APPROXIMATE CONTROLLABILITY OF DELAYED SEMILINEAR CONTROL SYSTEMS

LIANWEN WANG

We deal with the approximate controllability of control systems governed by delayed semilinear differential equations $\dot{y}(t) = Ay(t) + A_1 y(t - \Delta) + F(t, y(t), y_t) + (Bu)(t)$. Various sufficient conditions for approximate controllability have been obtained; these results usually require some complicated and limited assumptions. Results in this paper provide sufficient conditions for the approximate controllability of a class of delayed semilinear control systems under natural assumptions.

1. Introduction

The main concern in this paper is the approximate controllability of the following delayed semilinear control system:

$$\dot{y}(t) = Ay(t) + A_1 y(t - \Delta) + F(t, y(t), y_t) + (Bu)(t), \quad t \ge a,$$
$$y_a = \xi, \tag{1.1}$$

in a real Hilbert space X with the norm $\| \cdot \|$. The meaning of all notations is listed as follows: $\Delta \ge 0$ is a system delay; $y(\cdot) : [a - \Delta, b] \to X$ is the state function; $\xi \in C([-\Delta, 0]; X)$, the Banach space of all continuous functions $\psi : [-\Delta, 0] \to X$ endowed with the norm $|\psi| = \sup\{\|\psi(\theta)\| : -\Delta \le \theta \le 0\}$; A is the generator of a C_0 semigroup $T(t)$ in X; A_1 is a bounded linear operator from X to X; $F : [a, b] \times X \times C([-\Delta, 0]; X) \to X$ is a nonlinear operator; $u(\cdot) \in L^2(a, b; U)$ is a control function; U is a Hilbert space; B is a bounded linear operator from $L^2(a, b; U)$ to $L^2(a, b; X)$. In addition, for any $y \in C([a - \Delta, b]; X)$ and $t \in [a, b]$, define $y_t \in C([-\Delta, 0]; X)$ by $y_t(\theta) = y(t + \theta)$ for $\theta \in [-\Delta, 0]$.

Denote the state function of (1.1) corresponding to a control $u(\cdot)$ by $y(\cdot; u)$. Then $y(b; u)$ is the state value at terminal time b. Introduce the set

$$R_b(F) = \{y(b; u) : u(\cdot) \in L^2(a, b; U)\}, \tag{1.2}$$

which is called the reachable set of system (1.1) at terminal time b, its closure in X is denoted $\overline{R_b(F)}$.

Definition 1.1. The system (1.1) is said to be approximately controllable on $[a, b]$ if $\overline{R_b(F)} = X$.

The following system is called the corresponding linear system of (1.1):

$$\dot{y}(t) = Ay(t) + A_1 y(t - \Delta) + (Bu)(t), \quad t \geq a,$$
$$y_a = \xi. \tag{1.3}$$

This is a special case of (1.1) with $F \equiv 0$. The reachable set of system (1.3) at terminal time b is denoted $R_b(0)$. Similarly, system (1.3) is said to be approximately controllable on $[a, b]$ if $\overline{R_b(0)} = X$.

For semilinear control systems without delays, approximate controllability has been extensively studied in the literature. We list only a few of them. Zhou [10] studied the approximate controllability for a class of semilinear abstract equations. Naito [6] established the approximate controllability for semilinear control systems under the assumption that the nonlinear term is bounded. Approximate controllability for semilinear control systems also can be found in Choi et al. [1], Fernandez and Zuazua [2], Li and Yong [4], Mahmudov [5], and many other papers. Most of them concentrate on finding conditions of F, A, and B such that semilinear systems are approximately controllable on $[a, b]$ if the corresponding linear systems are approximately controllable on $[a, b]$.

For semilinear delayed control systems, some papers are devoted to the approximate controllability. For example, Klamka [3] provided some approximate controllability results. Naito and Park [7] dealt with approximate controllability for delayed Volterra systems. In [9] Ryu et al. studied approximate controllability for delayed Volterra control systems. The purpose of this paper is to study the approximate controllability of control system (1.1). We obtain the approximate controllability of system (1.1) if the corresponding linear system is approximate controllable and other natural assumptions such as the local Lipschitz continuity for F and the compactness of operator W are satisfied.

2. Basic assumptions

We start this section by introducing the fundamental solution $S(t)$ of the following system:

$$\dot{y}(t) = Ay(t) + A_1 y(t - \Delta), \quad t \geq a,$$
$$y_a = \xi. \tag{2.1}$$

We already know that (2.1) has a unique solution, denoted by $y^\xi(t)$, for each $\xi \in C([-\Delta, 0]; X)$. Hence, we can define an operator $S(t)$ in X by

$$S(t)\xi(0) = \begin{cases} y^\xi(t + a), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{2.2}$$

$S(t)$ is called the fundamental solution of (2.1). It is easy to check that $S(t)$ is the unique solution of the following operator equation:

$$S(t) = T(t) + \int_0^t T(t-s)A_1 S(s-\Delta)ds. \tag{2.3}$$

Let $K := \max\{\|T(t)\| : 0 \le t \le b\}$. By (2.3) we have

$$\|S(t)\| \le K + \int_0^t K\|A_1\|\,\|S(s-\Delta)\|ds \le K + K\|A_1\| \int_\Delta^t \|S(s)\|ds. \tag{2.4}$$

Gronwall's inequality implies that

$$\|S(t)\| \le K \exp\left(K\|A\|(b-\Delta)\right) := M_1, \quad 0 \le t \le b. \tag{2.5}$$

Throughout the paper we impose the following condition on F.

(H1) $F : [a,b] \times X \times C([-\Delta,0];X) \to X$ is locally Lipschitz continuous in y, η uniformly in $t \in [a,b]$; that is, for any $r > 0$, there is a constant $L(r)$ such that

$$\|F(t,y_1,\eta_1) - F(t,y_2,\eta_2)\| \le L(r)(\|y_1 - y_2\| + |\eta_1 - \eta_2|) \tag{2.6}$$

for any $t \in [a,b]$, $\|y_1\| \le r$, $\|y_2\| \le r$, $|\eta_1| \le r$, and $|\eta_2| \le r$.

With a minor modification of [8], we can prove that system (1.1) has a unique mild solution $y(\cdot;u) \in C([a-\Delta,b];X)$ for any control $u(\cdot) \in L^2(a,b;U)$ under assumption (H1). This mild solution is defined as a solution of the integral equation:

$$y(t;u) = S(t-a)\xi(0) + \int_a^t S(t-s)\left[F(s,y(s;u),y_s) + (Bu)(s)\right]ds, \quad t \ge a, \tag{2.7}$$
$$y_a = \xi.$$

Similarly, for any $z(\cdot) \in L^2(a,b;X)$, the following integral equation:

$$x(t;z) = S(t-a)\xi(0) + \int_a^t S(t-s)\left[F(s,x(s;z),x_s) + z(s)\right]ds, \quad t \ge a, \tag{2.8}$$
$$x_a = \xi,$$

has a unique mild solution $x(\cdot;z)$. Therefore, we can define an operator W from $L^2(a,b;X)$ to $C([a,b];X)$ by

$$(Wz)(\cdot) = x(\cdot;z). \tag{2.9}$$

Regarding the operator W, we assume that

(H2) W is a compact operator.

Remark 2.1. (H2) is the case if, for instance, $T(t)$, the semigroup generated by A, is a compact semigroup.

The following assumption (H3) was introduced by Naito in [6]. Define a linear operator φ from $L^2(a,b;X)$ to X by

$$\varphi p = \int_a^b S(b-s)p(s)ds \quad \text{for } p(\cdot) \in L^2(a,b;X). \tag{2.10}$$

Let the kernel of the operator φ be N; that is, $N = \{p : \varphi p = 0\}$. Then N is a closed subspace of $L^2(a, b; X)$. Denote its orthogonal space in $L^2(a, b; X)$ by $N^\perp$. Let G be the projection operator from $L^2(a, b; X)$ into $N^\perp$ and let $R[B]$ be the range of B. We assume that

(H3) for any $p(\cdot) \in L^2(a, b; X)$, there is a function $q(\cdot) \in \overline{R[B]}$ such that $\varphi p = \varphi q$.

Remark 2.2. (H3) is valid for many control systems, see [6] for detailed discussion.

It follows from assumption (H3) that $\{x + N\} \cap \overline{R[B]} \neq \varnothing$ for any $x \in N^\perp$. Therefore, the operator P from $N^\perp$ to $\overline{R[B]}$ defined by

$$Px = x^*, \tag{2.11}$$

where $x^* \in \{x + N\} \cap \overline{R[B]}$ and $\|x^*\| = \min\{\|y\| : y \in \{x + N\} \cap \overline{R[B]}\}$, is well defined. It is proved in [6] that P is bounded.

3. Lemmas

This section provides two lemmas that will be used to prove the main theorem.

LEMMA 3.1. *Assume that $a(t)$ is continuous on $[a, b]$, $b(t)$ is nonnegative and integrable on $[a, b]$, and $x(t)$ is a nonnegative continuous function satisfying the following inequality:*

$$x(t) \leq a(t) + \int_a^t b(s) x^\alpha(s) ds, \quad 0 \leq \alpha < 1,\ t \in [a, b]. \tag{3.1}$$

If the equation

$$y(t) = a(t) + \int_a^t b(s) y^\alpha(s) ds \tag{3.2}$$

has a unique solution $\bar{y}(t)$ on $[a, b]$, then

$$x(t) \leq \bar{y}(t), \quad t \in [a, b]. \tag{3.3}$$

Proof. Let $C[a, b]$ be the Banach space of all continuous functions on $[a, b]$ endowed with the maximum norm. Define an operator E from $C[a, b]$ to $C[a, b]$ by

$$(Ey)(t) = a(t) + \int_a^t b(s) y^\alpha(s) ds. \tag{3.4}$$

Construct a sequence $\{y_n\}$ as follows:

$$y_0(t) = x(t), \quad y_{n+1}(t) = (Ey_n)(t), \quad n = 0, 1, \dots. \tag{3.5}$$

We have

$$x(t) = y_0(t) \leq y_1(t) \leq \cdots, \qquad \|x\| = \|y_0\| \leq \|y_1\| \leq \cdots. \tag{3.6}$$

Note that

$$\|Ey\| \leq \|a\| + \|y\|^\alpha \int_0^t b(s) ds, \tag{3.7}$$

then we can find a number $d > 0$ such that

$$\|Ey\| \le \|y\| \quad \text{for } \|y\| \ge d. \tag{3.8}$$

If $\|y_n\| \le d$ holds for any integer $n = 0, 1, \dots$, then $\|y_n\|$ is bounded. Otherwise, it follows from (3.6) that a sufficiently large integer N exists such that

$$\|y_0\| \le \cdots \le \|y_N\| \le d, \qquad \|y_n\| > d \quad \text{for } n > N. \tag{3.9}$$

Thus

$$\max\{\|y_n\| : n \ge 0\} \le \max\{d, \|y_{N+1}\|\} := m. \tag{3.10}$$

Consequently,

$$0 \le x(t) = y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le \cdots \le m. \tag{3.11}$$

Therefore,

$$\lim_{n \to \infty} y_n(y) = \bar{y}(t). \tag{3.12}$$

Note that $\bar{y}(t)$ is the unique solution of (3.2). The conclusion of Lemma 3.1 follows from (3.11). □

LEMMA 3.2. *Assume that (H1) is fulfilled. Furthermore, for any $y \in X$ and $\eta \in C([-\Delta, 0]; X)$*

$$\|F(t, y, \eta)\| \le M(1 + \|y\|^\alpha + |\eta|^\alpha), \quad 0 \le \alpha < 1, \, t \in [a, b]. \tag{3.13}$$

Then the mild solution $x(t; z)$ of (2.8) has the estimate

$$|x_t| \le H(\|z\|), \tag{3.14}$$

where $H(r)$ is an increasing function and $H(r) = O(r)$ as $r \to \infty$.

Proof. Recall that

$$M_1 = \max\{\|S(t)\| : 0 \le t \le b\}. \tag{3.15}$$

It follows from $\|x(s)\| \le |x_s|$ and (2.8) that

$$
\begin{aligned}
\|x(t)\| &\le M_1 |\xi| + M_1 M \int_a^t (1 + \|x(s)\|^\alpha + |x_s|^\alpha) \, ds + M_1 \int_a^t \|z(s)\| \, ds \\
&\le M_1 |\xi| + M_1 M (b - a) + M_1 \sqrt{b - a} \|z\| + 2 M_1 M \int_a^t |x_s|^\alpha \, ds.
\end{aligned}
\tag{3.16}
$$

For any $\theta \in [-\Delta, 0]$, we have

$$\|x(t + \theta)\| \le M_1 |\xi| + M_1 M (b - a) + M_1 \sqrt{b - a} \|z\| + 2 M_1 M \int_a^{t+\theta} |x_s|^\alpha \, ds. \tag{3.17}$$

Hence

$$|x_t| \le M_1|\xi| + M_1 M(b-a) + M_1\sqrt{b-a}\|z\| + 2M_1 M \int_a^t |x_s|^\alpha ds. \qquad (3.18)$$

Note that for any two constants V_1 and V_2, the following equation

$$y(t) = V_1 + V_2 \int_0^t y^\alpha(s)ds \qquad (3.19)$$

has a unique solution

$$y(t) = \left[(1-\alpha)V_2 t + V_1^{1-\alpha}\right]^{1/(1-\alpha)}. \qquad (3.20)$$

Applying Lemma 3.1 to (3.18), we obtain

$$|x_t| \le \left[2(1-\alpha)MM_1(b-a) + (M_1|\xi| + M_1\sqrt{b-a}\|z\| + MM_1(b-a))^{1-\alpha}\right]^{1/(1-\alpha)}$$
$$:= H(\|z\|).$$
$$(3.21)$$

Clearly, the function $H(r)$ satisfies all requirements of Lemma 3.2 and the proof of the lemma is complete.   □

4. Approximate controllability

The following theorem is the main result of this paper.

THEOREM 4.1. *Assume that linear system (1.3) is approximately controllable on $[a,b]$. If (H1), (H2), (H3), and (3.13) are fulfilled, then system (1.1) is approximately controllable on $[a,b]$.*

Proof. Note that system (1.3) is approximately controllable on $[a,b]$ by the assumption, then $\overline{R_b(0)} = X$. To prove the approximate controllability of (1.1); that is, $\overline{R_b(F)} = X$, it is sufficient to show that

$$R_b(0) \subset \overline{R_b(F)}. \qquad (4.1)$$

That means for any $\epsilon > 0$ and $x^b \in R_b(0)$, there exists a $\nu \in R_b(F)$ such that $\|\nu - x^b\| < \epsilon$.

By the definition of reachable set $R_b(0)$ of system (1.3), there is a control $u(\cdot) \in L^2(a,b;U)$ such that

$$x^b = S(b-a)\xi(0) + \int_a^b S(b-s)(Bu)(s)ds. \qquad (4.2)$$

Let $\bar{z}_0 = Bu$, $z_0 = G\bar{z}_0$. Then $z_0 \in N^\perp$. Define an operator J from $N^\perp$ to $N^\perp$ by

$$J\nu = z_0 - GTP\nu, \quad \nu \in N^\perp, \qquad (4.3)$$

where Γ is the operator from $L^2(a,b;X)$ to $L^2(a,b;X)$ defined by

$$(\Gamma z)(t) := F(t,(Wz)(t),(Wz)_t) = F(t,x(t;z),x_t). \tag{4.4}$$

For any $v \in N^{\perp}$, we have $Pv \in L^2(a,b;X)$, $\Gamma Pv \in L^2(a,b;X)$, and $G\Gamma Pv \in N^{\perp}$. Therefore, J is well defined.

Since W is compact by assumption (H2), for any bounded sequence $z_n(\cdot) \in L^2(a,b;X)$; that is, $\|z_n\| \leq r_1$ for some $r_1 > 0$, there is a subsequence $z_{n_k}(\cdot)$ of $z_n(\cdot)$ such that $(Wz_{n_k})(\cdot)$ converges to $\bar{x}(\cdot)$ in $C([a,b];X)$ as $k \to \infty$. So, Wz_{n_k} is bounded in $C([a,b];X)$; that is, $\|Wz_{n_k}\| \leq r_2$ for some constant $r_2 > 0$. (H1) implies that a constant $L(r) > 0$ exists such that

$$\begin{aligned}
\|F(t,(Wz_{n_k})(t),&(Wz_{n_k})_t) - F(t,\bar{x}(t),\bar{x}_t)\| \\
&\leq L(r)\left(\|(Wz_{n_k})(t) - \bar{x}(t)\| + |(Wz_{n_k})_t - \bar{x}_t|\right),
\end{aligned} \tag{4.5}$$

where $r = \max(r_1,r_2)$. Hence, we have

$$\begin{aligned}
\|\Gamma z_{n_k} &- F(\cdot,\bar{x}(\cdot),\bar{x}.)\|^2 \\
&= \|F(\cdot,(Wz_{n_k})(\cdot),(Wz_{n_k}).) - F(\cdot,\bar{x}(\cdot),\bar{x}.)\|^2 \\
&\leq L^2(r)(b-a)\left(\sup_{a \leq t \leq b} \|(Wz_{n_k})(t) - \bar{x}(t)\| \right. \\
&\qquad\qquad \left. + \sup_{a \leq t \leq b} |(Wz_{n_k})_t - \bar{x}_t| \right)^2 \longrightarrow 0
\end{aligned} \tag{4.6}$$

as $k \to \infty$. Therefore, Γ is compact and J is compact as well.

From Lemma 3.2, for any $z(\cdot) \in L^2(a,b;X)$, we have

$$\|F(t,x(t;z),x_t)\| \leq M(1 + \|x(t;z)\|^{\alpha} + |x_t|^{\alpha}) \leq M(1 + 2H^{\alpha}(\|z\|)). \tag{4.7}$$

Note that $H(r)$ is increasing and P is a bounded operator, then

$$\begin{aligned}
\|z_0 - G\Gamma Pv\| &\leq \|z_0\| + \|G\Gamma Pv\| \\
&\leq \|z_0\| + M\sqrt{b-a} + 2M\sqrt{b-a}H^{\alpha}(\|P\|\|v\|).
\end{aligned} \tag{4.8}$$

Taking into account

$$\lim_{\|v\| \to \infty} \frac{\|z_0\| + M\sqrt{b-a} + 2M\sqrt{b-a}H^{\alpha}(\|P\|\|v\|)}{\|v\|} = 0, \tag{4.9}$$

then

$$\lim_{\|v\| \to \infty} \frac{\|z_0 - G\Gamma Pv\|}{\|v\|} = 0. \tag{4.10}$$

Therefore, we can find a sufficiently large number $\bar{r}$ such that

$$\|z_0 - G\Gamma Pv\| \leq \bar{r} \quad \text{for } \|v\| \leq \bar{r}. \tag{4.11}$$

This means that J maps the bounded closed set $D(\bar{r}) = \{v : \|v\| \leq \bar{r}, v \in N^\perp\}$ of $N^\perp$ into itself. Consequently, a fixed point of operator J exists due to the Schauder fixed point theorem; that is, there is a $v^* \in D(\bar{r})$ such that

$$Jv^* = z_0 - G\Gamma Pv^* = v^*. \tag{4.12}$$

On account of

$$Pv^* \in (v^* + N) \cap \overline{R[B]}, \tag{4.13}$$

we have

$$\int_a^b S(b-s)(Pv^*)(s)ds = \int_a^b S(b-s)v^*(s)ds. \tag{4.14}$$

Note that G is the projection operator from $L^2(0,T;X)$ into $N^\perp$, then we have

$$\int_a^b S(b-s)Gp(s)ds = \int_a^b S(b-s)p(s)ds \quad \text{for } p(\cdot) \in L^2(a,b;X),$$

$$\int_a^b S(b-s)(Bu)(s)ds = \int_a^b S(b-s)[F(s,x(s;Pv^*),x_s) + v^*(s)]ds \tag{4.15}$$

$$= \int_a^b S(b-s)[F(s,x(s;Pv^*),x_s) + (Pv^*)(s)]ds.$$

Finally,

$$x^b = S(b-a)\xi(0) + \int_a^b S(b-s)[F(s,x(s;Pv^*),x_s) + (Pv^*)(s)]ds = x(b;Pv^*). \tag{4.16}$$

Observe that $Pv^* \in \overline{R[B]}$, then there is a sequence $u_n(\cdot) \in L^2(a,b;U)$ such that $Bu_n \to Pv^*$ as $n \to \infty$. W is continuous due to its compactness, then

$$WBu_n \longrightarrow WPv^* \quad \text{in } C([a,b];X). \tag{4.17}$$

This implies

$$x(b;Bu_n) \longrightarrow x(b;Pv^*) = x^b \tag{4.18}$$

as $n \to \infty$. Since $x(b;Bu_n) = y(b;u_n) \in R_T(F)$, we obtain $x^b \in \overline{R_T(F)}$ and complete the proof of the theorem. □

Remark 4.2. If $A_1 = 0$, (H3) implies the approximate controllability of (1.3) on $[a,b]$ (see [6]). Therefore, Naito's result in [6] is a special case of Theorem 4.1 when $A_1 = 0$, $\Delta = 0$, and $F(t,x(t),x_t) = F(x(t))$. In particular, we improve Naito's result by weakening the uniform Lipschitz continuity and the uniform boundedness imposed on the nonlinear term.

5. Example

Let $X = L^2(0,\pi)$ and $e_n(x) = \sin(nx)$ for $n = 1,\dots$. Then $\{e_n : n = 1,2,\dots\}$ is an orthogonal base for X. Define $A : X \to X$ by $Ay = y''$ with domain

$$D(A) = \{y \in X : y \text{ and } y' \text{ are absolutely continuous, } y'' \in X, \ y(0) = y(\pi) = 0\}. \quad (5.1)$$

Then

$$Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, \quad y \in D(A). \quad (5.2)$$

It is well known that A is the infinitesimal generator of an analytic group $T(t)$, $t \geq 0$, in X and is given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} \langle y, e_n \rangle e_n, \quad y \in X. \quad (5.3)$$

$T(t)$ is compact because it is an analytic semigroup. Define an infinite dimensional space U by

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n e_n, \ \sum_{n=2}^{\infty} u_n^2 < \infty \right\} \quad (5.4)$$

with the norm defined by

$$\|u\|_U = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}. \quad (5.5)$$

Define a mapping B from U to X as follows:

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n. \quad (5.6)$$

Consider the following delayed semilinear heat equation:

$$\begin{aligned}
\frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + y(t-\Delta,x) + F\big(y(t,x), y(t-\Delta,x)\big) \\
&\quad + Bu(t,x), \quad 0 < t < b, \ 0 < x < \pi, \\
y(t,0) &= y(t,\pi) = 0, \quad 0 \leq t \leq b, \\
y(t,x) &= \xi(x), \quad -\Delta \leq t \leq 0, \ 0 \leq x \leq \pi.
\end{aligned} \quad (5.7)$$

Then system (5.7) can be written to the abstract form (1.1). (H2) holds because $T(t)$ is a compact semigroup. Following the same arguments as in [6] we can prove that (H3) is valid and that the corresponding linear system is approximately controllable on $[0,b]$. By Theorem 4.1, system (5.7) is approximately controllable on $[0,b]$ if F is locally Lipschitz continuous and condition (3.13) is satisfied.

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Lianwen Wang: Department of Mathematics and Computer Science, Central Missouri State University, Warrensburg, MO 64093, USA

E-mail address: lwang@cmsu1.cmsu.edu