COMPLEMENTARITY PROBLEM FOR η-MONOTONE-TYPE MAPS

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The purpose of this paper is to establish certain existence theorems for complementarity problem in Banach space. Complementarity problem is defined for η-monotone operators and some results have been established. The result for complementarity problem was known for monotone operators.

1. Introduction

The theory of complementarity problem (CP) has a unique place in optimization theory due to its wide range of applications to various problems like equilibrium problems in economics and certain nonlinear optimization problems. Variational inequality problems are also very important due to their wide range of applications to various problems, that is, constrained mechanics problems, fluid flow through porus media, transportation problems, various engineering applications, and many others. In 1964, R. W. Cottle mentioned the complementarity problem in his Ph.D. thesis. Stampacchia discovered variational inequality in the year 1966. Both the theories of variational inequality and complementarity problem are related, and these two problems are essentially the same as was shown by Karamardian [2] in the year 1971. Generally, people working in applied mathematics or engineering sciences use variational inequality and their problem is infinite-dimensional, whereas people working in economic and operations research use complementarity problem and their problem is finite-dimensional. Many people have worked on various aspects like existence, uniqueness, and algorithmic approach of the solution under different conditions on the operator as in [1, 2]. In this paper, we prove the existence of solution of complementarity problems in Banach space. We define the complementarity problem for η-monotonicity and establish some results.

Before going to the results we mention the preliminaries.

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2. Preliminaries

Let X be a reflexive real Banach space and let X^* be its dual. In particular, X could be $\mathbb{R}^n$; in that case $X^* = \mathbb{R}^n$. The value of $f \in X^*$ at $x \in X$ is denoted by (f,x). Let K be a closed convex cone in X.

Let $\eta : K \times K \mapsto X^*$, $T : K \mapsto X^*$, then we define the following monotonicities:

T is said to be *η-monotone* if $(Tx - Ty, \eta(x,y)) \geq 0$ for all $x, y \in K$, *strictly η-monotone* if $(Tx - Ty, \eta(x,y)) > 0$ for all $x, y \in K$, $x \neq y$, η; is said to be antisymmetric if $\eta(y,x) = -\eta(x,y)$. η; is said to be positive homogeneous if $\eta(\lambda y, x) = \lambda\eta(y,x)$.

The nonlinear variational inequality (NVI) problem with respect to η is defined as follows.

Find x such that

$$x \in K : (Tx, \eta(y,x)) \geq 0 \quad \forall y \in K. \tag{2.1}$$

Another NVI can be stated as follows.

Find x such that

$$x \in K : (Ty, \eta(y,x)) \geq 0 \quad \forall y \in K. \tag{2.2}$$

We refer to the above variational inequalities as NVI(1) and NVI(2), and their solution sets as NVIS(1) and NVIS(2), respectively. If $\eta(y,x) = y - x$, then the above variational inequalities reduce to usual variational inequalities.

Let K be a closed convex cone in X. Let K^* be the subset of X^* defined by

$$K^* = \{y \in X^* : (y, \eta(x,x)) \geq 0 \ \forall x \in K\}. \tag{2.3}$$

Then

$$x \in K, \ Tx \in K^* \quad (Tx, \eta(x,x)) = 0 \tag{2.4}$$

will be called the generalized complementarity problem (GCP). Let C denote the set of all solutions of GCP.

3. Main results

Theorem 3.1. *If η is antisymmetric, that is, $\eta(y,x) = -\eta(x,y)$ and T is strictly η-monotone, then NVIS(1) is empty or singleton.*

Proof. Assume that $x_1, x_2 \in NVIS(1)$. Then

$$(Tx_1, \eta(x_2,x_1)) \geq 0, \qquad (Tx_2, \eta(x_1,x_2)) \geq 0. \tag{3.1}$$

From (3.1), η being antisymmetric we have $(Tx_1, \eta(x_1,x_2)) \leq 0$. Hence $(Tx_1 - Tx_2, \eta(x_1, x_2)) \leq 0$. Since T is strictly η-monotone, this is impossible unless $x_1 = x_2$ and this completes the proof. □

Theorem 3.2. *Let T be η-monotone, $\eta(x,x) = 0$, η is positive homogeneous and hemicontinuous, then $NVIS(1) = NVIS(2)$.*

Proof. Let $x \in NVIS(1)$. Since T is η-monotone, $(Ty,\eta(y,x)) \geq (Tx,\eta(y,x)) \geq 0 \Rightarrow x \in NVIS(2)$. Let $x \in NVIS(2)$ and $y \in K$. Since K is convex, for $0 < t < 1$, $x_t = (1-t)x + ty = x + t(y-x) \in K$. Hence $(Tx_t,\eta(x_t,x)) \geq 0 \Rightarrow (Tx_t,\eta(x+t(y-x),x)) \geq 0$.

Since $\eta(x_t,x) = \eta(x+t(y-x),x) = \eta(x,x) + t(\eta(y-x,x)) = \eta(x,x) + t\eta(y,x) + t\eta(x,x) = t\eta(y,x)$, we get $(Tx_t,t\eta(y,x)) \geq 0$. Taking $t \to 0$, we get the result. □

Theorem 3.3. *Let K be a closed convex cone, η is antisymmetric, that is, $\eta(y,x) = -\eta(x,y)$. Then $C = NVIS(1)$.*

Proof. Let $x \in$ NVIS(1). Take $y = x$. Then $(Tx,\eta(x,x)) \geq 0$. Since η is antisymmetric, $(Tx,\eta(x,x)) \leq 0 \Rightarrow (Tx,\eta(x,x)) = 0$. Let $x \in C$. Then $Tx \in K^*$ and $(Tx,\eta(x,x)) = 0$. Hence the proof is completed. □

Remark 3.4. It is natural to ask for which class of functions η, result similar to Hartman and Stampacchia will hold, that is, the following theorem will hold.

Theorem 3.5. *If K is a compact convex subset of $\mathbb{R}^n$ and T is a continuous map, then there exists a point $x_0 \in K$ such that $(Tx_0,\eta(y,x_0)) \geq 0$ for all $y \in K$.*

This problem has been solved by Yang [3, Theorem 8] in the following manner.

Theorem 3.6 (Yang). *Let K be a compact convex subset of $\mathbb{R}^n$ and let*
(i) *T be continuous on K,*
(ii) *$h : K \to R$ defined by $h(y) = (Tx,\eta(y,x))$ be convex for each fixed $x \in K$.*
Then there exists $x_0 \in K$ such that $(Tx_0,\eta(y,x_0)) \geq 0$ for all $y \in K$.

From the above theorem we establish the solution for the generalized complementarity problem as follows.

Theorem 3.7. *Let K be a convex closed cone. Let*
(1) *T be continuous on K,*
(2) *the map $h : K \to R$ defined by $h(y) = (Tx,\eta(y,x))$ be convex for each fixed $x \in K$,*
(3) *η be positive homogeneous and hemicontinuous,*
(4) *$\exists x \in K$ with $Tx \in \operatorname{int} K^*$.*
Then the complementarity problem $(Tx,\eta(x,x)) = 0$ has a solution.

Proof. Let

$$\begin{aligned} D_u &= \{x \in K : (Tx,\eta(x,x)) \leq 0\}, \\ D_u^0 &= \{x \in K : (Tx,\eta(x,x)) < 0\}, \\ S_u &= \{x \in K : (Tx,\eta(x,x)) = 0\}. \end{aligned} \tag{3.2}$$

It is easy to show that D_u is convex compact. Hence by Theorem 3.6 (Yang), there exists $x_u \in D_u$ such that $(Tx_u,\eta(y,x_u)) \geq 0$, for all $y \in D_u$. Since $x_u \in D_u$, $(Tx_u,\eta(x_u,x_u)) \geq 0$. Hence the complementarity problem has a solution. Now two cases can occur.

Case 1. Let $x_u \in D_u^0 \Rightarrow (Tx_u,\eta(\lambda x_u,x_u)) < 0 \Rightarrow \exists \lambda > 1$ such that $\lambda x_u \in S_u \subset D_u$. Hence $(Tx_u,\eta(x_u,x_u)) \leq (Tx_u,\eta(\lambda x_u,x_u)) = \lambda(Tx_u,\eta(x_u,x_u))$. Since $(Tx_u,\eta(x_u,x_u)) < 0$, this is impossible unless $(Tx_u,\eta(x_u,x_u)) = 0$.

Case 2. Let $x_u \in S_u$ for all $u \in K$. Then by our assumption, we have a solution for the complementarity problems. □

Note that the result holds if η is linear. Certainly, the result is not true if η is any continuous convex function.

For example, if $X = R$, $K = [0,1]$, $Tx = -1$, and $\eta(y,x) = |y - x|$. Then for any $x_0 \in K$, if we choose $y \in K$ such that $y \neq x_0$, we have $(Tx_0, \eta(y,x)) = -|y - x|$. But for the result to be true, there would exist $x_0 \in K$ such that $-|y - x_0| \geq 0$ for all $y \in K$ and this cannot happen.

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