Journal of Applied Mathematics and Stochastic Analysis, **12**:3 (1999), 293-300.

EXISTENCE, UNIQUENESS AND CONTINUABILITY OF SOLUTIONS OF IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

D.D. BAINOV
Medical University of Sofia
P.O. Box 45
1504 Sofia, Bulgaria

I.M. STAMOVA
Technical University of Sliven
Sliven, Bulgaria

(Received May, 1996; Revised October, 1998)

We consider an initial value problem for impulsive differential-difference equations, and obtain sufficient conditions for the existence, uniqueness, and continuability of solutions of such problem.

Key words: Existence, Uniqueness, Continuability, Impulsive Differential-Difference Equations.

AMS subject classifications: 34A37.

1. Introduction

The necessity to study impulsive differential equations is due to the fact that these equations are a useful mathematical machinery in modelling of many processes and phenomena studied in theory of optimal control, biology, mechanics, biotechnology, medicine, electronics, radio engineering, etc.

Processes, which can be adequately modelled by impulsive differential-difference equations, are characterized by a per saltum changing of their state as well as by the fact that the processes under consideration depend on their pre-history at each moment of time.

Impulsive differential-difference equations are natural generalization of impulsive differential equations. Their theory is analytically more attractive than the theory of impulsive ordinary differential equations.

At the present time the theory of such equations undergoes rapid development [1-6].

In the present paper we consider the problems of existence, uniqueness, and continuability of solutions of nonlinear system of impulsive differential-difference equations. The impulsive moments occur when the integral curve of the system meets given hypersurfaces situated in the extended phase space.

2. Statement of the Problem. Preliminary Notes

Let R^n be the n-dimensional Euclidean space with elements $x = (x_1, \ldots, x_n)^T$ and norm $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$; $h = \text{const} > 0$; $\Omega \subseteq R^n$, $\Omega \neq \emptyset$; $(t_0, x_0) \in R \times \Omega$.

We consider the following initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), t > t_0, t \neq \tau_k(x(t)), \\ x(t) = \varphi_0(t), t \in [t_0 - h, t_0], \\ x(t_0 + 0) = x_0, \\ \Delta x(t) = I_k(x(t)), t = \tau_k(x(t)), t > t_0, k = 1, 2, \ldots, \end{cases} \tag{1}$$

where $f: (t_0, \infty) \times \Omega \times \Omega \to R^n$; $\tau_k: \Omega \to (t_0, \infty)$; $I_k: \Omega \to R^n$, $k = 1, 2, \ldots$, $\Delta x(t) = x(t+0) - x(t-0)$; $\varphi_0: [t_0 - h, t_0] \to \Omega$.

Let $\tau_0(x) \equiv t_0$ for $x \in \Omega$.

Introduce the following conditions:

(H_1) $\tau_k \in C[\Omega, (t_0, \infty)], k = 1, 2, \ldots$.

(H_2) $t_0 < \tau_1(x) < \tau_2(x) < \ldots$, $x \in \Omega$.

(H_3) $\tau_k(x) \to \infty$ as $k \to \infty$, uniformly on $x \in \Omega$.

Under the assumption that conditions (H_1), (H_2) and (H_3) hold, we define the following notations:

$$G_k = \{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) < t < \tau_k(x)\},\ k = 1, 2, \ldots$$

$$F_k = \{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) \leq t < \tau_k(x)\},\ k = 1, 2, \ldots$$

$$\sigma_k = \{(t, x) \in [t_0, \infty) \times \Omega : t = \tau_k(x)\},\ \text{i.e.,}\ \sigma_k,\ k = 1, 2, \ldots$$

are hypersurfaces with equations $t = \tau_k(x)$.

We denote by $PC(t_0)$ the space of all functions $\varphi: [t_0 - h, t_0] \to \Omega$ having points of discontinuity at $\theta_1, \theta_2, \ldots, \theta_s \in (t_0 - h, t_0)$ of the first kind and are left continuous at these points.

Let $\varphi_0 \in PC(t_0)$. We shall denote by $x(t) = x(t; t_0, x_0, \varphi_0)$ the solution of problem (1), and by $J^+(t_0, x_0, \varphi_0)$-the maximal interval of type $[t_0, \beta)$ where the solution $x(t; t_0, x_0, \varphi_0)$ is defined.

We will give a precise description of solution $x(t)$ to problem (1):

1. If $t_0 - h \leq t \leq t_0$, then solution $x(t)$ of problem (1) coincides with the function $\varphi_0(t)$.
2. Let $t_1, t_2, \ldots (t_0 < t_1 < t_2 < \ldots)$ be the moments of time at which the integral curve $(t, x(t))$ of problem (1) meets the hypersurfaces $\{\sigma_k\}_{k=1}^{\infty}$, i.e., each of the points $t_1, t_2, \ldots$, is a solution of one of the equations $t = \tau_k(x(t))$, $k = 1, 2, \ldots$.

Let $t_l^h = t_l + h$, $l = 0, 1, 2, \ldots$ and $\theta_r^h = \theta_r + h$, $r = 1, 2, \ldots, s$.

It is easy see that $\{t_l^h\}_{l=0}^{\infty} \cap \{\theta_r^h\}_{r=1}^{s} = \emptyset$. We shall note that in general it is possible that the following relation is valid:

$$\{t_k\}_{k=1}^{\infty} \cap (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s}) \neq \emptyset.$$

We form the sequence $\{\tau_i\}_{i=0}^{\infty}$ observing the following rules:

a) $\{\tau_i\}_{i=0}^{\infty} = \{t_k\}_{k=0}^{\infty} \cup \{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s}$.

b) $\tau_0 \equiv t_0$.

c) The sequence $\{\tau_i\}_{i=0}^{\infty}$ is monotone increasing.

d) If $\tau_{i_j} \in \{t_k\}_{k=1}^{\infty} \cap (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s})$ for some $i_j = 1,2,\ldots$, then the moment τ_{i_j} take part in the sequence $\{\tau_i\}_{i=1}^{\infty}$ exactly once.

2.1: For $\tau_0 < t \leq \tau_1$, the solution $x(t)$ of problem (1) coincides with the solution of the problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), \\ x(t) = \varphi_0(t), t \in [t_0 - h, t_0], \\ x(t_0 + 0) = x_0. \end{cases} \tag{2}$$

2.2: For $\tau_i < t \leq \tau_{i+1}$, $i = 1,2,\ldots$, one of the following three cases may occur:

a) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \backslash (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s})$, $\tau_i = t_k$ and i_k is the number of the hypersurface met by the integral curve $(t, x(t))$ at the moment t_k, then the solution $x(t)$ coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h)), \tag{3}$$

$$y(t_k) = x(t_k) + I_{i_k}(x(t_k)). \tag{4}$$

b) If $\tau_i \in (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s}) \backslash \{t_k\}_{k=1}^{\infty}$, then the solution $x(t)$ of problem (1) coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h+0)), \tag{5}$$

$$y(\tau_i) = x(\tau_i). \tag{6}$$

c) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \cap (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s})$ and $\tau_i = t_k$, then the solution $x(t)$ coincides with the solution of the problem (5), (4).

3. If the point $x(t_k) + I_{i_k}(x(t_k)) \notin \Omega$, then the solution $x(t)$ of problem (1) is not defined for $t > t_k$.

4. The function $x(t)$ is piecewise continuous on $J^+(t_0, x_0, \varphi_0)$, left continuous at points $t_k \in J^+(t_0, x_0, \varphi_0)$, and $x(t_k + 0) = x(t_k) + I_{i_k}(x(t_k))$, $k = 1, 2, \ldots$.

Introduce the following conditions:

(H_4) The function f is continuous on $(t_0, \infty) \times \Omega \times \Omega$.

(H_5) The function f is locally Lipschitz continuous with respect to its second and third arguments on $(t_0, \infty) \times \Omega \times \Omega$.

(H_6) There exists a constant $M > 0$ such that

$$|f(t,x,y)| \leq M < \infty \text{ for } (t,x,y) \in (t_0, \infty) \times \Omega \times \Omega.$$

(H_7) $(I + I_k): \Omega \to \Omega$, $k = 1,2,\ldots$, where I is the identity on Ω.

(H_8) $(t, x + I_k(x)) \in F_{k+1}$, for $(t,x) \in \sigma_k$, $k = 1,2,\ldots$.

(H_9) The functions τ_k's are Lipschitz continuous with respect to $x \in \Omega$ with Lipschitz constants L_k, $0 < L_k \leq \frac{1}{M}$, $k = 1, 2, \ldots$.

(H_{10}) For each $(t_0, x_0, \varphi_0) \in R \times \Omega \times PC(t_0)$, the solution $\phi(t; t_0, x_0, \varphi_0)$ of initial value problem (2) without impulses does not leave the domain Ω for $t \in J$, where

$$J = \begin{cases} (t_0, \infty) \text{ if the } \tau_k\text{'s are finitely many,} \\ \cup (t_{k-1}, t_k] \text{ if the } \tau_k\text{'s are infinitely many.} \end{cases}$$

We shall note that for the impulsive systems with variable impulsive perturbations a phenomenon the so-called "beating" may occur, i.e., a phenomenon for which the integral curve $(t, x(t))$ meets several of infinitely many times one and the same hypersurface.

In this paper, we exclude the case of "beating".

Efficient sufficient conditions which guarantee the absence of the "beating" for impulsive functional differential equations were found by D. D. Bainov and A. B. Dishliev in [1] and [2].

In the sequel, we shall use the following lemma:

Lemma 1: [1, 2] *Let conditions* (H_1)*-*(H_4)*,* (H_6)*-*(H_{10}) *hold and* $(t_0, x_0) \in F_k$*,* $k = 1, 2, \ldots$.

Then the integral curve $(t, x(t))$ *of the solution of problem* (1) *meets successively each one of the hypersurfaces* σ_k*,* $\sigma_{k+1}, \ldots$ *exactly once.*

The already formulated lemma guarantees the absence of "beating" of solution to problem (1) as well as the validity of the following properties:

1. For each $(t_0, x_0, \varphi_0) \in \sigma_k \times PC(t_0)$, $k = 1, 2, \ldots$ there exists a constant $\beta > t_0$ such that $(t, \phi(t; t_0, x_0, \varphi_0)) \in G_{k+1}$, $t \in (t_0, \beta)$, where $\phi(t; t_0, x_0, \varphi_0)$ is a solution of problem (2).
2. If $(t_0, x_0, \varphi_0) \in G_{k+1} \times PC(t_0)$, $k = 1, 2, \ldots$ and $x(t; t_0, x_0, \varphi_0)$ is a solution of problem (1), then

$$\{(t, x(t; t_0, x_0, \varphi_0)): t \geq t_0\} \cap \{\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k\} = \phi.$$

3. Main Results

Theorem 1: *Let conditions* (H_1)*-*(H_4)*,* (H_6)*-*(H_{10}) *hold.*

Then:

1. *For each point* $(t_0, x_0) \in R \times \Omega$ *and for each function* $\varphi_0 \in PC(t_0)$*, there exists a solution* $x(t) = x(t; t_0, x_0, \varphi_0)$ *to the initial value problem* (1) *defined on* $J^+(t_0, x_0, \varphi_0)$.
2. $J^+(t_0, x_0, \varphi_0) = [t_0, \infty)$.
3. *If, moreover, condition* (H_5) *is met then the solution* $x(t; t_0, x_0, \varphi_0)$ *is unique.*

Proof of Assertion 1: The validity of (H_4), (H_{10}) as well as the existence theorem applied to problem (2) (cf. Hale [7]) imply that for each point $(t_0, x_0) \in R \times \Omega$ and for each function $\varphi_0 \in PC(t_0)$ there exists a solution $\Phi_1(t)$ of problem (2) for $t \geq t_0$. Moreover, $\Phi_1(t) = \varphi_0(t)$ as $t \in [t_0 - h, t_0]$, $\Phi_1(t_0) = x_0$ and this solution does not leave the domain Ω. Let t_1 be the first moment at which the integral curve $(t, \Phi_1(t))$ reaches some of the hypersurfaces $\{\sigma_k\}_{k=1}^{\infty}$. Conditions of Lemma 1 are fulfilled and therefore $\tau_1 = t_1 > t_0$. Moreover, the order number of the first hypersurfaces reached by the integral curve is $i_1 = 1$. Setting $x(t; t_0, x_0, \varphi_0) = \Phi_1(t)$ as

$t \in [t_0, t_1]$ we have $\Phi_1(t_1+0) = I_1(\Phi_1(t_1)) + \Phi_1(t_1) = \Phi_1^+$.

Now the above mentioned existence theorem applied to the problem (2) in the interval (t_1, τ_2) ensures that there exists a solution $\Phi_2(t)$ such that $\Phi_2(t) = \Phi_1(t)$ for $t_1 - h \le t \le t_1$ and $\Phi_2(t_1) = \Phi_1^+$. The solution $x(t; t_0, x_0, \varphi_0)$ of problem (1) can be extended to the moment $t = \tau_2$ by setting $x(t; t_0, x_0, \varphi_0) = \Phi_2(t)$ for $t_1 < t \le \tau_2$.

In the same way, let us denote by $\Phi_i(t)$ the solutions of problem (2) in the intervals $(\tau_{i-1}, \tau_i]$, $i = 3, 4, \ldots$, respectively. On each of the intervals $(\tau_i, \tau_{i+1}]$, we have only one of the next two cases:

a. $\tau_i = t_k$. Then, for $t = t_k$, we have

$$\Phi_i(t_k+0) = \Phi_i(t_k) + I_k(\Phi_i(t_k)) = \Phi_i^+.$$

It follows from the existence theorem for problem (2) on the interval $(t_k, \tau_{i+1}]$ that there exists a solution $\Phi_{i+1}(t)$ such that $\Phi_{i+1}(t) = \Phi_i(t)$ for $\tau_i - h \le t \le t_k$ and $\Phi_{i+1}(t_k) = \Phi_i^+$. Thus the solution $x(t; t_0, x_0, \varphi_0)$ of problem (1) can be extended to the moment τ_{i+1}, $i = 2, 3, \ldots$, when setting $x(t; t_0, x_0, \varphi_0) = \Phi_{i+1}(t)$, $t \in (t_k, \tau_{i+1}]$.

b. $\tau_i \in (\{t_l^h\}_{l=0}^{\infty} \cup \{\theta_r^h\}_{r=1}^{s}) \backslash \{t_k\}_{k=2}^{\infty}$. Then, by virtue of theorem on continuability of solutions of problem (2) (cf. Hale, [7]), the solution $x(t; t_0, x_0, \varphi_0)$ of the initial value problem (1) can be defined on interval $(\tau_i, \tau_{i+1}]$ by setting $x(t; t_0, x_0, \varphi_0) = \Phi_i(t)$, $t \in (\tau_i, \tau_{i+1}]$.

Finally, by means of condition (H_3), solution $x(t; t_0, x_0, \varphi_0)$ of problem (1) is defined for $t \in J^+(t_0, x_0, \varphi_0)$.

Proof of Assertion 2: Conditions of Lemma 1 are satisfied and $i_1 = 1$. Therefore, $i_2 = 2$, $i_3 = 3$, ..., where i_k is the order number of the hypersurface that the integral curve $(t, x(t; t_0, x_0, \varphi_0))$ reaches at moment t_k, $k = 1, 2, \ldots$.

Thus we conclude that $i_k \to \infty$ as $k \to \infty$. Now, condition (H_3) leads to

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} \tau_{i_k}(x_k) = \lim_{k\to\infty} \tau_k(x_k) = \infty, \tag{7}$$

where $x_k = x(t_k; t_0, x_0, \varphi_0)$.

Since the solution $x(t) = x(t; t_0, x_0, \varphi_0)$ is defined on each of the intervals $(t_k, t_{k+1}]$, $k = 1, 2, \ldots$, then from (7) and Assertion 1, we conclude that it can be continued for all $t \ge t_0$, i.e., $J^+(t_0, x_0, \varphi_0) = [t_0, \infty)$.

Proof of Assertion 3: Validity of condition (H_5) ensures that the above defined solutions $\Phi_1(t)$, $\Phi_2(t)$,... are unique and therefore the solution $x(t; t_0, x_0, \varphi_0)$ of problem (1) is unique.

Let us consider now an initial value problem for the system of differential-difference equations with impulse effects at fixed moments of time:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)),\ t > t_0, t \ne \tau_k, \\ x(t) = \varphi_0(t), t \in [t_0 - h, t_0], \\ x(t_0+0) = x_0, \\ \Delta x(\tau_k) = x(\tau_k+0) - x(\tau_k-0) = I_k(x(\tau_k)), \tau_k > t_0, k = 1, 2, \ldots, \end{cases} \tag{8}$$

where $t_0 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} < \ldots$ and $\lim_{k\to\infty} \tau_k = \infty$. In the present case, $\tau_k(x) \equiv \tau_k$, $k = 1, 2, \ldots$ and σ_k are hyperplanes in R^{n+1}.

Theorem 2: *Let conditions* (H_4), (H_6)-(H_8) *are met.*

Then for each point $(t_0, x_0) \in R \times \Omega$ *and for each function* $\varphi_0 \in PC(t_0)$, *there exists a solution* $x(t; t_0, x_0, \varphi_0)$ *of problem* (8) *that is defined on interval* $[t_0 - h, \omega)$ *and it can be continued to the right from* ω.

If, in addition, condition (H_5) *is met, then the solution of* (8) *is unique.*

The proof of Theorem 2 is a simple consequence of Theorem 1.

Now we consider an initial value problem for the linear system of differential-difference equations with impulse effects at fixed moments:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)x(t-h), t > t_0, t \neq \tau_k, \\ x(t) = \varphi_0(t), t \in [t_0 - h, t_0], \\ x(t_0 + 0) = x_0, \\ \Delta x(\tau_k) = B_k x(\tau_k), \tau_k > t_0, k = 1, 2, \dots \end{cases} \tag{9}$$

where $A(t)$, $B(t)$, and B_k, $k = 1, 2, \dots$ are $n \times n$ matrices.

Theorem 3: *Let the matrix functions* $A(t)$ *and* $B(t)$ *are continuous for* $t > t_0$, $t \neq \tau_k$, $k = 1, 2, \dots$ *with points of discontinuity at* τ_1, τ_2, ... *where they are left continuous.*

Then for each point $(t_0, x_0) \in R \times \Omega$ *and for each function* $\varphi_0 \in PC(t_0)$ *there exists a unique solution* $x(t) = x(t; t_0, x_0, \varphi_0)$ *of problem* (9) *that is defined for all* $t > t_0$.

Theorem 3 is a consequence of the theorem on existence and uniqueness for the solutions of a linear system of differential-difference equations [7].

The problem on left-continuability of solutions will be considered now for systems of type (8) only.

Assume that $x(t)$ is a solution of (8) defined on interval (γ, ω).

If $\gamma \neq \tau_k$, then the problem on continuability of $x(t)$ on the left of γ can be solved in the same way as for differential-difference equations without impulses. In this case the solution $x(t)$ is continuable on the left of γ and $J^- = J^-(t_0, x_0, \varphi_0) = (\alpha, t_0)$.

A straightforward calculation shows that the solution $x(t)$ of problem (8) satisfies the equation:

$$x(t) = \begin{cases} x_0 + \sum\limits_{t_0 \leq \tau_k < t} I_k(x(\tau_k)) + \int\limits_{t_0}^{t} f(s, x(s), x(s-h))ds, \ t \in J^+, \\ x_0 - \sum\limits_{t \leq \tau_k < t_0} I_k(x(\tau_k)) + \int\limits_{t_0}^{t} f(s, x(s), x(s-h))ds, t \in J^-. \end{cases}$$

The solution of linear system (9) can be extended to the left of τ_k if the below conditions are met:

$$\det(E + B_k) \neq 0, \ k = 1, 2, \dots, \tag{10}$$

where E is the $n \times n$ identity matrix.

Let $U_k(t, s)$ $(t, s \in (\tau_{k-1}, \tau_k])$ be the Cauchy matrix [8] for the linear system

$$\dot{x}(t) = A(t)x(t),\ \tau_{k-1} < t \leq \tau_k,\ k = 1, 2, \ldots.$$

Then by virtue of Theorem 3, the solution of the initial problem (9) can be decomposed as:

$$x(t; t_0, x_0, \varphi_0) = x(t) = W(t, t_0 + 0)x_0 + \int_{t_0}^{t} W(t,s)B(s)x(s-h)ds,\ \ t > t_0, \tag{11}$$

where

$$W(t,s) = \begin{cases} U_k(t,s) \text{ as } t, s \in (\tau_{k-1}, \tau_k], \\ U_{k+1}(t, \tau_k + 0)(E + B_k)U_k(\tau_k, s) \\ \text{as } \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}, \\ U_k(t, \tau_k)(E + B_k)^{-1}U_{k+1}(\tau_k + 0, s) \\ \text{as } \tau_{k-1} < t \leq \tau_k < s \leq \tau_{k+1}, \\ U_{k+1}(t, \tau_k + 0)\prod_{j=k}^{i+1}(E + B_j)U_j(\tau_j, \tau_{j-1} + 0)(E + B_i)U_i(\tau_i, s) \\ \text{as } \tau_{i-1} < s \leq \tau_i < \tau_k < t \leq \tau_{k+1}, \\ U_i(t, \tau_i)\prod_{j=i}^{k-1}(E + B_j)^{-1}U_{j+1}(\tau_j + 0, \tau_{j+1})(E + B_k)^{-1}U_{k+1}(\tau_k + 0, s) \\ \text{as } \tau_{i-1} < t \leq \tau_i < \tau_k < s \leq \tau_{k+1}, \end{cases} \tag{12}$$

is the solving operator of the system

$$\begin{cases} \dot{x}(t) = A(t)x(t),\ t \neq \tau_k, \\ \Delta x(\tau_k) = B_k x(\tau_k). \end{cases}$$

Now, (11), (12), and the linearity of operator $U_k(t,s)$ imply that the space L of all solutions of the problem (9) is an n-dimensional linear space.

Acknowledgements

The present investigation was supported by the Bulgarian Ministry of Education and Science under Grant MM-702.

References

[1] Bainov, D.D. and Dishliev, A.B., Quasiuniqueness, uniqueness and continuability of the solutions of impulsive functional differential equations, *Rendiconti di Matematica* **15** (1995), 391-404.

[2] Bainov, D.D. and Dishliev, A.B., The phenomenon "beating" of the solutions of impulsive functional differential equations, *Commun. in Appl. Anal.* **1** (1997), 435-441.

[3] Bainov, D.D., Kulev, G., and Stamova, I.M., Global stability of the solutions of impulsive differential-difference equations, *SUT J. of Math.* **31** (1995), 55-71.

[4] Bainov, D.D. and Stamova, I.M., On the practical stability of the solutions of impulsive differential-difference equations with variable impulsive perturbations, *J. of Math. Anal. and Appl.* **200** (1996), 272-288.

[5] Bainov, D.D. and Stamova, I.M., Global stability of the solutions of impulsive differential-difference equations with variable impulsive perturbations, *COMPEL, The Int. J. for Comp. and Math. in Electrical and Electronic Eng.* **1** (1997), 3-16.

[6] Bainov, D.D. and Stamova, I.M., Application of Lyapunov's direct method to the investigation of the stability of the solutions of impulsive differential-difference equations with variable impulsive perturbations, *Appl. Anal.* **63** (1996), 253-269.

[7] Hale, J., *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin 1977.

[8] Hartman, P., *Ordinary Differential Equations*, John Wiley, New York, London, Sydney 1964.