\omega) \in A(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2 : \Omega \times C \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \quad \forall \omega \in \Omega. \tag{3.10}$$

As above, f_2 is a measurable function and weakly lower semicontunuous function. Then the marginal functions

$$r_2(\omega) := \inf_{x \in A(\omega)} f_2(\omega, x),$$
$$A^1(\omega) := \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\} \tag{3.11}$$

are measurable. Since $A^1(\omega) = A(A(\omega), \{x_n^1(\omega)\})$, it follows that $A^1(\omega)$ is a weakly compact and convex. Moreover, we also note that $r_2(\omega) = r(A(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $A^1(\omega)$. Moreover, we proceed as above using Lemmas 3.1 and 2.7 to obtain that

$$\begin{gathered} T(\omega,x(\omega)) \cap I_{A^1}(x(\omega)) \neq \varnothing \quad \forall x(\omega) \in A^1 = A(A(\omega),\{x_n^1(\omega)\}), \\ r_C(A^1) \leq \lambda r(A(\omega),\{x_n^1(\omega)\}) \leq \lambda r_C(A(\omega)). \end{gathered} \tag{3.12}$$

By induction, for each $m \geq 1$, we take a sequence $\{x_n^m(\omega)\}_n \subseteq A^{m-1}$ such that $r_C(A^m) \leq \lambda^m r_C(A(\omega))$ and $\lim_n d(x_n^m(\omega),T(\omega,x_n^m(\omega))) = 0$ for each fixed $\omega \in \Omega$, where $A^m := A(C, \{x_n^m(\omega)\})$. Since $\operatorname{diam} R_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m\to\infty} \operatorname{diam} R_m(\omega) = 0$. Note that $\{R_m(\omega)\}$ is a descending sequence of weakly compact subset of C for each $\omega \in \Omega$. Thus we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega),\{z(\omega)\}) \leq \operatorname{diam} R_m(\omega) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{3.13}$$

Therefore, by Lemma 2.2, $z(\omega)$ is measurable. Finally, we will show that $z(\omega)$ is a fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z(\omega),T(\omega,z(\omega))) &\leq \|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega),T(\omega,x_n^m(\omega))) \\ &\quad + H(T(\omega,x_n^m(\omega)),T(\omega,z(\omega))) \\ &\leq 2\|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega),T(\omega,x_n^m(\omega))) \\ &\leq 2\operatorname{diam} R_m(\omega) + d(x_n^m(\omega),T(\omega,x_n^m(\omega))). \end{aligned} \tag{3.14}$$

Taking the upper limit as $n \to \infty$,

$$d(z(\omega),T(\omega,z(\omega))) \leq 2\operatorname{diam} R_m(\omega). \tag{3.15}$$

Finally, taking limit in m in both sides, we obtain $z(\omega) \in T(\omega,z(\omega))$. □

Theorem 3.3. *Let C be a nonempty closed bounded convex separable subset of Banach spaces X such that $\epsilon_\alpha(X) < 1$, and $T : \Omega \times C \to KC(X)$ a multivalued nonexpansive random operator and 1-χ-contractive nonexpansive mapping, such that for each $\omega \in \Omega, T(\omega,C)$ is a bounded set, which satisfies the inwardness condition, that is, for each $\omega \in \Omega, T(\omega,x) \subset I_C(x), \forall x \in C$.*

Then T has a random fixed point.

Proof. Following from Theorem 3.2 and using Lemma 2.8. □

Corollary 3.4. *Let C be a nonempty closed bounded convex subset of Banach spaces X such that $\epsilon_\beta(X) < 1$. If $T : C \to KC(X)$ is a multivalued nonexpansive and 1-χ-contractive nonexpansive mapping, such that $T(C)$ is a bounded set, which satisfies the inwardness condition, that is, for each $Tx \subset I_C(x), \forall x \in C$.*

Then T has a fixed point.

Corollary 3.5 (Domínguez Benavides and Ramírez, cf. [5, Theorem 3.4]). *Let X be Banach spaces such that $\varepsilon_\alpha(X) < 1$, and C a nonempty closed bounded convex subset of X. If $T : C \to KC(X)$ is nonexpansive and 1-χ-contractive nonexpansive mapping, such that $T(C)$ is a bounded set, which satisfies $Tx \subset I_C(x)$ $\forall x \in C$, then T has a fixed point.*

Acknowledgment

The first author would like to thank The Thailand Research Fund for financial support.

References

[1] J. M. Ayerbe Toledano, T. Domínguez Benavides, and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications, vol. 99, Birkhäuser, Basel, 1997.

[2] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1974.

[3] T. Domínguez Benavides, G. López Acedo, and H.-K. Xu, *Random fixed points of set-valued operators*, Proceedings of the American Mathematical Society **124** (1996), no. 3, 831–838.

[4] T. Domínguez Benavides and P. Lorenzo Ramírez, *Fixed-point theorems for multivalued nonexpansive mappings without uniform convexity*, Abstract and Applied Analysis **2003** (2003), no. 6, 375–386.

[5] ———, *Fixed point theorems for multivalued nonexpansive mappings satisfying inwardness conditions*, Journal of Mathematical Analysis and Applications **291** (2004), no. 1, 100–108.

[6] K. Goebel, *On a fixed point theorem for multivalued nonexpansive mappings*, Annales Universitatis Mariae Curie-Skłodowska **29** (1975), 69–72.

[7] S. Itoh, *A random fixed point theorem for a multivalued contraction mapping*, Pacific Journal of Mathematics **68** (1977), no. 1, 85–90.

[8] T. C. Lim, *Remarks on some fixed point theorems*, Proceedings of the American Mathematical Society **60** (1976), 179–182.

[9] P. Lorenzo Ramírez, *Some random fixed point theorems for nonlinear mappings*, Nonlinear Analysis **50** (2002), no. 2, 265–274.

[10] K.-K. Tan and X.-Z. Yuan, *Some random fixed point theorems*, Fixed Point Theory and Applications (Halifax, NS, 1991) (K.-K. Tan, ed.), World Scientific, New Jersey, 1992, pp. 334–345.

[11] D.-H. Wagner, *Survey of measurable selection theorems*, SIAM Journal on Control and Optimization **15** (1977), no. 5, 859–903.

[12] H.-K. Xu, *Some random fixed point theorems for condensing and nonexpansive operators*, Proceedings of the American Mathematical Society **110** (1990), no. 2, 395–400.

[13] ———, *Random fixed point theorems for nonlinear uniformly Lipschitzian mappings*, Nonlinear Analysis **26** (1996), no. 7, 1301–1311.

[14] ———, *Multivalued nonexpansive mappings in Banach spaces*, Nonlinear Analysis **43** (2001), no. 6, 693–706.

[15] X.-Z. Yuan and J. Yu, *Random fixed point theorems for nonself mappings*, Nonlinear Analysis **26** (1996), no. 6, 1097–1102.

S. Plubtieng: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
E-mail address: somyotp@nu.ac.th

P. Kumam: Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand
E-mail address: pooom.kum@kmutt.ac.th