ON SOME STOCHASTIC PARABOLIC DIFFERENTIAL EQUATIONS IN A HILBERT SPACE

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We consider some stochastic difference partial differential equations of the form $du(x,t,c) = L(x,t,D)u(x,t,c)dt + M(x,t,D)u(x,t-a,c)dw(t)$, where $L(x,t,D)$ is a linear uniformly elliptic partial differential operator of the second order, $M(x,t,D)$ is a linear partial differential operator of the first order, and $w(t)$ is a Weiner process. The existence and uniqueness of the solution of suitable mixed problems are studied for the considered equation. Some properties are also studied. A more general stochastic problem is considered in a Hilbert space and the results concerning stochastic partial differential equations are obtained as applications.

1. Introduction

Consider the stochastic linear system

$$du(t,c) = Au(t,c)dt + \sum_{i=1}^{n}\sum_{j=1}^{k} b_{ij}\big(B_i u(t-c_j,c)\big)dw_{ij}(t), \tag{1.1}$$

where A is a linear closed operator generating the strongly continuous semigroup $Q(t)$ on a separable Hilbert space H, and w_{ij} are mutually independent Wiener processes on a separable Hilbert space K with covariance operators W_{ij}, positive nuclear operators in the space $L(K,K)$ of continuous linear mapping of K into itself.

It is assumed that A is defined on $S_1 \subset H$ into H and S_1 is dense in H (see [4]).

It is assumed also that $B_1,\dots,B_n$ are linear closed operators defined on $S_2 \supset S_1$, $S_2 \subset H$, and with values in H.

$b_{ij}(\cdot)$ are elements of $L(H,L(K,H))$, (see [1, 2, 4]). We will study the existence and uniqueness of mild solutions, in other words, the existence and uniqueness of a solution of the equation

$$u(t,c) = Q(t)u_0 + \sum_{i=1}^{n}\sum_{j=1}^{k}\int_0^t Q(t-\theta)b_{ij}\big(B_i u(\theta-c_j,c)\big)dw_{ij}(\theta). \tag{1.2}$$

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We write $\|u\|$ for the Hilbert space norm of u, and $\|b_{ij}(u)\|$ for the norms of $b_{ij}(u)$ in $L(H,L(K,H))$. We write $\operatorname{tr} W_{ij}$ for the trace of W_{ij}. The processes $w_{ij}(t)$ are defined on a probability space (Ω,F,P). We denote by $E[u]$ the expectation of u. We suppose that the initial condition u_0 is independent of

$$w_{ij}(t)-w_{ij}(s), \quad t\geq s>0, \tag{1.3}$$

for all $i=1,\dots,n$, $j=1,\dots,k$.

We suppose also that

$$E\Big[||u_0||^2\Big]<\infty, \tag{1.4}$$

and that there is a number $\gamma\in(0,1)$ such that

$$||Q(t)b_{ij}(B_i f)||\leq\frac{\alpha}{t^{\gamma/2}}\|f\|, \tag{1.5}$$

where α is a positive constant and $f\in S_2$.

For $f\in H$, we suppose that

$$||BQ(t)b_{ij}(f)||\leq\frac{\alpha}{t^{\gamma/2}}\|f\|. \tag{1.6}$$

In Section 2, we will study the uniqueness and existence of w_{ij} adapted solution $u(t,c)$ of (1.2) in the space $C(0,T;L_2(\Omega,H))$, where $C([0,T],\Lambda)$ denotes the space of continuous functions mapping $[0,T]$ into $\Lambda\subset K$.

In Section 3, we study a mixed problem (initial and boundary value problem) of some stochastic difference partial differential equations.

2. Uniqueness and existence of mild solutions

Let $u(t,c)$ satisfy the condition

$$u(t,c)=F(t), \quad -T_0<t<0, \tag{2.1}$$

where F is a given function in the space $C([-T_0,0],L_2(\Omega,H)\cap S_2)$.

We assume that

$$F(0)=u_0. \tag{2.2}$$

We prove now the following theorem.

Theorem 2.1. *Let $u\in C([0,T],L_2(\Omega,H))\cap S_2$ be the solution of (1.2). If $F(t)=0$ on $[-T_0,0]$, then $u(t)=0$ for all $t\geq 0$.*

Proof. The solution of the above equation can be written in the form

$$u(t,c)=\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{\gamma_j(t)}^{t}Q(t-\theta)b_{ij}\big(B_i u(\theta-c_j,c)\big)dw_{ij}(\theta), \tag{2.3}$$

where

$$\gamma_j(t) = \begin{cases} t, & t \le c_j, \\ c_j, & t > c_j. \end{cases} \tag{2.4}$$

Thus,

$$E\left[\|u(t,c)\|^2\right] \le \sum_{i=1}^{n}\sum_{j=1}^{k} \operatorname{tr} W_{ij} \int_{\gamma_j(t)}^{t} \frac{\alpha^2}{(t-\theta)^\gamma} E\left[\|u(\theta - c_j, c)\|^2\right] d\theta. \tag{2.5}$$

So, there is a positive constant λ such that

$$E\left[\|u(t,c)\|^2\right] \le \frac{\lambda M t^{1-\gamma}}{1-\gamma}, \tag{2.6}$$

where

$$M = \sup_{\theta,c} E\|u(\theta,c)\|^2. \tag{2.7}$$

It is easy to see that

$$E\left[\|u(t,c)\|^2\right] \le \frac{1}{1-\gamma} t^{2(1-\gamma)} \lambda^2 M \beta(1-\gamma, 2-\gamma), \tag{2.8}$$

where $\beta(m,n)$ is the β function. Now for every $r = 1,2,\ldots$, we can prove that

$$E\left[\|u(t,c)\|^2\right] \le \frac{\lambda^r M t^{r(1-\gamma)} (\Gamma(1-\gamma))^r}{\Gamma(r(1-\gamma)) + 1}, \tag{2.9}$$

where Γ is the gamma function.

Taking the limit as $r \to \infty$, we get the required result.

Now to prove the existence of solutions, we suppose that

$$B_1 = B_2 = \cdots = B_k = B \tag{2.10}$$

and $u_0 \in S_2$. □

Theorem 2.2. *There exists a unique mild solution $u \in C([0,T], L_2(\Omega, H)) \cap S_2$ of (1.2).*

Proof. We apply the method of successive approximation. To do this, we set

$$\begin{aligned} u_{r+1}(t,c) = Q(t)u_0 &+ \sum_{i=1}^{n}\sum_{j=1}^{k} \int_0^{\gamma_j(t)} Q(t-\theta) b_{ij}(BF(\theta - c_j))\, dw_{ij}(\theta) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{k} \int_{\gamma_j(t)}^{t} Q(t-\theta) b_{ij}(Bu_r(\theta - c_j, c))\, dw_{ij}(\theta). \end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned} v_{r+1}(t,c) = BQ(t)u_0 + \sum_i \sum_j \int_0^{\gamma_j(t)} BQ(t-\theta)b_{ij}\big(BF(\theta - c_j)\big)dw_{ij}(\theta) \\ + \sum_i \sum_j \int_{\gamma_j(t)}^{t} BQ(t-\theta)b_{ij}\big(v_r(\theta - c_j, c)\big)dw_{ij}(\theta), \end{aligned} \tag{2.12}$$

where

$$v_r(t,c) = Bu_r(t,c). \tag{2.13}$$

The zero approximation is taken to be zero.

It is easy to see that

$$E\Big[\|v_1(t,c)\|^2\Big] \le E\Big[\|BQ(t)u_0\|^2\Big] + \sum_{i=1}^{n}\sum_{j=1}^{k} \operatorname{tr} W_{ij} \int_0^{\gamma_j(t)} \frac{\alpha^2}{(t-\theta)^{\gamma}} \|b_{ij}\|^2 E\Big[\|BF(\theta - c_j)\|^2\Big] d\theta. \tag{2.14}$$

Using the method of Theorem 2.1, we can prove that there exists a positive number λ such that

$$E\Big[\|v_{r+1}(t,c) - v_r(t,c)\|^2\Big] \le \frac{\lambda^r t^{r(1-\gamma)}\big(\Gamma(1-\gamma)\big)^r}{\Gamma\big(r(1-\gamma)+1\big)}. \tag{2.15}$$

Since v can be written in the form

$$v(t,c) = \sum_{r=0}^{\infty}\big[v_{r+1}(t,c) - v_r(t,c)\big], \tag{2.16}$$

it follows that

$$E\Big[\|v(t,c)\|^2\Big] \le \sum_r \frac{1}{r^2(1-\gamma)^2} \sum_r r^2(1-\gamma)^2 E\Big[\|v_{r+1}(t,c) - v_r(t,c)\|^2\Big]. \tag{2.17}$$

So v represents the solution of the equation

$$\begin{aligned} v(t,c) = BQ(t)u_0 + \sum_i \sum_j \int_0^{\gamma_j(t)} BQ(t-\theta)b_{ij}\big(BF(\theta - c_j)\big)dw_{ij}(\theta) \\ + \sum_i \sum_j \int_{\gamma_j(t)}^{t} BQ(t-\theta)b_{ij}\big(v(\theta - c_j, c)\big)dw_{ij}(\theta). \end{aligned} \tag{2.18}$$

Using (1.2) and (2.18), we deduce the existence of the solution of (1.2) in the space

$$C(0,T;L_2(\Omega,H)) \cap S_2. \tag{2.19}$$

The uniqueness of this solution follows from Theorem 2.1. □

We give now conditions for the second moment of $u(t,c)$ to decay exponentially. To state the third theorem, we need the following conditions.

C_1: there are positive numbers α and μ such that

$$||Q(t)|| \leq \alpha e^{-\mu t}, \quad t > 0. \tag{2.20}$$

This exponential stability of the semigroup is equivalent to the requirement that for all $\lambda > -\mu$,

$$||(\lambda I + A)^{-1}|| \leq \alpha(\lambda + \mu)^{-1}. \tag{2.21}$$

C_2: $\|BQ(t)f\| \leq (\alpha/t^{\gamma/2})e^{-\mu t}\|f\|$, $t > 0$.
C_3: $\|Q(t)Bf\| \leq (\alpha/t^{\gamma/2})e^{-\mu t}\|f\|$, $t > 0$.

Theorem 2.3. *Assume conditions C_1, C_2, and C_3 then for sufficiently large μ, constants a and b can be found such that*

$$E\left[||u(t,c)||^2\right] \leq aE\left[||u_0||^2\right]e^{-bt}, \quad a > 0, b > 0. \tag{2.22}$$

Proof. Using conditions C_1, C_2, and C_3, and (1.2), we get

$$h(t,c) \leq \lambda_1 + \lambda_2 \sum_{j=1}^{k} \int_0^t \frac{h(\theta - c_j, c)}{(t-\theta)^\gamma} d\theta, \tag{2.23}$$

where $\lambda_1 = \alpha^2 E[\|u_0\|^2]$, $\lambda_2 > \alpha^2 \operatorname{tr} W_{ij}$, λ_2 is a positive constant, and $h(t,c) = e^{2\mu t}E[\|u(t, c)\|^2]$.

Let $\{h_r\}$ be a sequence of functions such that

$$h_{r+1}(t,c) \leq \lambda_1 + \lambda_2 \sum_{j=1}^{k} \int_0^t \frac{h_r(\theta - c_j, c)}{(t-\theta)^\gamma} d\theta, \tag{2.24}$$

where the zero approximation is taken to be zero. As $r \to \infty$, we get

$$h(t,c) \leq \lambda_1 \sum_r \frac{\lambda_2^r t^{r(1-\gamma)} (\Gamma(1-\gamma))^r}{\Gamma(r(1-\gamma)+1)}. \tag{2.25}$$

Using the properties of Mittag-Leffler function, we get

$$h(t,c) \leq C_1 \exp\left[t\lambda_2^{1/(1-\gamma)} \left(\Gamma(1-\gamma)^{1/(1-\gamma)}\right)\right] + \frac{C_2}{1+t^{(1-\gamma)}}, \tag{2.26}$$

where C_1 and C_2 are positive constants. Thus for a sufficiently large μ, we get the required result. □

3. Stochastic parabolic differential equations

Let $C^m(S)$ be the set of all continuous functions in S together with all their m-partial derivatives. Denote by $C_0^m(S)$ the subset of $C^m(S)$ consisting of all functions which have a compact support. Let $W^m(S)$ be a Sobolev space. In other words, $W^m(S)$ is the complete space of $C^m(S)$ with respect to the norm

$$\|f\|_m = \left[\sum_{|\alpha|\leq m}\int_S |D^\alpha f(x)|^2\,dx\right], \tag{3.1}$$

where $x = (x_1,\dots,x_n)$,

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j}, \quad j = 1,\dots,n,\ |\alpha| = \alpha_1+\cdots+\alpha_n \tag{3.2}$$

and $\alpha = (\alpha_1,\dots,\alpha_n)$ is an n-dimensional multi-index. We denote by $W_0^m(S)$ the complete space of $C_0^m(S)$ with respect to the norm defined by (3.1).

Let r_b be the cylinder; $r_b = (x,t) : x \in S,\ 0 < t < b,\ 0 < b < \infty$, and let Γ_b be the lateral boundary

$$\Gamma_b = \{(x,t) : x \in \partial S,\ 0 < t < b\}. \tag{3.3}$$

We consider the parabolic stochastic partial differential equations

$$\begin{aligned} du(x,t,c) = {} & \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u(x,t,c)}{\partial x_i\partial x_j}dt \\ & + \sum_{i=1}^{n}\sum_{r=1}^{k}\left[b_{ir}(x,t)\frac{\partial}{\partial x_i} + b_{0r}(x,t)\right]u(x,t-c_r,c)\,dw_{ir}(t), \end{aligned} \tag{3.4}$$

with the initial and boundary conditions

$$u(x,0,c) = u_0(x), \tag{3.5}$$

$$u(x,t,c)|_{\Gamma_b} = 0. \tag{3.6}$$

It is assumed that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \delta\sum_{i=1}^{n}\xi_i^2, \tag{3.7}$$

where $\delta > 0$, $(x,t) \in \bar{\Omega}_b$ and $\bar{\Omega}_b$ is the closure of Ω_b, and Ω_b is an open bounded domain in the $n+1$ dimensional Euclidean space. It is assumed also that all the coefficients a_{ij}, b_{ir}, and b_{or} are continuous on $\bar{\Omega}_b$ and satisfy a uniform Hölder condition in $t \in [0,b]$.

The mixed problem (3.4), (3.5), (3.6) can be written in the abstract form

$$du(t,c) = Au(t,c)dt + \sum_{i=1}^{n}\sum_{r=1}^{k} b_{ir}(B_i u(t-c_r,c))\,dw_{ir}(t) + \sum_r b_{or}u(t-c_r,c)\,dw_{or}(t), \tag{3.8}$$

where A is the operator with domain $G = W^2(S) \cap W_0^1(S)$ given by

$$Au = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u. \tag{3.9}$$

Let $L_2(S)$ be the space of all square integrable functions on S. The space $H = L_2(S)$ is a Hilbert space and G is dense in H.

The operators $B_1, \ldots, B_n$ with domains $W^1(S) \cap W_0^1(S)$ are given by

$$B_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \tag{3.10}$$

and b_{ir}, b_{or} are the continuous functions defined on $\bar{\Omega}_b$.

Since A is uniformly elliptic on $\bar{\Omega}_b$, it follows that the semigroup $Q(t)$ exists with the properties (1.5) and (1.6) (see [3, 5, 6]).

Consequently, Theorems 2.1, 2.2, and 2.3 can be applied for the considered abstract mixed problem.

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