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INVARIANT MEASURES FOR CHEBYSHEV MAPS[1]

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Let $T_\lambda(x) = \cos(\lambda \arccos x)$, $-1 \le x \le 1$, where $\lambda > 1$ is not an integer. For a certain set of λ's which are irrational, the density of the unique absolutely continuous measure invariant under T_λ is determined exactly. This is accomplished by showing that T_λ is differentially conjugate to a piecewise linear Markov map whose unique invariant density can be computed as the unique left eigenvector of a matrix.

Key words: Chebyshev Map, Absolutely Continuous Invariant Measure, Markov Map.

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1. Chebyshev Maps

The function $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$, $n = 0, 1, 2, \ldots$ defines the nth Chebyshev polynomial, which is a solution of the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0. \tag{1}$$

Polynomial T_n, $n \ge 2$, transforms each of the intervals $[\frac{i}{n-1}, \frac{i+1}{n-1}]$, $i = -(n-1), \ldots, (n-1)-1$, onto $[-1, 1]$. It is easy to show (see [1]) that the unique absolutely continuous invariant measure for all T_n's is

$$d\mu(x) = \frac{1}{\pi\sqrt{1-x^2}}dx.$$

In this note, we consider the family of Chebyshev maps $T_\lambda(x) = \cos(\lambda \arccos x)$, $-1 \le x \le 1$, where $\lambda > 1$ is not an integer. T_λ is a solution of the same differential equation (1) with n replaced by λ, but T_λ is no longer a polynomial and we, therefore, refer to it as a Chebyshev map. The first monotonic branch of T_λ is not onto, but all the others are (see Figure 1).

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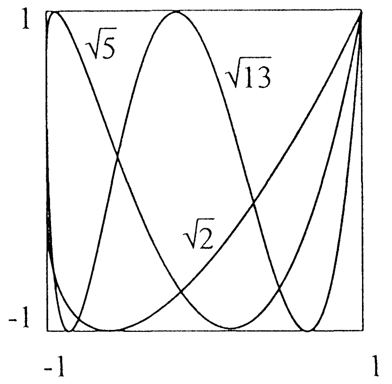

Figure 1: Chebyshev maps for $\lambda = \sqrt{2}$, $\sqrt{5}$, $\sqrt{13}$.

The main result of this note characterizes a set of λ's, not integers, for which the unique absolutely continuous measure invariant under T_λ can be determined exactly. This is accomplished by showing that T_λ is differentially conjugate to a piecewise linear Markov map whose unique invariant density can be computed as the unique left eigenvector of a matrix.

A related problem of finding an absolutely continuous invariant measure for a piecewise linear map with two branches ($1 < \lambda < 2$) was considered in [3].

2. Differentiable Conjugacy

For $\lambda > 1$, let $\Lambda_\lambda: [0,1] \to [0,1]$ be a piecewise linear continuous map having slope $\mp \lambda$ and defined by joining points $(0,0)$, $(\frac{1}{\lambda}, 1)$, $(\frac{2}{\lambda}, 0)$, $(\frac{3}{\lambda}, 1)$,.... Depending on λ, either of the two situations shown in Figure 2 can occur, where $[\lambda]$ is the largest integer less than or equal to λ.

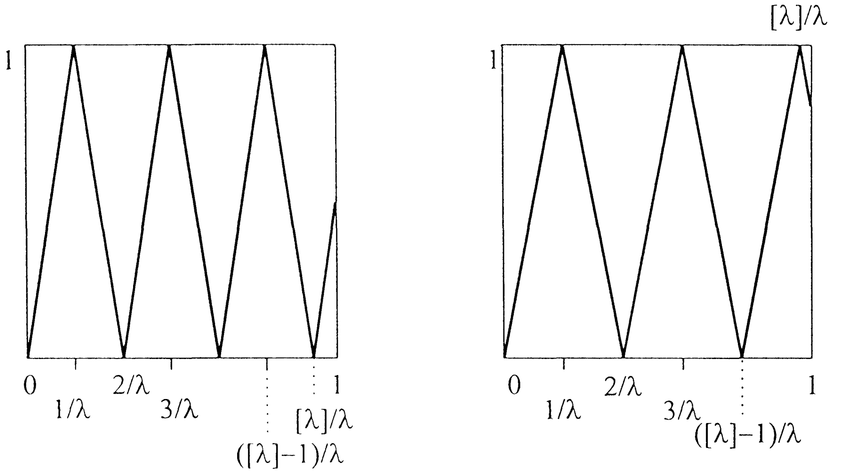

Figure 2: Two possible shapes of Λ_λ.

Let $h:[0,1] \to [-1,1]$ be defined by $h(x) = \cos(\pi x)$. Both h and h^{-1} are continuous differentiable.

Proposition 1: *For any $\lambda > 1$, we have $\Lambda_\lambda = h^{-1} \circ T_\lambda \circ h$, i.e., T_λ is differentially conjugated to the triangle map Λ_λ.*

Proof: It is enough to show that $T_\lambda \circ h = h \circ \Lambda_\lambda$. Let us consider x in the intervals $[\frac{k}{\lambda}, \frac{k+1}{\lambda}]$, for $0 \le k \le [\lambda] - 1$ and $[\frac{[\lambda]}{\lambda}, 1]$, for $k = [\lambda]$. For k even, we have $\Lambda_\lambda(x) = \lambda(x - \frac{k}{\lambda})$. Thus

$$h(\Lambda_\lambda(x)) = \cos(\pi\lambda x - \pi k) = \cos(\pi \lambda x).$$

For k odd, we have $\Lambda_\lambda(x) = 1 - \lambda(x - \frac{k}{\lambda})$. Therefore,

$$h(\Lambda_\lambda(x)) = \cos(\pi - \pi\lambda x + \pi k) = \cos(\pi\lambda x).$$

On the other hand, we always have

$$T_\lambda(h(x)) = \cos(\lambda \arccos(\cos(\pi x))) = \cos(\lambda \pi x).$$

This proves the claimed conjugation. □

Remark 1: For $[\lambda]$ even, we have $\Lambda_\lambda(1) = \lambda - [\lambda]$. For odd $[\lambda]$, we have $\Lambda_\lambda(1) = 1 - (\lambda - [\lambda])$.

3. The Measure Invariant Under a Chebyshev Map

Definition: Let $\tau:[0,1] \to [0,1]$ be a piecewise monotonic map for which there exist points $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = 1$ such that for $i = 0, 1, \ldots, n-1$, $\tau \mid_{(a_i, a_{i+1})}$ is a homeomorphism onto some interval $(a_{j(i)}, a_{k(i)})$. Then τ is called Markov with respect to the partition $(a_1, a_2, \ldots, a_n)$.

The main tool in exploring absolutely continuous invariant measures of τ is the Frobenius-Perron operator P_τ on the space of Lebesgue integrable functions $L^1([0,1])$:

$$(P_\tau f)(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{\mid \tau'(\tau_i^{-1}(x)) \mid},$$

where τ_i^{-1}, $i = 1, \ldots, n$, are inverse branches of τ. A function f satisfies $P_\tau f = f$ if and only if f is the density of an absolutely continuous τ-invariant measure. For more detailed information on P_τ, see [1].

Let $\mathcal{P}$ be the partition $(0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{[\lambda]}{\lambda}, 1)$. Clearly, if λ is a positive integer, Λ_λ is a Markov map with respect to $\mathcal{P}$. Below we characterize the set of λ's, not integers, for which Λ_λ is a Markov map with respect to $\mathcal{P}$. It is easy to see that Λ_λ is Markov if and only if $\Lambda_\lambda(1) = \frac{m}{\lambda}$, where m is an integer satisfying $1 \le m \le [\lambda]$.

Proposition 2: *If λ is not an integer, then Λ_λ is Markov with respect to $\mathcal{P}$ if and only if:*

***Case* 1:**

$$\lambda = n + \sqrt{n^2 + m}, \text{ for } [\lambda] = 2n \text{ and } 1 \le m \le 2n; \tag{2}$$

***Case* 2:**

$$\lambda = (n+1) + \sqrt{(n+1)^2 - m}, \; for \; [\lambda] = 2n+1 \; and \; 1 \le m \le 2n. \tag{3}$$

In both cases, λ is irrational. In (3), m cannot be equal to $2n+1$ since then λ would be an integer.

Proof: We use the values of $\Lambda_\lambda(1)$ from Remark 1. In Case 1, we have $\Lambda_\lambda(1) = \lambda - 2n = \frac{m}{\lambda}$, or $\lambda^2 - 2\lambda n - m = 0$, whose positive solution is given in (2). In Case 2, we have $\Lambda_\lambda(1) = 1 - (\lambda - 2n - 1) = \frac{m}{\lambda}$, or $\lambda^2 - 2\lambda(n+1) + m = 0$. The only positive solution is given in (3). □

Remark 2: There are no non-integer solutions to (3) for $[\lambda] = 1$. Thus, the smallest $[\lambda]$ we can actually consider is $[\lambda] = 2$.

When Λ_λ is Markov with respect to $\mathcal{P}$, then its Frobenius-Perron operator restricted to the piecewise constant functions on the partition $\mathcal{P}$, can be represented by the $\lambda_1 \times \lambda_1$ matrix $\mathbb{M}$, where $\lambda_1 = [\lambda] + 1$. For $[\lambda] = 2n$ and $\lambda - [\lambda] = \frac{m}{\lambda}$ (Case 1), we have

$$\mathbb{M} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & 0 & \cdots & 0 \end{bmatrix},$$

where the upper $\lambda_1 - 1$ rows consists of $\frac{1}{\lambda}$'s and there are exactly m $\frac{1}{\lambda}$'s at the beginning of the last row. If $[\lambda] = 2n+1$ and $1 - (\lambda - [\lambda]) = \frac{m}{\lambda}$ (Case 2), the matrix $\mathbb{M}$ is similar, except that the last row starts with m zeros and ends with $(\lambda_1 - m)$ $\frac{1}{\lambda}$'s.

Let $f = (f_1, f_2, \ldots, f_{\lambda_1})$ represent a piecewise constant function on the partition $(0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{[\lambda]}{\lambda}, 1)$. We consider the equation $f\mathbb{M} = f$ with the normalizing condition which makes f a density of a probability measure:

$$\sum_{i=1}^{[\lambda]} f_i \frac{1}{\lambda} + f_{\lambda_1}\left(1 - \frac{[\lambda]}{\lambda}\right) = 1. \tag{4}$$

Case 1: $[\lambda] = 2n$ and $\lambda - [\lambda] = \frac{m}{\lambda}$. Then, $f\mathbb{M} = f$ reduces to

$$f_1 = f_2 = \ldots = f_m$$

$$f_{m+1} = f_{m+2} = \ldots = f_{\lambda_1}$$

$$f_1 = \frac{m}{\lambda} f_1 + \frac{\lambda_1 - m}{\lambda} f_{\lambda_1}$$

$$f_{\lambda_1} = \frac{m}{\lambda} f_1 + \frac{\lambda_1 - 1 - m}{\lambda} f_{\lambda_1}.$$

Since $\lambda = [\lambda] = \frac{m}{\lambda}$, the last two equations coincide and we obtain:

$$f_1 = \frac{\lambda_1 - m}{\lambda - m} f_{\lambda_1}.$$

Substituting into (4), we get

$$\frac{m}{\lambda} \frac{\lambda_1 - m}{\lambda - m} f_{\lambda_1} + \frac{[\lambda] - m}{\lambda} f_{\lambda_1} + (1 - \frac{[\lambda]}{\lambda}) f_{\lambda_1} = 1,$$

which gives

$$f_{\lambda_1} = \frac{\lambda(\lambda - m)}{m(\lambda_1 - m) + (\lambda - m)^2} \text{ and } f_1 = \frac{\lambda(\lambda_1 - m)}{m(\lambda_1 - m) + (\lambda - m)^2}. \tag{5}$$

Case 2: $[\lambda] = 2n + 1$ and $1 - (\lambda - [\lambda]) = \frac{m}{\lambda}$. Considerations analogous to that of Case 1 lead to:

$$f_{\lambda_1} = \frac{\lambda(\lambda - m)}{m([\lambda] - m) + (\lambda - m)^2} \text{ and } f_1 = \frac{\lambda([\lambda] - m)}{m([\lambda] - m) + (\lambda - m)^2}. \tag{6}$$

We have proved the following theorem.

Theorem 1: *If $\lambda > 1$ is such that Λ_λ is a Markov map with respect to $\mathcal{P}$, then the unique invariant density of Λ_λ is: for $[\lambda] = 2n$*

$$f_\lambda(x) = \begin{cases} \dfrac{\lambda(\lambda_1 - m)}{m(\lambda_1 - m) + (\lambda - m)^2} & \text{for } 0 \le x < \frac{m}{\lambda}; \\[2ex] \dfrac{\lambda(\lambda - m)}{m(\lambda_1 - m) + (\lambda - m)^2} & \text{for } \frac{m}{\lambda} \le x \le 1; \end{cases}$$

and for $[\lambda] = 2n + 1$:

$$f_\lambda(x) = \begin{cases} \dfrac{\lambda([\lambda] - m)}{m([\lambda] - m) + (\lambda - m)^2} & \text{for } 0 \le x < \frac{m}{\lambda}; \\[2ex] \dfrac{\lambda(\lambda - m)}{m([\lambda] - m) + (\lambda - m)^2} & \text{for } \frac{m}{\lambda} \le x \le 1. \end{cases}$$

Now, we can use the differentiable conjugacy h of Proposition 1 to find invariant densities for Markov Chebyshev maps. By Proposition 2 of [2], the T_λ-invariant density is

$$F_\lambda(x) = f_\lambda(h^{-1}(x)) \,|\, (h^{-1})' \,| = f_\lambda(\tfrac{1}{\pi} \arccos x) \frac{1}{\pi\sqrt{1 - x^2}}.$$

Thus the following theorem holds.

Theorem 2: *If $\lambda > 1$ is such that T_λ is a Markov map with respect to $\mathcal{P}$, then the unique invariant density of T_λ is*

$$F_\lambda(x) = \begin{cases} \dfrac{f_1}{\pi\sqrt{1-x^2}}, & \textit{for } 0 \le \frac{1}{\pi}\arccos x < \frac{m}{\lambda}; \\ \dfrac{f_{\lambda_1}}{\pi\sqrt{1-x^2}}, & \textit{for } \frac{m}{\lambda} \le \frac{1}{\pi}\arccos x \le 1; \end{cases}$$

where m satisfies $\lambda - [\lambda] = \frac{m}{\lambda}$ for $[\lambda] = 2n$ or $1 - (\lambda - [\lambda]) = \frac{m}{\lambda}$ for $[\lambda] = 2n+1$ and constants f_1, f_{λ_1} are given by formulas (5) or (6), respectively.

Remark 3: For a non-integer λ, we always have $|T'_\lambda(-1+t)| = O((\sqrt{1-t})^{-1})$, as $t \to 0^+$, which explains the lack of singularity of F_λ at $\frac{m}{\lambda}$.

4. Examples

Example 1: Using Maple V, release 5, we have calculated some values of λ and corresponding values of f_1 and f_{λ_1}. They are presented in the tables below.

Case 1:

(n,m)	λ	f_1	f_{λ_1}
$(1,1)$	2.414213562	1.207106781	.8535533903
$(5,7)$	10.65685425	1.030330086	.9419417381
$(100,107)$	200.5335765	1.002319740	.9973462776

Case 2:

(n,m)	λ	f_1	f_{λ_1}
$(1,1)$	3.732050808	.7886751346	1.077350270
$(5,7)$	11.38516481	.9642383460	1.057086015
$(100,107)$	201.4689006	.9976664394	1.002643103

For some $\lambda > 1$, the map Λ_λ may not be Markov with respect to the partition $\mathcal{P}$ but will be Markov with respect to some finer partition. In all such cases, it is possible to find an invariant density of Λ_λ and then use it to find the invariant density of the Chebyshev map T_λ. Below, we present two simple examples of such situations.

Example 2: Let us look for λ such that $[\lambda] = 1$ and $\Lambda_\lambda(1) = \frac{1}{\lambda^2}$, see Figure 3a). Then $\lambda(1 - \frac{1}{\lambda}) = 1 - \frac{1}{\lambda^2}$, or $\lambda^3 - 2\lambda^2 + 1 = 0$. The non-integer positive solution is $\lambda = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$. It is easy to see that Λ_λ is Markov with respect to the partition $\mathcal{Q} = (0, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1)$ and the corresponding Frobenius-Perron matrix is

$$\mathbb{M} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda} \end{bmatrix}.$$

Its left invariant vector is $f = (f_1, f_2, f_3) = (0, 1, \lambda) f_2$ and after the normalization the Λ_λ-invariant density is $f = (0, \frac{\lambda+1}{2\lambda-1}, \frac{2\lambda+1}{2\lambda-2}) \approx (0, 1.171,\ 1.894)$.

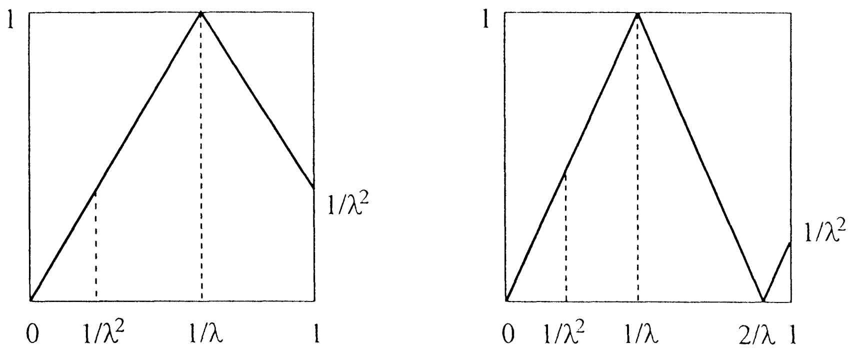

Figure 3

Example 3: Let us look for λ such that $[\lambda] = 2$ and $\Lambda_\lambda(1) = \frac{1}{\lambda^2}$, see Figure 3b). Then, $\lambda(1 - 2\frac{1}{\lambda}) = \frac{1}{\lambda^2}$, or $\lambda^3 - 2\lambda^2 - 1 = 0$. The only real solution is $\lambda \approx 2.206$. It is easy to see that Λ_λ is Markov with respect to the partition $\mathbb{Q} = (0, \frac{1}{\lambda^2}, \frac{1}{\lambda}, \frac{2}{\lambda}, 1)$. The corresponding Frobenius-Perron matrix is

$$\mathbb{M} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 & 0 & 0 \end{bmatrix}.$$

Its left invariant vector is $f = (f_1, f_2, f_3, f_4) = (\lambda^2 - \lambda - 1, \lambda - 1, 1, 1) f_3$ and after normalization, the Λ_λ-invariant density is $f = \frac{1}{3\lambda - 4}(\lambda^2 - \lambda + 1, \lambda^2 - \lambda, \lambda, \lambda) \approx$ (1.398, 1.016, .843, .843).

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