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A NOTE ON CONTROLLABILITY OF SEMILINEAR INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

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Sufficient conditions for controllability of semilinear integrodifferential systems in a Banach space are established. The results are obtained by using the Schaefer fixed-point theorem.

Key words: Controllability, Semilinear Integrodifferential System, Fixed-Point Theorem.

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1. Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators. Naito [8, 9] studied the controllability of semilinear systems whereas Yamamoto and Park [13] considered the same problem for parabolic equation with uniformly bounded nonlinear term. Lasiecka and Triggiani [5] studied exact controllability of abstract semilinear equations. Chukwu and Lenhart [3] discussed the controllability of nonlinear systems in abstract spaces and Naito [10] established the controllability for nonlinear Volterra integrodifferential systems. Do [4] and Zhou [14] investigated the approximate controllability for a class of semilinear abstract equations. Recently Balachandran et al. [1, 2] established sufficient conditions for the controllability of nonlinear integrodifferential systems in Banach spaces by using Schauder's fixed-point theorem. The purpose of this paper is to study the controllability of semilinear integrodifferential systems in Banach spaces by suitably applying the Schaefer fixed-point theorem.

2. Preliminaries

Consider the semilinear integrodifferential system

$$\dot{x}(t) = A[x(t) + \int_0^t F(t-s)x(s)ds] + (Bu)(t) + f(t, x(t), \int_0^t g(t,s,x(s))ds), \ t \in J = [0,b],$$

$$x(0) = x_0, \tag{1}$$

where the state $x(\cdot)$ takes values in a Banach space X and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Here A is the generator of a strongly continuous semigroup, B is a bounded linear operator from U into X, and $g: J \times J \times X \to X$ and $f: J \times X \times X \to X$ are given functions. $F(t): Y \to Y$ and for $x(\cdot)$ continuous in Y, $AF(\cdot)x(\cdot) \in L^1([0,b],X)$. $F(t) \in B(X), t \in J$ and for some $x \in X$, $F'(t)x$ is continuous in $t \in [0,b]$, where $B(X)$ is the space of all bounded linear operators on X, and Y is the Banach space formed from $D(A)$, the domain of A endowed with the graph norm.

We need the following fixed point theorem due to Schaefer [12].

Schaefer Theorem: *Let S be a convex subset of a normed linear space E and $0 \in S$. Let $F: S \to S$ be a completely continuous operator and let*

$$\zeta(F) = \{x \in S; x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

The system (1) has a mild solution of the following form [11]:

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\left[(Bu)(s) + f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)\right]ds, \tag{2}$$

where $R(t)$ is a resolvent operator [6].

In order to study the controllability problem of (1), we consider the following system as in [7]:

$$\dot{x}(t) = \lambda A\left[x(t) + \int_0^t F(t-s)x(s)ds\right] + \lambda(Bu)(t)$$

$$+ \lambda f\left(t, x(t), \int_0^t g(t, sx(s))ds\right), \quad \lambda \in (0,1), \quad t \in J, \tag{3}$$

$$x(0) = x_0.$$

Then for system (3), there exists a mild solution of the form

$$x(t) = \lambda R(t)x_0 + \lambda \int_0^t R(t-s)\left[(Bu)(s) + f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)\right]ds.$$

Definition: System (1) is said to be *controllable on the interval J* if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$.

We assume the following hypotheses:

(i) The resolvent operator $R(t)$ is compact such that

$$\max_{t > 0} \| R(t) \| \leq M_1,$$

where $M_1 > 0$.

(*ii*) The linear operator W from $L^2(J,U)$ into X, defined by

$$Wu = \int_0^b R(b-s)Bu(s)ds$$

has an invertible operator W^{-1} defined on $L^2(J,U)/kerW$ and there exist positive constants M_2, M_3 such that $\| B \| \leq M_2$ and $\| W^{-1} \| \leq M_3$.

(*iii*) For each $t \in J$, the function $g(t,s,\cdot): X \rightarrow X$ is continuous and for each $x \in X$, the function $g(\cdot,\cdot,x): J \times J \rightarrow X$ is strongly measurable.

(*iv*) For each $t \in J$, the function $f(t,\cdot,\cdot): X \times X \rightarrow X$ is continuous and for each $x, y \in X$, the function $f(\cdot,x,y): J \rightarrow X$ is strongly measurable.

(*v*) For every positive integer k, there exists $h_k \in L^1(0,b)$ such that for a.a. $t \in J$,

$$\sup_{\| x \| \leq k} \left\| f(t,x(t), \int_0^t g(t,s,x(s))ds) \right\| \leq h_k(t).$$

(*vi*) There exists a continuous function $m: J \times J \rightarrow [0,\infty)$ such that

$$\| g(t,s,x) \| \leq m(t,s)\Omega(\| x \|), \quad t,s \in J, \quad x \in X,$$

where $\Omega: [0,\infty) \rightarrow (0,\infty)$ is a continuous nondecreasing function.

(*vii*) There exists a continuous function $p: J \rightarrow [0,\infty)$ such that

$$\| f(t,x,y) \| \leq p(t)\Omega_0(\| x \| + \| y \|), \; t \in J \quad x,y \in X,$$

where $\Omega_0: [0,\infty) \rightarrow (0,\infty)$ is a continuous nondecreasing function.

(*viii*)

$$\int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)},$$

where $c = M_1(\| x_0 \|) + M_1Nb$, $\widehat{m}(t) = \max\{M_1p(t), Lm(t,t)\}$,

$$N = M_2M_3\left(\| x_1 \| + M_1 \| x_0 \| + M_1 \int_0^b p(s)\Omega_0(\| x \| + L\int_0^s m(s,\tau)\Omega(\| x \|)d\tau)ds \right).$$

3. Main Result

Theorem: *If the hypotheses* (*i*)-(*viii*) *are satisfied, then system* (1) *is controllable on* J.

Proof: Using hypothesis (*ii*) for an arbitrary function $x(\cdot)$, define the control

$$u(t) = W^{-1}\left[x_1 - R(b)x_0 - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds\right](t).$$

We shall now show that when using this control the operator defined by

$$(Fx)(t) = R(t)x_0 + \int_0^t R(t-s)[(Bu)(s)$$

$$+ f(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau)]ds, t \in J,$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(b) = x_1$, which means that the control u steers the semilinear integrodifferential system from the initial state x_0 to x_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

First, we obtain a priori bounds for the following equation:

$$x(t) = \lambda R(t)x_0 + \lambda \int_0^t R(t-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s,x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds](\eta)d\eta$$

$$+ \lambda \int_0^t R(t-s)f(s,x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds.$$

We have, from the assumptions,

$$\| x(t) \| \leq M_1 \| x_0 \| + \int_0^t \| R(t-\eta) \| M_2 M_3[\| x_1 \| + M_1 \| x_0 \|$$

$$+ M_1 \int_0^b p(s)\Omega_0(\| x(s) \| + \int_0^s m(s,\tau)\Omega(\| x(\tau) \|)d\tau)ds]d\eta$$

$$+ M_1 \int_0^t p(s)\Omega_0(\| x(s) \| + \int_0^s m(s,\tau)\Omega(\| x(\tau) \|)d\tau)ds$$

$$\leq M_1 \| x_0 \| + \int_0^t M_1 N ds$$

$$+ M_1 \int_0^t p(s)\Omega_0(\| x(s) \| + \int_0^s m(s,\tau)\Omega(\| x(\tau) \|)d\tau)ds$$

$$\le M_1 \| x_0 \| + M_1 Nb + M_1 \int_0^t p(s)\Omega_0(\| x(s) \|$$

$$+ \int_0^s m(s,\tau)\Omega(\| x(\tau) \|)d\tau)ds.$$

Denoting by $v(t)$ the right-hand side of the above inequality, we have $v(0) = M_1 \| x_0 \| + M_1 Nb$, $\| x(t) \| \le v(t)$ and

$$v'(t) = M_1 p(t)\Omega_0(\| x(t) \| + \int_0^t m(t,\tau)\Omega(\| x(\tau) \|)d\tau)$$

$$\le M_1 p(t)\Omega_0(v(t) + \int_0^t m(t,\tau)\Omega(v(\tau))d\tau).$$

Let

$$w(t) = v(t) + \int_0^t m(t,\tau)\Omega(v(\tau))d\tau.$$

Then $w(0) = v(0) = c$, $v(t) \le w(t)$, and

$$w'(t) = v'(t) + m(t,t)\Omega(v(t)) \le M_1 p(t)\Omega_0(w(t)) + m(t,t)\Omega(w(t))$$

$$\le \widehat{m}(t)[\Omega_0(w(t)) + \Omega(w(t))].$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \le \int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J,$$

which in turn implies that there is a constant K such that $w(t) \le K$, $t \in J$, and hence $\| x(t) \| \le K$, $t \in J$, where K depends only on b and on the functions m, Ω_0, and Ω.

Second, we must prove that the operator $F: C = C(J,X) \to C$ defined by

$$(Fx)(t) = R(t)x_0 + \int_0^t R(t-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s,x(s), \int_0^s g(s,\tau,x(\tau))d\tau)ds](\eta)d\eta$$

$$+\int_0^t R(t-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds$$

is a completely continuous operator.

Let $B_k = \{x \in C\colon \| x \| \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \le b$,

$$\| (Fx)(t_1) - (Fx)(t_2) \| \le \| R(t_1) - R(t_2) \| \; \| x_0 \|$$

$$+ \| \int_0^{t_1} [R(t_1-\eta) - R(t_2-\eta)]BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds](\eta)d\eta \|$$

$$+ \| \int_{t_1}^{t_2} R(t_2-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds](\eta)d\eta \|$$

$$+ \| \int_0^{t_1} [R(t_1-s) - R(t_2-s)]f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds \|$$

$$+ \| \int_{t_1}^{t_2} R(t_2-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds \|$$

$$\le \| R(t_1) - R(t_2) \| \; \| x_0 \|$$

$$+ \int_0^{t_1} \| R(t_1-\eta) - R(t_2-\eta) \| M_2M_3[\| x_1 \| + M_1e^{\omega b} \| x_0 \|$$

$$+ M_1 \int_0^b e^{\omega(b-s)}h_k(s)ds]d\eta$$

$$+ \int_{t_1}^{t_2} \| R(t_2-\eta) \| M_2M_3[\| x_1 \| + M_1e^{\omega b} \| x_0 \|$$

$$+ M_1 \int_0^b e^{\omega(b-s)}h_k(s)ds]d\eta$$

$$+ \int_0^{t_1} \| [R(t_1 - s) - R(t_2 - s)] \| h_k(s) ds$$

$$+ \int_{t_1}^{t_2} \| R(t_2 - s) \| h_k(s) ds.$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$, since the compactness of $R(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus, F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next, we show that $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem, to show that F maps B_k into a precompact set in X.

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(F_\epsilon x)(t) = R(t)x_0 + \int_0^{t-\epsilon} R(t-\eta) BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) ds](\eta) d\eta$$

$$+ \int_0^{t-\epsilon} R(t-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) ds$$

$$= R(t)x_0 + R(\epsilon) \int_0^{t-\epsilon} R^{-1}(\epsilon) R(t-\eta) BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) ds](\eta) d\eta$$

$$+ R(\epsilon) \int_0^{t-\epsilon} R^{-1}(\epsilon) R(t-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) ds.$$

Since $R(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$\| (Fx)(t) - (F_\epsilon x)(t) \| \leq \int_{t-\epsilon}^t \| R(t-\eta) BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)ds](\eta) \,\|\, d\eta$$

$$+ \int_{t-\epsilon}^t \| R(t-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau) \| ds$$

$$\leq \int_{t-\epsilon}^t \| R(t-\eta) \| M_2M_3[\, \| x_1 \| + M_1e^{\omega b} \| x_0 \|$$

$$+ M_1 \int_0^b e^{\omega(b-s)}h_k(s)ds]d\eta$$

$$+ \int_{t-\epsilon}^t \| R(t-s) \| h_k(s)ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fx)(t): x \in B_k\}$. Hence the set $\{(Fx)(t): x \in B_k\}$ is precompact in X.

It remains to show that $F: C \rightarrow C$ is continuous. Let $\{x_n\}_0^\infty \subseteq C$ with $x_n \rightarrow x$ in C. Then there is an integer r such that $\| x_n(t) \| \leq r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$.

By (iv),

$$f(t,x_n(t),\int_0^t g(t,s,x_n(s))ds) \rightarrow f(t,x(t),\int_0^t g(t,s,x(s))ds)$$

for each $t \in J$ and since

$$\| f(t,x_n(t),\int_0^t g(t,s,x_n(s))ds) - f(t,x(t),\int_0^t g(t,s,x(s))ds) \| \leq 2h_r(t),$$

we have by dominated convergence theorem,

$$\| Fx_n - Fx \| = \sup_{t \in J} \| \int_0^t R(t-\eta)BW^{-1}[\int_0^b R(b-s)$$

$$[f(s,x_n(s),\int_0^s g(s,\tau,x_n(\tau))d\tau) - f(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau)]ds](\eta)d\eta$$

$$+ \int_0^t R(t-s)[f(s,x_n(s),\int_0^s g(s,\tau,x_n(\tau))d\tau)$$

$$- f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)]ds \,\|$$

$$\leq \int_0^b \| R(t-\eta) \| M_2 M_3 [M_1 \int_0^b e^{\omega(b-s)} \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau)$$

$$- f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| ds] d\eta$$

$$+ \int_0^b \| R(t-s) \| \; \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau)$$

$$- f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| ds \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\zeta(F) = \{x \in C : x = \lambda F x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator F has a fixed point in C. This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus system (1) is controllable on J.

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