DAVIS-TYPE THEOREMS FOR MARTINGALE DIFFERENCE SEQUENCES

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We study Davis-type theorems on the optimal rate of convergence of moderate deviation probabilities. In the case of martingale difference sequences, under the finite pth moments hypothesis ($1 \le p < \infty$), and depending on the normalization factor, our results show that Davis' theorems either hold if and only if $p > 2$ or fail for all $p \ge 1$. This is in sharp contrast with the classical case of i.i.d. centered sequences, where both Davis' theorems hold under the finite second moment hypothesis (or less).

1. Introduction

Let $(X_n)_{n\ge1}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ and, for each $n \ge 1$, denote by $\mathcal{F}_n$ the σ-algebra generated by $X_1, X_2, \ldots, X_n$. We say that $(X_n)_{n\ge1}$ is martingale difference sequence if $S_n := X_1 + \cdots + X_n$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\ge1}$, that is, $E[S_n|\mathcal{F}_{n-1}] = S_{n-1}$ a.s. for $n \ge 1$ (here $S_0 = 0$ and $\mathcal{F}_0$ is the trivial σ-algebra). Obviously, any i.i.d. centered sequence $(X_n)_{n\ge1}$ from L^1 is a martingale difference sequence, but the converse is false (consider, e.g., ergodic dynamical systems of positive entropy). Let $p \ge 1$; we say that the martingale difference sequence $(X_n)_{n\ge1}$ is L^p-bounded if it has finite pth moments, that is, $\|X_n\|_p \le C$ for some constant $C > 0$ and any $n \ge 1$.

Davis' first and second theorems refer to the rate of convergence of the *moderate deviations* probabilities $P[|S_n| > \varepsilon(n\log n)^{1/2}]$ or $P[|S_n| > \varepsilon(n\log\log n)^{1/2}]$, for $\varepsilon > 0$; they were studied for L^2-bounded i.i.d. sequences in Davis [1, 2] (see also [3, 4, 5]). In contrast with the i.i.d. case, in this paper, we prove that Davis' first theorem for L^p-bounded martingale difference sequences holds if and only if $p > 2$. Depending on the normalization factor therein, we prove that Davis' second theorem for L^p-bounded martingale difference sequences holds if and only if $p > 2$, or fails for any $p \ge 1$. Our results complement the optimal rates of convergence in the Baum-Katz-type theorems for martingale difference sequences obtained in [6, 7], that is, the rate of convergence of the *large deviations* probabilities $P[|S_n| > \varepsilon n^{\alpha}]$, with $1/2 < \alpha \le 1$. Moreover, sharp precise asymptotics (as $\varepsilon \searrow 0$) are known for i.i.d. centered sequences in both Baum-Katz and Davis' theorems (see [4, 5]).

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In the martingale difference case, our proofs of Davis' theorems will also provide, as consequence, such asymptotics.

2. Davis' first theorem

Let $\varepsilon > 0$ and let $\delta = \delta(p)$ be a function of $p \geq 1$. Consider the series

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\big[\,|S_n| > \varepsilon (n \log n)^{1/2}\big]. \tag{2.1}$$

Assume $(X_n)_{n\geq 1}$ is an i.i.d. sequence and $\delta = 1$. *Davis' first theorem* says that series (2.1) is convergent for any $\varepsilon > 0$ if and only if $(X_n)_{n\geq 1}$ is an L^2-bounded centered sequence (see [1, 3, 4]).

In the martingale difference sequences case, we have the following result.

Theorem 2.1. (i) *For any $p > 2$ and L^p-bounded martingale difference sequence $(X_n)_{n\geq 1}$, series (2.1) is convergent for any $0 \leq \delta < p/2 - 1$ and any $\varepsilon > 0$.*

(ii) *For any $p \geq 1$ there exists an L^p-bounded martingale difference sequence $(X_n)_{n\geq 1}$ such that series (2.1) diverges for any $\delta > p/2 - 1$ and any $\varepsilon > 0$.*

(iii) *There exist a probability space $(\Omega, \mathcal{F}, P)$ and an $L^{p-\lambda}$-bounded martingale difference sequence $(X_n)_{n\geq 1}$ (for all $0 < \lambda \leq p - 1$) such that series (2.1) diverges for $\delta \geq p/2 - 1$ and any $\varepsilon > 0$.*

Proof. Throughout the paper, $C > 0$ denotes a generic numerical constant. The sign $\sim$ between two series means that they are either both convergent or both divergent. In the sequel we use that each of the series

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\alpha}}{n^{\beta}}, \qquad \sum_{n=3}^{\infty} \frac{(\log\log n)^{\alpha}}{n(\log n)^{\beta}} \tag{2.2}$$

is convergent if and only if $\beta > 1$ or $\beta = 1$ and $\alpha < -1$ (by the integral test).

The idea in proving Theorem 2.1(i) is to obtain direct sharp estimates for the two-sided deviation probabilities $P[|X_n| - (\log n)^a]$ for some $a > 0$ to be specified later, instead of using standard remainder term estimates for the central limit theorem. To this aim, we are going to use the following truncated processes from [6, Theorem 3.2] adapted to our problem:

$$\begin{aligned} X_n^1 &:= X_n \mathbf{1}_{\{|X_n| \leq (\log n)^a\}} - E\big[X_n \mathbf{1}_{\{|X_n| \leq (\log n)^a\}} \mid \mathcal{F}_{n-1}\big], \\ X_n^2 &:= X_n \mathbf{1}_{\{|X_n| > (\log n)^a\}} - E\big[X_n \mathbf{1}_{\{|X_n| > (\log n)^a\}} \mid \mathcal{F}_{n-1}\big], \end{aligned} \tag{2.3}$$

where $(\mathcal{F}_n)_{n\geq 1}$ is the filtration generated by $(X_n)_{n\geq 1}$, and $\mathcal{F}_0$ is the trivial σ-algebra. It is immediate that $(X_n^1)_{n\geq 1}$ and $(X_n^2)_{n\geq 1}$ are martingale difference sequences with respect to $(\mathcal{F}_n)_{n\geq 1}$ and $X_n = X_n^1 + X_n^2$. Put $S_n^i = X_1^i + \cdots + X_n^i$; $i = 1, 2$.

Let $p > 2$; as $\|X_k\|_p \leq C$, we have, for any $k \geq 1$ and $x > 0$,

$$F_k(x) := P\big[\,|X_k| > x\big] \leq C x^{-p}. \tag{2.4}$$

We obtain

$$
\begin{aligned}
E\left|X_k^2\right|^2 &\le -\int_{(\log n)^a}^{+\infty} x^2 dF_k(x) \\
&= -\lim_{N\to+\infty}\left[N^2F_k(N)-(\log n)^{2a}F_k\left((\log n)^a\right)+2\int_{(\log n)^a}^{N} xF_k(x)dx\right] \\
&\le C(\log n)^{(2-p)a}.
\end{aligned}
\tag{2.5}
$$

The second line is obtained by integration by parts and by using estimation (2.4) in the proper integral in (2.5).

By a general property for the martingale difference sequences, we have

$$
E\left|S_n^2\right|^2=\sum_{k=1}^{n}E\left|X_k^2\right|^2. \tag{2.6}
$$

Combining (2.5) and (2.6) gives

$$
\begin{aligned}
P\left[\left|S_n^2\right|>\frac{\varepsilon}{2}(n\log n)^{1/2}\right] &= P\left[\left|S_n^2\right|^2>\frac{\varepsilon^2}{4}n\log n\right] \\
&\le C\varepsilon^{-2}(n\log n)^{-1}E\left|S_n^2\right|^2 \\
&\le C\varepsilon^{-2}(\log n)^{(2-p)a-1}.
\end{aligned}
\tag{2.7}
$$

Let $\alpha>2$; by [6, Theorem 3.6], and taking into account that $|X_n^1|\le 2(\log n)^a$, we obtain

$$
E\left|S_n^1\right|^\alpha \le Cn^{\alpha/2-1}\sum_{k=1}^{n}E\left|X_k^1\right|^\alpha \le Cn^{\alpha/2}(\log n)^{a\alpha}. \tag{2.8}
$$

Hence

$$
P\left[\left|S_n^1\right|^\alpha>\left(\frac{\varepsilon}{2}\right)^\alpha(n\log n)^{\alpha/2}\right]\le C\varepsilon^{-\alpha}(n\log n)^{-\alpha/2}E\left|S_n^1\right|^\alpha\le C\varepsilon^{-\alpha}(\log n)^{a\alpha-\alpha/2}. \tag{2.9}
$$

Let $0\le\delta<p/2-1$. In the sequel take $\alpha>\alpha_0:=2(p-2)(1+\delta)/(p-2-2\delta)$, and note that $\alpha_0\ge 2$. By (2.7) and (2.9) we obtain that series (2.1) is dominated by

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(\log n)^\delta}{n}P\left[\left|S_n^1\right|>\frac{\varepsilon}{2}(n\log n)^{1/2}\right]+\sum_{n=2}^{\infty}\frac{(\log n)^\delta}{n}P\left[\left|S_n^2\right|>\frac{\varepsilon}{2}(n\log n)^{1/2}\right] \\
&\quad\le C\varepsilon^{-\alpha}\sum_{n=2}^{\infty}\frac{(\log n)^{a\alpha-\alpha/2+\delta}}{n}+C\varepsilon^{-2}\sum_{n=2}^{\infty}\frac{(\log n)^{(2-p)a-1+\delta}}{n}=:A+B.
\end{aligned}
\tag{2.10}
$$

Part (i) of Theorem 2.1 is proved if we choose $\delta/(p-2)<a<(\alpha-2\delta-2)/2\alpha$. Indeed, the first inequality ensures the convergence of series B, and the second inequality ensures the convergence of series A. Such an a exists, because the compatibility inequality $\delta/(\text{p-2})<(\alpha-2\delta-2)/2\alpha$ is equivalent to $\alpha>\alpha_0$.

For part (ii) of Theorem 2.1, let $p \geq 1$ and $\delta > p/2 - 1$. We need to construct a martingale difference sequence $(X_n)_{n\geq 1}$ with $\|X_n\|_p \leq C$ for any $n \geq 1$, and such that series (2.1) diverges. To this aim, consider $X_n = Z \cdot Y_n$, where $(Y_n)_{n\geq 1}$ is an i.i.d. bounded centered sequence, and Z is independent of $(Y_n)_{n\geq 1}$ with

$$P[|Z| > n] = Cn^{-c} \tag{2.11}$$

for $n \geq 1$ (where C is a normalization factor), for some $c > 0$ to be specified later. If $c > p$, then $(X_n)_{n\geq 1}$ has finite pth moments, as

$$E|X_n|^p \leq CE|Z|^p \leq C\sum_{n=1}^{\infty} n^p \left(P[|Z| > n] - P[|Z| > n+1]\right) \sim \sum_{n=1}^{\infty} n^{p-c-1}. \tag{2.12}$$

By independence and the central limit theorem, we have

$$\begin{aligned} P[|S_n| > \varepsilon(n\log n)^{1/2}] &\geq P[|Y_1 + \cdots + Y_n| > n^{1/2}] \cdot P[|Z| > \varepsilon(\log n)^{1/2}] \\ &\geq CP[|Z| > \varepsilon(\log n)^{1/2}], \end{aligned} \tag{2.13}$$

hence

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P[|S_n| > \varepsilon(n\log n)^{1/2}] \geq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\delta - c/2}. \tag{2.14}$$

The latter series in (2.14) diverges if $\delta \geq c/2 - 1$. Hence, for series (2.1) to diverge, it suffices to choose $p < c \leq 2\delta + 2$. Such c exists because the compatibility inequality $p < 2\delta + 2$ is equivalent to $\delta > p/2 - 1$.

Let $\delta \geq p/2 - 1$; to prove Theorem 2.1(iii) consider Z with finite second moment and such that

$$P[|Z| > n] \geq \frac{C}{n^p} \tag{2.15}$$

for $n \geq 1$. Define as in the proof of part (ii) $X_n = Z \cdot Y_n$ and note that $(X_n)_{n\geq 1}$ has finite moments of order $p - \lambda$, for all $0 < \lambda \leq p - 1$. As such,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P[|S_n| > \varepsilon(n\log n)^{1/2}] \\ &\quad \geq C\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P[|Z| > \varepsilon(\log n)^{1/2}] \geq C\sum_{n=2}^{\infty} \frac{(\log n)^{\delta - p/2}}{n} = +\infty. \end{aligned} \tag{2.16}$$

□

Remark 2.2. In particular, from Theorem 2.1, we deduce the following. If $1 \le p \le 2$, then series (2.1) diverges for any $\delta \ge 0$ and some L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$. For any $p > 2$ and L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$, series (2.1) converges for some $\delta > 0$: if $p > 3$ one can take $\delta = 1/2$, and if $p > 4$ one can take $\delta = 1$, and so forth.

The asymptotics, as $\varepsilon \searrow 0$, in Theorem 2.1(i) are given below.

Corollary 2.3. *For any $p > 2$ and L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$, it holds that*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\alpha_0} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left[\,|S_n| > \varepsilon (n \log n)^{1/2}\right] < +\infty \tag{2.17}$$

for any $0 \le \delta < p/2 - 1$, and where $\alpha_0 := 2(p-2)(1+\delta)/(p-2-2\delta)$.

Proof. In formula (2.10), as $\varepsilon \searrow 0$, series B behaves like ε^{-2} and series A behaves like ε^{α}, for $\alpha > \alpha_0$. Notice that $\alpha_0 \ge 2\delta + 2 \ge 2$, hence the normalization factor ε^{α_0} makes the limit in formula (2.17) finite. □

Remark 2.4. For L^2-bounded centered i.i.d. sequences, the optimal normalization factor in Corollary 3.2 is $\varepsilon^{2\delta+2}$ (see [4, Theorem 3]), in which case the limit in (2.17) is strictly positive. Moreover, note that $\alpha_0 \ge 2\delta + 2 \ge 2$ and $\lim \alpha_0 = 2\delta + 2$ as $p \to \infty$, hence the latter result can be viewed as a limiting case of our general result (2.17).

3. Davis' second theorem

Let $\varepsilon > 0$ and let $\delta = \delta(p)$ be a function of $p \ge 1$. Consider the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\delta}} P\left[\,|S_n| > \varepsilon (n \log\log n)^{1/2}\right]. \tag{3.1}$$

Assume $(X_n)_{n\ge1}$ is an i.i.d. sequence. If $\delta = 0$, *Davis' second theorem* says that series (3.1) converges when $\varepsilon > \sigma\sqrt{2}$ if and only if $(X_n)_{n\ge1}$ is an L^2-bounded centered sequence, and where $\sigma^2 := E[X_1^2]$. If $\delta = 1$, series (3.1) is convergent for any $\varepsilon > 0$ if $(X_n)_{n\ge1}$ is centered and satisfies slightly less than a second moment, but the necessary and sufficient moment condition is not known (see [2, 3, 4, 5]).

In the martingale difference sequences case, we have the following result.

Theorem 3.1. (i) *For any $p > 2$ and L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$, series (3.1) is convergent for $\delta \ge 1$ and any $\varepsilon > 0$.*

(ii) *For any $1 \le p < 2$ there exists an L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$ such that series (3.1) diverges for $\delta = 1$ and any $\varepsilon > 0$.*

(iii) *For any $p \ge 1$ there exists an L^p-bounded martingale difference sequence $(X_n)_{n\ge1}$ such that series (3.1) diverges for any $0 \le \delta < 1$ and $\varepsilon > 0$.*

(iv) *There exist a probability space $(\Omega, \mathcal{F}, P)$ and an $L^{2-\lambda}$-bounded martingale difference sequence $(X_n)_{n\ge1}$ (for all $0 < \lambda \le 1$) such that series (3.1) diverges for $0 \le \delta \le 1$ and any $\varepsilon > 0$.*

Proof. To prove (i), first remark that formulas (2.7) and (2.9) can be proved the same way as we did in Theorem 2.1(i) when replacing $(\log n)^a$ by $(\log\log n)^a$. As such, with the same notations therein, we obtain

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P\left[|S_n|>\frac{\varepsilon}{2}(n\log\log n)^{1/2}\right]\\ \leq C\varepsilon^{-\alpha}\sum_{n=3}^{\infty}\frac{(\log\log n)^{a\alpha-\alpha/2}}{n(\log n)^{\delta}}+C\varepsilon^{-2}\sum_{n=3}^{\infty}\frac{(\log\log n)^{(2-p)a-1}}{n(\log n)^{\delta}}.\end{aligned}\tag{3.2}$$

From (3.2) we deduce that series (3.1) with $\delta>1$ is convergent regardless of the values of a and α; also, series (3.1) with $\delta=1$ is convergent if $0<a<(\alpha-2)/2\alpha$. Indeed, in the latter case, the second series in the second line of (3.2) is convergent if $a>0$ and the first series in the second line of (3.2) is convergent if $a<(\alpha-2)/2\alpha$. Such a exists because $\alpha>2$.

For parts (ii) and (iii) of Theorem 3.1, within the same construction of the counterexample of martingale difference sequence in Theorem 2.1(ii), take Z with $P[|Z|>n]=Cn^{-c}$ for some $c>p$. Note that $(X_n)_{n\geq1}$ have finite pth-order moments. As

$$\begin{aligned}P[|S_n|>\varepsilon(n\log\log n)^{1/2}]\geq P[|Y_1+\cdots+Y_n|>n^{1/2}]\cdot P[|Z|>\varepsilon(\log\log n)^{1/2}]\\ \geq C\cdot P[|Z|>\varepsilon(\log\log n)^{1/2}],\end{aligned}\tag{3.3}$$

series (3.1) dominates the following analog of (2.14):

$$C\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P[|Z|>\varepsilon(\log\log n)^{1/2}]\geq C\sum_{n=3}^{\infty}\frac{(\log\log n)^{-c/2}}{n(\log n)^{\delta}}.\tag{3.4}$$

The latter series in (3.4) diverges for any $0\leq\delta<1$ and $c>0$, and this proves (iii); it also diverges for $\delta=1$ and $c<2$, and this proves (ii) for $1\leq p<2$.

To prove (iv) consider Z with finite second moment and such that $P[|Z|>n]\geq C/n^2$ for $n\geq1$. Define as above $X_n=Z\cdot Y_n$ and note that $(X_n)_{n\geq1}$ has finite moments of order $2-\lambda$, for all $0<\lambda\leq1$. As such,

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P[|S_n|>\varepsilon(n\log\log n)^{1/2}]\geq C\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P[|Z|>\varepsilon(\log\log n)^{1/2}]\\ \geq C\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}(\log\log n)}=+\infty,\end{aligned}\tag{3.5}$$

as $0\leq\delta\leq1$. □

The asymptotics, as $\varepsilon\searrow0$, in Theorem 3.1(i) are given below.

Corollary 3.2. *For any $p>2$, $\delta\geq1$ and L^p-bounded martingale difference sequence $(X_n)_{n\geq1}$, it holds that*

$$\lim_{\varepsilon\searrow0}\varepsilon^2\sum_{n=3}^{\infty}\frac{1}{n(\log n)^{\delta}}P[|S_n|>\varepsilon(n\log n\log n)^{1/2}]<+\infty.\tag{3.6}$$

Proof. In formula (3.2), as $\varepsilon \searrow 0$, the second series behaves like ε^{-2} and the first series behaves like ε^{α}, for all $\alpha > 2$. Hence the normalization factor ε^2 makes the limit in formula (3.6) finite. □

Remark 3.3. For L^2-bounded centered i.i.d. sequences, the optimal normalization factor in Corollary 3.2 is precisely ε^2 (see [5]), that is, the limit in (3.6) is strictly positive. In other words, our Corollary 3.2 gives sharp rates of convergence for series (3.1).

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