LOCAL VOLATILITY IN THE HESTON MODEL: A MALLIAVIN CALCULUS APPROACH

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We implement the Heston stochastic volatility model by using multidimensional Ornstein-Uhlenbeck processes and a special Girsanov transformation, and consider the Malliavin calculus of this model. We derive explicit formulas for the Malliavin derivatives of the Heston volatility and the log-price, and give a formula for the local volatility which is approachable by Monte-Carlo methods.

1. Introduction

Within the last ten years there have been many published and unpublished contributions on how to apply Malliavin calculus in the context of mathematical finance. The purpose of this paper is on one side to give the mathematical background for the application of Malliavin calculus in the framework of the Heston stochastic volatility model, and on the other side to provide an applicable formula for the local volatility. The Heston model is one of the most applied stochastic volatility models. It is more or less characterized by a volatility process which satisfies the following stochastic differential equation

$$dv_t = \kappa(\nu - v_t)\,dt + \theta\sqrt{v_t}\,dW_t. \tag{1.1}$$

Processes satisfying (1.1) are also said to be of Cox-Ingersoll-Ross type (see [1]). The problems in applying Malliavin calculus methods to this process come mainly from the fact that the square root is not differentiable in 0 and is not even locally Lipschitz continuous. The standard results for diffusions with Lipschitz coefficients cannot be used here (see [11]). Although existence and path-wise uniqueness of such a process is guaranteed by general theorems (see [8, Chapter IV, Theorems 2.3 and 2.4]), the lack of a concrete representation leads to problems when computations have to be done. As an example I should mention that until the result of Deelstra and Delbaen [2] appeared, it was not even clear whether the standard Euler scheme for (1.1) converges. Deelstra and Delbaen's result however, does not imply that the corresponding Euler scheme for the Malliavin derivative converges. This paper shows how one can avoid these problems by working with a multidimensional Brownian motion instead of a one-dimensional one, and considering the corresponding multidimensional Malliavin calculus. We show that

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for special coefficients the process (ν_t) can be represented as the square of a multidimensional Ornstein-Uhlenbeck process. In the case of general coefficients, we apply a Girsanov transformation in order to adjust the drift term. The set-up of this paper is as follows. We will first present some background on Malliavin calculus and then consider the Malliavin calculus of Ornstein-Uhlenbeck processes as well as their squares. After this we setup our version of the Heston stochastic volatility model and show that in fact, Malliavin calculus is applicable to this model. In the last section, we apply our results to give a formula for the local volatility in the Heston model. By local volatility we mean the conditional expectation of the volatility, given the spot price of the stock, that is,

$$\mathbb{E}\left(\sqrt{\nu_T}|S_T = x\right). \tag{1.2}$$

Sometimes also the expression $\mathbb{E}(\nu_T|S_T = x)$ is referred to as local volatility. Both expressions appear in the literature. Using our formula, Monte-Carlo methods can be applied to compute the local volatility. This is of practical relevance for the calibration of the model, which means the adaptation of model coefficients to observed market data.

2. Preliminaries: a short introduction into Malliavin calculus

In this section, we summarize the construction of the Malliavin derivative operator and shortly revise its main properties. Though changed by its appearance, the material presented here has mainly been taken from the classical references [11, 14].

Assume we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is defined an m-dimensional Brownian motion $\mathbb{W}$. We would like to differentiate functionals of the form

$$F : \Omega \longrightarrow \mathbb{R} \tag{2.1}$$

or at least those of a certain nice subclass. For this we first assume that the functional is given by

$$F = f\left(\mathbb{W}_{t_1}, \ldots, \mathbb{W}_{t_l}\right), \tag{2.2}$$

where $f \in C_b^\infty((\mathbb{R}^m)^l)$ is a smooth function with bounded derivatives of all orders. Given $h \in L^2([0,T], \mathbb{R}^m)$, we have that $\int_0^\cdot h(s)ds \in C_0([0,T], \mathbb{R}^m)$, where the integral is computed component wise and the dot indicates that the upper bound of the integral is taken as a variable. The subspace of $C_0([0,T], \mathbb{R}^m)$ generated by this kind of functions is called the Cameron-Martin space. The directional derivative of F in direction $\int_0^\cdot h(s)ds$ at ω is given by

$$\begin{aligned} D_h F(\omega) &:= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \tilde{F}\left(\mathbb{W}(\omega) + \epsilon \cdot \int_0^\cdot h(s)ds\right) \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f\left(\mathbb{W}_{t_1}(\omega) + \int_0^{t_1} h(s)ds, \ldots, \mathbb{W}_{t_l}(\omega) + \int_0^{t_l} h(s)ds\right) \\ &= \sum_{i=1}^m \nabla_i f\left(\mathbb{W}_{t_1}(\omega), \ldots, \mathbb{W}_{t_l}(\omega)\right)^\top \cdot \int_0^{t_i} h(s)ds, \end{aligned} \tag{2.3}$$

where $\tilde{F}$ is the functional on the m-dimensional Wiener space through which F factorizes and ∇_i denotes the gradient with respect to the ith m-dimensional argument in f. Now for fixed ω, consider the linear bounded functional on $L^2[0,T]$ given by

$$h \longmapsto D_h F(\omega). \tag{2.4}$$

By the Riesz representation theorem, there is an element $DF(\omega)$ in $L^2([0,T],\mathbb{R}^m)$ which is considered as a row vector such that

$$D_h F(\omega) = < h, DF(\omega)^\top >_{L^2([0,T],\mathbb{R}^m)} = \int_0^T DF(\omega)(s)h(s)ds \quad \forall h \in L^2([0,T],\mathbb{R}^m). \tag{2.5}$$

In the following we denote $DF(\omega)(s)$ with $D_s F(\omega)$. We now consider ω as a variable. The assumption that f has bounded derivatives of all orders ensures that for all $p \geq 1$, we have $DF \in L^p(\Omega, L^2([0,T],\mathbb{R}^m))$ when considered as an $L^2([0,T],\mathbb{R}^m)$-valued functional in ω.

Assume now that the functional F is not necessarily cylindrical but there exists a sequence of cylindrical functionals F_i such that (F_i) converges to F in $L^p(\Omega)$ and (DF_i) converges to G in $L^p(\Omega, L^2([0,T],\mathbb{R}^m))$. Then we define

$$DF := G = \lim_{i\to\infty} DF_i. \tag{2.6}$$

Using the Cameron-Martin theorem it is not hard to show that if one has another sequence $(\tilde{F}_i)$ converging to F in $L^p(\Omega)$ such that $(D\tilde{F}_i)$ converges to $\tilde{G}$ in $L^p(\Omega, L^2([0,T],\mathbb{R}^m))$, then $G = \tilde{G}$ in L^p. This basically shows that the operator

$$D : L^p(\Omega) \longrightarrow L^p\left(\Omega, L^2([0,T],\mathbb{R}^m)\right) \tag{2.7}$$

defined on the cylindrical functionals is closable.

Definition 2.1. For $p \geq 1$, define the Malliavin derivative operator

$$D : L^p(\Omega) \longrightarrow L^p\left(\Omega, L^2([0,T],\mathbb{R}^m)\right) \tag{2.8}$$

as the closure of the operator above. For F in the domain of D, define

$$\|F\|_{1,p} := \|F\|_{L^p(\Omega)} + \|DF\|_{L^p(\Omega,(L^2[0,T],\mathbb{R}^m))}. \tag{2.9}$$

Then the domain of D is precisely the closure of the cylindrical functionals under the norm above. It will be denoted with $\mathbb{D}_{1,p}$.

Example 2.2. The following is easy to verify (see [12, Example 4.8]). Assume $m = 1$ and $f \in L^2[0,T]$, then $F = \int_0^t f(u)dW_u \in \mathbb{D}_{1,2}$ and

$$D_s F = D_s \int_0^t f(u)dW_u = f(s) \cdot 1_{\{s\leq t\}}. \tag{2.10}$$

In particular for an Ornstein-Uhlenbeck process given by

$$\sigma_t = e^{-\beta t}\left(z + \int_0^t \delta e^{\beta u} dW_u\right) \tag{2.11}$$

for positive constants β and δ, one has $\sigma_t \in \mathbb{D}_{1,2}$ and

$$D_s\sigma_t = \delta e^{-\beta(t-s)} \cdot 1_{\{s\le t\}}. \tag{2.12}$$

If the functional F is vector-valued, then the Malliavin derivative is computed component wise and considered as a matrix in the same way as the Jacobian matrix in standard calculus. In the following we restrict ourselves to the case where $p = 2$. Then we are dealing with Hilbert spaces.

Definition 2.3. The adjoint operator $\delta = D^*$, where

$$\delta : L^2(\Omega, \mathbb{R}^m) \longrightarrow L^2(\Omega), \tag{2.13}$$

is called the Skorohod integral. Denote its domain with $\mathrm{dom}(\delta)$.

The word "integral" is motivated by the following proposition.

Proposition 2.4. *The class $L_a^2(\Omega \times [0,T], \mathbb{R}^m)$ of adapted square integrable processes is contained in $\mathrm{dom}(\delta)$ and on this class the Skorohod integral coincides with the Itô integral.*

The following formula is called the integration by parts formula of Malliavin calculus.

Proposition 2.5. *If $F \in \mathbb{D}_{1,2}$ and $u \in \mathrm{dom}(\delta)$, then*

$$\mathbb{E}\left(\int_0^T D_tF \cdot u(\cdot,t)dt\right) = \mathbb{E}(F \cdot \delta(u)). \tag{2.14}$$

Proof. This follows directly from the definition of δ as the adjoint operator of D. □

Another useful formula for computing Malliavin derivatives is the following chain rule (see [10, Lemma 2.1]).

Proposition 2.6. *Let $\phi : \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable function and let $F = (F_1, \ldots, F_k)$ such that $F_i \in \mathbb{D}_{1,2}$. Then $\phi(F) \in \mathbb{D}_{1,2}$ if and only if $\phi(F) \in L^2(\Omega)$ and $\nabla\phi(F)^\top DF \in L^2(\Omega \times [0,T], \mathbb{R}^m)$, and in this case*

$$D_t\phi(F) = \nabla\phi(F)^\top \cdot D_tF. \tag{2.15}$$

Example 2.7. Consider the Ornstein-Uhlenbeck process of Example 2.2 and take $\phi(x) = x^2$. Clearly $\sigma_t^2 = \phi(\sigma_t) \in L^2(\Omega)$ and furthermore considered as a function of ω and s,

$$\phi'(\sigma_t)D_s\sigma_t = 2\sigma_t\delta e^{-\beta(t-s)} \cdot 1_{\{s\le t\}} \in L^2(\Omega \times [0,T]). \tag{2.16}$$

Therefore $\sigma_t^2 \in \mathbb{D}_{1,2}$ and $D_s\sigma_t^2 = 2\delta e^{-\beta(t-s)}\sigma_t \cdot 1_{\{s\le t\}}$.

The integration by parts formula is the key point in the proof of the following proposition about conditional expectations.

Proposition 2.8. *Let $F \in \mathbb{D}_{1,2}$ be $\mathbb{R}^k$-valued, let $G \in \mathbb{D}_{1,2}$ be $\mathbb{R}$-valued, and let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent measure such that there exists a process (η_s) satisfying*

$$Z_T := \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \eta_s d\mathbb{W}_s - \frac{1}{2}\int_0^T \eta_s^2 ds \right) \in \mathbb{D}_{1,2}. \tag{2.17}$$

Consider the Malliavin calculus under $\mathbb{P}$ and assume that D_tG is nondegenerate $\mathbb{P}$-almost sure for almost all $t \in [0,T]$ and there exists a process $(u_t) \in \mathrm{dom}(\delta)$ such that

(1) $\mathbb{E}(\int_0^T (D_tG)u_t dt | \sigma(Z_T,F,G)) = 1$,

(2) $\mathbb{E}(\int_0^T (D_tZ_T)u_t dt | \sigma(Z_T,F,G)) = 0$,

(3) $\mathbb{E}(\int_0^T (D_tF)u_t dt | \sigma(Z_T,F,G)) = (0,\dots,0)^\top$.

Let $\phi : \mathbb{R}^k \to \mathbb{R}$ be measurable and of at most linear growth at infinity. Then the following formula holds

$$\mathbb{E}_{\mathbb{Q}}(\phi(F)|G=0) = \frac{\mathbb{E}_{\mathbb{Q}}(1_{\{G>0\}}\phi(F)\delta(u))}{\mathbb{E}_{\mathbb{Q}}(1_{\{G>0\}}\delta(u))}. \tag{2.18}$$

Proof. We first consider the case where $\mathbb{P} = \mathbb{Q}$. This is more or less the case considered in [5]. We repeat the main arguments. Denote with δ_0 the Dirac distribution and with H the Heaviside function $1_{\{x>0\}}$. At least formally we have $H' = \delta_0$ and

$$\mathbb{E}(\phi(F)|G=0) = \frac{\mathbb{E}(\phi(F)\delta_0(G))}{\mathbb{E}(\delta_0(G))}. \tag{2.19}$$

Using our assumptions on the process u, the chain rule, and the integration by parts formula, we can write

$$\begin{aligned}
\mathbb{E}(\phi(F)\delta_0(G)) &= \mathbb{E}\left(\phi(F)\delta_0(G)\cdot\mathbb{E}\left(\int_0^T (D_tG)u_t dt|\sigma(F,G)\right)\right)\\
&= \mathbb{E}\left(\int_0^T \phi(F)D_t(H(G))\cdot u_t dt\right)\\
&= \mathbb{E}\left(\int_0^T D_t(\phi(F)H(G))u_t dt\right) - \mathbb{E}\left(\int_0^T \nabla\phi(F)(D_tF)H(G)u_t dt\right)\\
&= \mathbb{E}(\phi(F)H(G)\delta(u)) - \mathbb{E}\left(H(G)\nabla\phi(F)\underbrace{\mathbb{E}\left(\int_0^T (D_tF)u_t dt|\sigma(F,G)\right)}_{=(0,\dots,0)^\top}\right)\\
&= \mathbb{E}(\phi(F)H(G)\delta(u)),
\end{aligned} \tag{2.20}$$

where we used condition (3). To establish the third equality above we used a special kind of product rule for the Malliavin derivative, which in this case is a consequence

of Proposition 2.6. The result for $\mathbb{Q} = \mathbb{P}$ then follows from

$$\begin{aligned}\mathbb{E}(\delta_0(G)) &= \mathbb{E}\left(\delta_0(G)\cdot\mathbb{E}\left(\int_0^T D_t G\cdot u_t dt|\sigma(F,G)\right)\right)\\ &= \mathbb{E}\left(\int_0^T D_t(H(G))\cdot u_t dt\right)\\ &= \mathbb{E}(H(G)\delta(u)).\end{aligned} \tag{2.21}$$

To prove the statement in the general case for a measure $\mathbb{Q}$ with $Z_T = d\mathbb{Q}/d\mathbb{P}$, we define the $\mathbb{R}^{k+1}$-valued functional $\bar{F}$ via

$$\bar{F} = \begin{pmatrix} Z_T \\ F \end{pmatrix}, \tag{2.22}$$

and instead of $\phi(y)$, we consider the function $\psi(x,y) := x\cdot\phi(y)$. The function $\psi(x,y)$ is not of linear growth at infinity, however one can verify that the approximation procedure presented in [5] works for ψ. Conditions (2) and (3) above imply that

$$\mathbb{E}\left(\int_0^T (D_t\bar{F})u_t dt|\sigma(Z_T,F,G)\right) = (0,\ldots,0)^\top, \tag{2.23}$$

and condition (1) is unchanged to the previous case. Therefore applying the already proven part of the statement for $\mathbb{P}$ utilizing $\bar{F}$, G, and ψ instead of F, G, and ϕ we obtain

$$\mathbb{E}(Z_T\phi(F)|G=0) = \frac{\mathbb{E}(1_{\{G>0\}}Z_T\phi(F)\delta(u))}{\mathbb{E}(1_{\{G>0\}}\delta(u))}, \tag{2.24}$$

and for $\mathbb{P}$ utilizing Z_T, G, and id (y) instead of F, G, and $\phi(y)$,

$$\mathbb{E}(Z_T|G=0) = \frac{\mathbb{E}(1_{\{G>0\}}Z_T\delta(u))}{\mathbb{E}(1_{\{G>0\}}\delta(u))}. \tag{2.25}$$

Using a general result on conditional expectations (see [12, Lemma 8.2.4]) we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\phi(F)|G=0) &= \frac{\mathbb{E}(Z_T\phi(F)|G=0)}{\mathbb{E}(Z_T|G=0)}\\ &= \frac{\mathbb{E}(1_{\{G>0\}}Z_T\phi(F)\delta(u))/\mathbb{E}(1_{\{G>0\}}\delta(u))}{\mathbb{E}(1_{\{G>0\}}Z_T\delta(u))/\mathbb{E}(1_{\{G>0\}}\delta(u))}\\ &= \frac{\mathbb{E}_{\mathbb{Q}}(1_{\{G>0\}}\phi(F)\delta(u))}{\mathbb{E}_{\mathbb{Q}}(1_{\{G>0\}}\delta(u))},\end{aligned} \tag{2.26}$$

which proves the general result. □

3. The Heston stochastic volatility model

In this section, we set up our version of the Heston stochastic volatility model. Given coefficients κ, θ, ν as in (1.1) as well as the initial volatility $x > 0$, we fix a natural number $n \geq 2$ and work in the framework of an $(n+1)$-dimensional Brownian motion $\mathbb{W} = (W^0, \ldots, W^n)^\top$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the volatility process ν_t only dimensions $1, 2, \ldots, n$ are needed, dimension 0 will be used exclusively for modeling the stock price. We first assume that there exist positive constants δ, β such that

$$\kappa := 2\beta, \qquad \nu := \frac{n\delta^2}{2\beta}, \qquad \theta := 2\delta. \tag{3.1}$$

We consider the n-dimensional Ornstein-Uhlenbeck process $\tilde{\sigma}_t$ which has as its components n independent Ornstein-Uhlenbeck processes $\tilde{\sigma}_t^i$ starting at time 0 at $z := \sqrt{x/n}$ satisfying

$$d\tilde{\sigma}_t^i = -\beta\tilde{\sigma}_t^i dt + \delta dW_t^i \tag{3.2}$$

for $i = 1, \ldots, n$. Clearly we have

$$d[\tilde{\sigma}_t, \tilde{\sigma}_t] = \delta^2 \cdot \mathrm{Id} \cdot dt, \tag{3.3}$$

where Id denotes the $n \times n$ identity matrix. We define a new process ν_t by applying the function

$$\begin{aligned} f &: \mathbb{R}^n \longrightarrow \mathbb{R}, \\ f(x_1, \ldots, x_n) &:= \sum_{i=1}^n x_i^2 \end{aligned} \tag{3.4}$$

on $\tilde{\sigma}_t$:

$$\nu_t := f(\tilde{\sigma}_t) = \sum_{i=1}^n (\tilde{\sigma}_t^i)^2 = \tilde{\sigma}_t^\top \tilde{\sigma}_t. \tag{3.5}$$

Denoting $\hat{\mathbb{W}}_t = (W_t^1, \ldots, W_t^n)$, we get by application of the Itô formula

$$\begin{aligned} d\nu_t &= \nabla f(\tilde{\sigma}_t)^\top d\tilde{\sigma}_t + \frac{1}{2}\Delta f(\tilde{\sigma}_t)\delta^2 dt \\ &= 2\tilde{\sigma}_t^\top(-\beta\tilde{\sigma}_t dt + \delta d\hat{\mathbb{W}}_t) + n\delta^2 dt \\ &= (n\delta^2 - 2\beta\nu_t)dt + 2\delta\tilde{\sigma}_t^\top d\hat{\mathbb{W}}_t \\ &= (n\delta^2 - 2\beta\nu_t)dt + 2\delta\sqrt{\nu_t} \cdot \frac{\tilde{\sigma}_t^\top}{\sqrt{\nu_t}} d\hat{\mathbb{W}}_t. \end{aligned} \tag{3.6}$$

To justify the last step, we need $\sqrt{\nu_t}$ to be strictly greater than zero almost sure for all $t \in [0, T]$. For $n \geq 2$ this is in fact true and the reason for this is more or less, that

Brownian motion in two- or higher-dimensional space never hits the origin at any time $t > 0$ (see [9, Proposition 3.22, page 161]). We define a one-dimensional stochastic process (W_t) via

$$dW_t = \frac{\tilde{\sigma}_t^{\top}}{\sqrt{v_t}} d\hat{\mathbb{W}}_t. \tag{3.7}$$

It follows immediately from the Lévy characterization (see [9, Theorem 3.16, page 157]) of Brownian motion, that W_t is a one-dimensional Brownian motion. Using this, we can write the equation for v_t as

$$dv_t = (n\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dW_t. \tag{3.8}$$

Substituting κ, θ, and ν from the beginning of this section we can write the last equation as

$$dv_t = \kappa(\nu - v_t)dt + \theta\sqrt{v_t}dW_t \tag{3.9}$$

with $v_0 = x$. If the coefficients κ, θ, and ν do not admit a representation via α, β, and n, then we proceed as follows. We choose $n = 2$, $\delta := \theta/2$, and $\beta := \kappa/2$. Then the procedure presented above yields a process v_t satisfying

$$dv_t = \left(\frac{1}{2}\theta^2 - \kappa v_t\right)dt + \theta\sqrt{v_t}dW_t. \tag{3.10}$$

We define the process (η_t) by

$$\eta_t := \frac{(1/2)\theta^2 - \kappa\nu}{\theta\sqrt{v_t}}. \tag{3.11}$$

(The following lemma was pointed out to me by M. Yor.)

Lemma 3.1. *If $\theta^2 \leq 2\kappa\nu$, then the process (η_t) satisfies the Novikov-condition*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \eta_t^2 dt\right)\right) < \infty. \tag{3.12}$$

Proof. It is not hard to show that a Cox-Ingersoll-Ross-type process (v_t) satisfying (1.1) is just a reparametrized Bessel process (see [6, Section I.5]). The condition $\theta^2 \leq 2\kappa\nu$ ensures that the dimension of the corresponding Bessel process is at least two (i.e., $\delta \geq 2$ or $\mu \geq 0$ in the notation of Revuz and Yor). The statement of the lemma then follows from an analogous result for Bessel processes (see [13, Example 1.22, page 450]). □

The condition $\theta^2 \leq 2\kappa\nu$ is a standard condition when working with Cox-Ingersoll-Ross-type processes in mathematical finance. The main implication of this condition is that the process never hits zero, which perfectly fits the application.

Applying the Girsanov theorem we see that under the equivalent measure $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \eta_s dW_s - \frac{1}{2}\int_0^T \eta_s^2 ds\right), \tag{3.13}$$

the process given by (3.10) satisfies

$$dv_t = \kappa(\nu - v_t)dt + \theta\sqrt{v_t}d\tilde{W}_t, \tag{3.14}$$

where $\tilde{W}_t = W_t + \int_0^t \eta_s ds$ is a $\mathbb{Q}$-Brownian motion. In terms of the n-dimensional $\mathbb{P}$-Brownian motion $(\hat{\mathbb{W}}_t)$ the density term is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \eta_s \frac{\tilde{\sigma}_s^\top}{\sqrt{v_s}} d\hat{\mathbb{W}}_s - \frac{1}{2}\int_0^T \eta_s^2 ds \right), \tag{3.15}$$

and therefore the zero component (W_t^0) of the $(n+1)$-dimensional $\mathbb{P}$-Brownian motion $(\mathbb{W}_t)$ is not affected by the Girsanov transformation and remains a Brownian motion under $\mathbb{Q}$. For notational reasons, we denote the zero component (W_t^0) with (B_t). Then (B_t) and $(\tilde{W}_t)$ are uncorrelated Brownian motions under $\mathbb{Q}$.

We now consider the financial market modeled on $(\Omega, \mathcal{F}, \mathbb{Q})$ consisting of one stock (S_t) following the dynamics

$$dS_t = S_t(b_t dt + \sqrt{v_t}(\mu dB_t + \rho d\tilde{W}_t)) \tag{3.16}$$

with respect to some initial condition $S_0 = x$ and v_t following the dynamics (3.14). Here $-1 \le \rho, \mu \le 1$ with $\rho^2 + \mu^2 = 1$ are chosen in order to allow correlation between stock and volatility. This model has been introduced by Heston [7] and has since been called the Heston model. Equation (3.16) can be solved easily:

$$S_t = S_0 \cdot \exp\left(\int_0^t b_s - \frac{1}{2}v_s ds + \mu\int_0^t \sqrt{v_s}dB_s + \rho\int_0^t \sqrt{v_s}d\tilde{W}_s \right). \tag{3.17}$$

From now on we assume the standard condition $\theta^2 \le 2\kappa\nu$ whenever we speak of the Heston model.

4. The Malliavin derivative of volatility and stock price in the Heston model

In this section, we show that the volatility process v_t and its square root $\sqrt{v_t}$ as well as the log-price $\log(S_t)$ in the Heston model belong to $\mathbb{D}_{1,2}$. Furthermore we compute their Malliavin derivatives. The Malliavin calculus considered in this section is the Malliavin calculus corresponding to the original measure $\mathbb{P}$ not to $\mathbb{Q}$. It is not clear whether the Malliavin derivatives with respect to the measure $\mathbb{Q}$, or equivalently with respect to the $\mathbb{Q}$-Brownian motion $(\tilde{W}_t)$, exist (see however [3]). We keep the notation of the previous section and denote with D the Malliavin derivative with respect to the $(n+1)$-dimensional Brownian motion $\mathbb{W}_t$, and with $\hat{D}$ the Malliavin derivative with respect to the n-dimensional Brownian motion $\hat{\mathbb{W}}_t$. Clearly $D = (D^0, \hat{D})$ where D^0 denotes the Malliavin derivative with respect to $(W_t^0) = (B_t)$.

Proposition 4.1. *The Heston volatility process v_t and its square root $\sqrt{v_t}$ modeled as in Section 3 belong to $\mathbb{D}_{1,2}$ and satisfy*

$$\begin{aligned} D_s v_t &= 2e^{-\beta(t-s)}\delta \cdot 1_{\{s\le t\}}(0,\tilde{\sigma}_t^1,\dots,\tilde{\sigma}_t^n), \\ D_s\sqrt{v_t} &= \frac{1}{\sqrt{v_t}}\delta e^{-\beta(t-s)} \cdot 1_{\{s\le t\}}(0,\tilde{\sigma}_t^1,\dots,\tilde{\sigma}_t^n). \end{aligned} \tag{4.1}$$

Furthermore $\tilde{\sigma}_T \in \mathbb{D}_{1,2}$ and

$$D_s\tilde{\sigma}_T = \delta e^{-\beta(T-s)} \cdot 1_{\{s\le T\}} \begin{pmatrix} 0 & & \\ \vdots & \text{Id} & \\ 0 & & \end{pmatrix}, \tag{4.2}$$

where Id *denotes the $n\times n$ identity matrix.*

Proof. It follows from Example 2.2 that

$$\hat{D}_s\tilde{\sigma}_t = \delta e^{-\beta(t-s)} \cdot 1_{\{s\le t\}} \cdot \text{Id}, \tag{4.3}$$

and from Example 2.7 that

$$\hat{D}_s v_t = 2e^{-\beta(t-s)}\delta\tilde{\sigma}_t^{\top} \cdot 1_{\{s\le t\}}. \tag{4.4}$$

Since the square root is obviously not Lipschitz continuous, we cannot just apply the chain rule to compute $\hat{D}_s\sqrt{v_t}$. We have to go one step back and consider $\sqrt{v_t}$ as the Euclidean norm of the n-dimensional Ornstein-Uhlenbeck process $(\tilde{\sigma}_t)$. The norm function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is obviously Lipschitz continuous. Therefore it follows from [11, Proposition 1.2.3] that $\sqrt{v_t} = \|\tilde{\sigma}_t\| \in \mathbb{D}_{1,2}$. Since the norm function is everywhere differentiable (except in the origin) with derivative

$$\nabla\|x\| = \frac{x}{\|x\|}, \tag{4.5}$$

it follows by an elementary approximation procedure that

$$\begin{aligned} \hat{D}_s\sqrt{v_t} = D_s||\tilde{\sigma}_t|| &= \frac{\tilde{\sigma}_t^{\top}}{\|\tilde{\sigma}_t\|}\hat{D}_s\tilde{\sigma}_t \\ &= \frac{\tilde{\sigma}_t^{\top}}{v_t}\delta e^{-\beta(t-s)}1_{\{s\le t\}}. \end{aligned} \tag{4.6}$$

The statements in the proposition therefore follow from $D = (D^0, \hat{D})$ and the fact that $\tilde{\sigma}_t$ and therefore also v_t and $\sqrt{v_t}$ do not depend on (W_t^0). □

We assume in the following that the drift term of the stock is equal to the deterministic constant interest rate. This corresponds to a risk neutral setting, that is, $\mathbb{P}$ is already an equivalent martingale measure for the market.

Proposition 4.2. *The terminal log-price $\zeta_T = \log(S_T)$ in the Heston model given by the logarithm of (3.17) for terminal time $t = T$ belongs to $\mathbb{D}_{1,2}$. Its partial Malliavin derivatives are given by*

$$\begin{aligned} D_s^0 \zeta_T &= \mu\sqrt{\nu_s}, \\ D_s^i \zeta_T &= \rho\tilde{\sigma}_s^i - 2\delta e^{\beta s}\int_s^T \tilde{\sigma}_t^i e^{-\beta t}dt + \delta e^{\beta s}\left(\mu\int_s^T e^{-\beta t}\frac{\tilde{\sigma}_t^i}{\sqrt{\nu_t}}dW_t^0 + \rho\int_s^T e^{-\beta t}dW_t^i\right) \end{aligned} \tag{4.7}$$

for all $1 \le i \le n$.

Proof. Defining the $\mathbb{R}^{n+1}$-valued process (u_t) as $u_t = (\mu\sqrt{\nu_t}, \rho\tilde{\sigma}_t^1, \ldots, \rho\tilde{\sigma}_t^n)^\top$, we can write ζ_T as

$$\zeta_T = rT - \frac{1}{2}\int_0^T \nu_t dt + \int_0^T u_t^\top d\mathbb{W}_t. \tag{4.8}$$

It then follows [11, (1.46), page 38] (see also [14, Proposition II.2, page 13] for the multidimensional case) that $\zeta_T \in \mathbb{D}_{1,2}$ with Malliavin derivative given by

$$D_s\zeta_T = -\frac{1}{2}\int_0^T D_s\nu_t dt + u_s^\top + \left(\int_0^T (D_s u_t)^\top d\mathbb{W}_t\right)^\top \tag{4.9}$$

for all $0 \le s \le T$. It follows from the second part of Proposition 4.1 that

$$\begin{aligned} D_s u_t &= 1_{\{s\le t\}}\begin{pmatrix} 0 & \mu\delta e^{-\beta(t-s)}\dfrac{\tilde{\sigma}_t^1}{\sqrt{\nu_t}} & & \cdots & \mu\delta e^{-\beta(t-s)}\dfrac{\tilde{\sigma}_t^n}{\sqrt{\nu_t}} \\ 0 & \rho\delta e^{-\beta(t-s)} & 0 & \cdots & 0 \\ & 0 & & & \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & \rho\delta e^{-\beta(t-s)} \end{pmatrix} \\ &= 1_{\{s\le t\}}\delta e^{-\beta(t-s)}\begin{pmatrix} 0 & \mu\dfrac{\tilde{\sigma}_t^1}{\sqrt{\nu_t}} & & \cdots & \mu\dfrac{\tilde{\sigma}_t^n}{\sqrt{\nu_t}} \\ 0 & \rho & 0 & \cdots & 0 \\ & 0 & & & \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & \rho \end{pmatrix} \end{aligned} \tag{4.10}$$

and therefore

$$\int_0^T (D_s u_t)^\top d\mathbb{W}_t = \begin{pmatrix} 0 \\ \mu \int_s^T \delta e^{-\beta(t-s)} \dfrac{\tilde{\sigma}_t^1}{\sqrt{v_t}} dW_t^0 + \rho \int_s^T \delta e^{-\beta(t-s)} dW_t^1 \\ \vdots \\ \mu \int_s^T \delta e^{-\beta(t-s)} \dfrac{\tilde{\sigma}_t^n}{\sqrt{v_t}} dW_t^0 + \rho \int_s^T \delta e^{-\beta(t-s)} dW_t^n \end{pmatrix}. \tag{4.11}$$

Applying the first part of Proposition 4.1 we get

$$\int_0^T D_s v_t dt = 2\delta e^{\beta s} \left(0, \int_s^T \tilde{\sigma}_t^1 e^{-\beta t} dt, \ldots, \int_s^T \tilde{\sigma}_t^n e^{-\beta t} dt\right) \tag{4.12}$$

which finally proves the correctness of the formulas stated in the proposition. □

One can also show that $S_T \in \mathbb{D}_{1,2}$. Basically one has to apply the chain rule Proposition 2.6 on $S_T = \exp(\zeta_T)$. Since the exponential function is not globally Lipschitz, checking the technical assumptions in Proposition 2.6 is not completely trivial, however it is possible, using explicit formulas for the distribution of S_T (which are known). Since we do not need the result for S_T in this paper we omit the tedious proof.

The results obtained in this section are necessary whenever one wants to apply Malliavin calculus techniques in the framework of the Heston model. We will see a nice application in the next section. For other applications in the framework of calibration of stochastic volatility models see, for example, [4].

5. The local volatility in the Heston model

We consider the Heston stochastic volatility model from Section 3. A natural question is, if we know the stock price S_T or equivalently the log-price ζ_T at terminal time T, what can we say about the volatility, or mathematically more precisely, what can we say about $\mathbb{E}(\sqrt{v_t}|S_T = x)$. The last expression is called local volatility and plays a major role in the calibration of stochastic volatility models. Sometimes the expression $\mathbb{E}(v_t|S_T = x)$ is also referred to as local volatility. Knowing the local volatility of a model and comparing it to observed volatility data from the market can help to calibrate the model, that is to determine the parameters of the model (here κ, θ, ν) in a way that the model fits the observed market data best. We will now develop a formula for this expression which is tractable by Monte-Carlo methods and does not use kernel estimation techniques. It was shown in general (see [5]) that the type of formula obtained in this paper is faster and more accurate than corresponding formulas using kernel estimation techniques.

Theorem 5.1. *Consider the Heston stochastic volatility model where stock price (S_t) and volatility (v_t) follow the dynamics*

$$\begin{aligned} dS_t &= S_t\big(r\,dt + \sqrt{v_t}\big(\mu dB_t + \rho d\tilde{W}_t\big)\big), \\ dv_t &= \kappa(\nu - v_t)\,dt + \theta\sqrt{v_t}\,d\tilde{W}_t, \end{aligned} \tag{5.1}$$

where $(B_t, \tilde{W}_t)$ is a two-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ where the coefficients of the model satisfy the standard assumption $\theta^2 \leq 2\kappa\nu$. Assume furthermore that the correlation coefficient satisfies $\rho \neq 1, -1$. Then the two following formulas hold:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\big(\sqrt{v_T}\,|S_T = x\big) &= \frac{\mathbb{E}_{\mathbb{Q}}\big(1_{\{S_T>x\}}\cdot\sqrt{v_T}\int_0^T (1/\sqrt{v_t})dB_t\big)}{\mathbb{E}_{\mathbb{Q}}\big(1_{\{S_T>x\}}\cdot\int_0^T (1/\sqrt{v_t})dB_t\big)}, \\ \mathbb{E}_{\mathbb{Q}}\big(v_T|S_T = x\big) &= \frac{\mathbb{E}_{\mathbb{Q}}\big(1_{\{S_T>x\}}\cdot v_T\int_0^T (1/\sqrt{v_t})dB_t\big)}{\mathbb{E}_{\mathbb{Q}}\big(1_{\{S_T>x\}}\cdot\int_0^T (1/\sqrt{v_t})dB_t\big)}. \end{aligned} \tag{5.2}$$

Proof. We are going to apply Proposition 2.8. Since $\theta^2 \leq 2\kappa\nu$, we can consider the equivalent measure $\mathbb{P} \sim \mathbb{Q}$ which is related to $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \eta_s dW_s - \frac{1}{2}\int_0^T \eta_s^2 ds\right), \tag{5.3}$$

where the process (η_s) is given by $\eta_s = ((1/2)\theta^2 - \kappa\nu)/\theta\sqrt{v_s}$ (see Lemma 3.1). As the discussion following Lemma 3.1 shows, there exists a two-dimensional $\mathbb{P}$-Ornstein-Uhlenbeck process $(\tilde{\sigma}_t)$ satisfying

$$d\tilde{\sigma}_t = -\begin{pmatrix} \frac{\kappa}{2} \\ \frac{\kappa}{2} \end{pmatrix} dt + \frac{\theta}{2} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \tag{5.4}$$

with a two-dimensional $\mathbb{P}$-Brownian motion $(W_t^1, W_t^2)^\top$ such that $v_t = \|\tilde{\sigma}_t\|$, and furthermore (B, W_t^1, W_t^2) is a three-dimensional $\mathbb{P}$-Brownian motion.

Let $\zeta_t = \log(S_t)$ denote the log-price process. Choose

$$\begin{aligned} F &= \tilde{\sigma}_T, \\ G &= \zeta_T - \log(x), \\ \phi(x_1,\ldots,x_n) &= \sqrt{\sum_{i=1}^n x_i^2} = \big|\big|(x_1,\ldots,x_n)\big|\big|. \end{aligned} \tag{5.5}$$

The function ϕ is clearly measurable and of linear growth at infinity. Propositions 4.1 and 4.2 show that F and G belong to $\mathbb{D}_{1,2}$. The same holds for Z_T. The latter follows after a long and tedious but straightforward check of the technical assumptions by application

of the chain rule Proposition 2.6. We now define the process (u_s) as follows:

$$u_s = \begin{pmatrix} \left(T \cdot D_s^0 \zeta_T\right)^{-1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \left(T\mu\sqrt{v_s}\right)^{-1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{5.6}$$

Since the first column of $D_s\tilde{\sigma}_T$ consists only of zeroes (see Proposition 4.1), it is clear that (u_s) satisfies the following three equations:

$$\begin{aligned} &\mathbb{E}\left(\int_0^T (D_s\zeta_T)u_s ds | \sigma(Z_T, \tilde{\sigma}_T, \zeta_T - \ln(x))\right) = 1, \\ &\mathbb{E}\left(\int_0^T (D_s\tilde{\sigma}_T)u_s ds | \sigma(Z_T, \tilde{\sigma}_T, \zeta_T - \ln(x))\right) = (0,0,0)^\top, \\ &\mathbb{E}\left(\int_0^T (D_s Z_T)u_s ds | \sigma(Z_T, \tilde{\sigma}_T, \zeta_T - \ln(x))\right) = 0. \end{aligned} \tag{5.7}$$

We can therefore apply Proposition 2.8 in order to compute $\mathbb{E}_{\mathbb{Q}}(\sqrt{v_T}|\zeta_T = \log(x))$. Since (u_s) is adapted, it follows from Proposition 2.4 that the Skorohod integral can be replaced by the Itô integral and therefore

$$\mathbb{E}_{\mathbb{Q}}\left(\sqrt{v_T}|\zeta_T = \log(x)\right) = \frac{\mathbb{E}_{\mathbb{Q}}\left(1_{\{\zeta_T > \log(x)\}} \cdot \sqrt{v_T}\int_0^T (1/\sqrt{v_t})dB_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(1_{\{\zeta_T > \log(x)\}} \cdot \int_0^T (1/\sqrt{v_t})dB_t\right)}, \tag{5.8}$$

where we used that the deterministic term $(T\mu)^{-1}$ cancels. Since obviously $S_T > x \Leftrightarrow \zeta_T > \log(x)$, one can now pass from the log-price process to the actual price process. This shows the validity of the first formula. In order to prove the second formula, one proceeds in exactly the same way as before by choosing the function $\psi(x_1,\ldots,x_n) = \sum_{i=1}^n x_i^2 = \|(x_1,\ldots,x_n)\|^2$ instead of ϕ. However one has to deal with the fact that this function is not of linear growth at infinity. Nevertheless the second result can be obtained by approximating the function ψ by a monotonic sequence of functions which are of linear growth at infinity and use the theorem of monotone convergence. We omit this tedious but straightforward part of the proof. □

The theorem above gives an easy-to-implement method to compute the expected local volatility with Monte-Carlo methods. In fact there are two possibilities to arrange the simulation. The first possibility is that one works under the reference measure $\mathbb{P}$ with the advantage that one only has to simulate Ornstein-Uhlenbeck processes and the drawback that one has to use the simulation to compute the density $d\mathbb{Q}/d\mathbb{P}$. The second possibility is that one works under the original measure $\mathbb{Q}$ with the advantage that the density

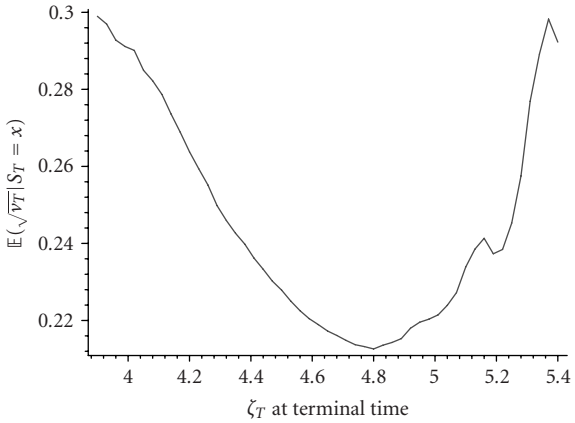

Figure 5.1. Local volatility in the Heston model: $\rho = -0.2$. The x-axis represents the logarithmic stock price ζ_T at terminal time and the y-axis represents local volatility $\mathbb{E}(\sqrt{v_T}|S_T = x)$.

process does not have to be simulated and the drawback that the simulation of the volatility process is more difficult (see, e.g., [2] for the convergence of the Euler scheme for Cox-Ingersoll-Ross-type processes). Within our own numerical experiments we observed that the first approach works better, however we are far from giving an analytical proof of this observation.

For the simulations presented in the following, we used the version of the Heston volatility discussed in Section 3 in the framework of two Ornstein-Uhlenbeck processes. We simulated $n = 10\,000$ paths of the logarithmic stock price ζ_t over the time interval $[0, 1]$. For the simulation of the Ornstein-Uhlenbeck processes we used a simple stochastic Euler scheme with time discretization $\Delta = 0.05$. As parameters for the model, we chose $\kappa = 2$, $\nu = 0.04$, $\theta = 0.4$ as well as $r = 0.025$ for the deterministic interest rate. Finally we chose $\zeta_0 = \log(100) \approx 4.6$ for the initial log-price of the stock (i.e., $S_0 = 100$) and $v_0 = 0.16$ for the initial variance (i.e., $\sqrt{v_0} = 0.4$ for the initial volatility). This setting needs 600 000 simulations of a standard normal distributed random variable. With the Maple 8 software and an Intel Pentium 4/2.6 GHz processor, each simulation took a little bit over one hour. For a more accurate simulation, one could of course spend more time on simulation. Nevertheless the result of the simulations is very promising. Figure 5.1 shows the result of a simulation where the correlation was chosen to be $\rho = -0.2$. This is a typical value for applications. One can see the typical volatility smile in the figure. In a future publication we will study the question of variance reduction. Here one has to make use of the other Malliavin derivatives $D_r^i \zeta_T$ for $i \neq 0$, which have not been used in the formula above. Methods from importance sampling can also be used to fasten up the simulation.

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