

CONTROLLABILITY OF SECOND ORDER SEMILINEAR ORDINARY DIFFERENTIAL SYSTEMS IN BANACH SPACES

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Sufficient conditions for controllability of semilinear second order ordinary differential systems in Banach spaces are established using the theory of strongly continuous cosine families. The results obtained are based on the Schaefer theorem.

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1. Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional spaces has been extensively studied. Several authors extended the concept to infinite dimensional systems in Banach spaces with bounded operators. Chukwu and Lenhart [3] studied controllability of nonlinear systems in abstract spaces. Naito [7, 8] studied controllability of semilinear systems and nonlinear Volterra integrodifferential systems. Quinn and Carmichael [11] showed that a controllability problem in Banach spaces can be converted into a fixed-point problem for a single-valued mapping. Balachandran et al. [1] established sufficient conditions for controllability of nonlinear integrodifferential systems in Banach spaces.

In many cases it is advantageous to treat second order abstract differential equations directly rather than convert them to first order systems. For example, Fitzgibbon [4] used the second order abstract differential equations for establishing the boundedness of solutions of an equation governing the transverse motion of an extensible beam. A useful tool for the study of abstract second order equations is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory [14, 15]. A motivation for second order systems can be found in [5, 6]. Recently, Park et al. [10] discussed controllability of second order nonlinear systems in Banach spaces with the help of the Schauder fixed point theorem. The purpose of this paper is to study controllability of semilinear second order differential systems in Banach spaces by using the Schaefer theorem.

2. Preliminaries

We consider the semilinear second order control system

$$\begin{aligned}
 x''(t) &= Ax(t) + f(t, x(t), x'(t)) + Bu(t), \quad t \in J = [0, T], \\
 x(0) &= x_0, \quad x'(0) = y_0,
 \end{aligned}
 \tag{1}$$

where the state $x(\cdot)$ takes values in a Banach space X , $x_0, y_0 \in X$, A is the infinitesimal generator of the strongly continuous cosine family $C(t)(t \in \mathbb{R})$, of bounded linear operators in X , f is a nonlinear mapping from $J \times X \times X$ to X , B is a bounded linear operator from U to X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U also being a Banach space.

Definition 1: [14] A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in the Banach space X is called a *strongly continuous cosine family* iff

- (i) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$;
- (ii) $C(0) = I$;
- (iii) $C(t)x$ is continuous in t on \mathbb{R} for each fixed $x \in X$.

Define the associated sine family $S(t)$, $t \in \mathbb{R}$, by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

Assume the following conditions on A .

- (H_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators from X into itself and the adjoint operator A^* is densely defined i.e., $\overline{D(A^*)} = X^*$ (see [2]).

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, is the operator $A: X \rightarrow X$ defined by

$$Ax = \left. \frac{d^2}{dt^2} C(t)x \right|_{t=0}, \quad x \in D(A),$$

where

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}.$$

Define

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}.$$

To establish our main theorem we need the following lemmas.

Lemma 1: [14] *Let (H_1) hold. Then,*

- (i) *there exist constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Ne^{\omega|t|} \quad \text{and} \quad \|S(t) - S(t^*)\| \leq N \left| \int_t^{t^*} e^{\omega|s|} ds \right| \text{ for } t, t^* \in \mathbb{R};$$

- (ii) $S(t)X \subset E$ and $S(t)E \subset D(A)$ for $t \in \mathbb{R}$;
- (iii) $\frac{d}{dt}C(t)x = AS(t)x$ for $x \in E$ and $t \in \mathbb{R}$;
- (iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$ for $x \in D(A)$ and $t \in \mathbb{R}$.

Lemma 2: [14] Let (H_1) hold, let $v: \mathbb{R} \rightarrow X$ be such that v is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then,

q is twice continuously differentiable and for $t \in \mathbb{R}$, $q(t) \in D(A)$,

$$\begin{aligned} q'(t) &= \int_0^t C(t-s)v(s)ds, \text{ and } q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) \\ &= Aq(t) + v(t). \end{aligned}$$

Lemma 3: (Schaefer Theorem [13]) Let S be a convex subset of a normed linear space Y and assume $0 \in S$. Let $F: S \rightarrow S$ be a completely continuous operator, and let

$$\xi(F) = \{x \in S: x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\xi(F)$ is unbounded or F has a fixed point.

We make the following assumptions:

- (H₂) $f(t, \cdot, \cdot): X \times X \rightarrow X$ is continuous for each $t \in J$ and the function $f(\cdot, x, y): J \rightarrow X$ is strongly measurable (see [12, p. 116]) for each $x, y \in X$.
- (H₃) For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|, \|y\| \leq k} \|f(t, x, y)\| \leq \alpha_k(t) \text{ for almost all } t \in J.$$

- (H₄) There exists a continuous function $m: J \rightarrow [0, \infty)$ such that

$$\|f(t, u, v)\| \leq m(t)\Omega(\|u\| + \|v\|), \quad t \in J, u, v \in X,$$

where $\Omega: (0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function and

$$M(T+1) \int_0^T m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

with

$$c = (M + M^*)\|x_0\| + (1+T)M\|y_0\| + (1+T)M\|B\|T\|W^{-1}\|$$

$$[\|x_1\| + M\|x_0\| + M\|y_0\| + MT \int_0^T m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds],$$

$M = \sup\{\|C(t)\| : t \in J\}$, and $M^* = \sup\{\|AS(t)\| : t \in J\}$.

- (H₅) $Bu(t)$ is continuous.
- (H₆) The linear operator $W: L^2(J, U) \rightarrow X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has a bounded invertible operator $W^{-1}: X \rightarrow L^2(J, U)/\ker W$.

(H_7) $C(t)$, $t > 0$ is compact.

Then the system (1) has a mild solution of the form (see [9])

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x'(s))ds + \int_0^t S(t-s)Bu(s)ds. \tag{2}$$

Definition 2: The system (1) is said to be *controllable* on J if for every $x_0 \in D(A)$, $y_0 \in E$ and $x_1 \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

3. Main Result

Theorem: Suppose (H_1)-(H_7) hold. Then system (1) is controllable on J .

Proof: Using (H_6), for an arbitrary function $x(\cdot)$ we define the control

$$u(t) = W^{-1}[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-s)f(s, x(s), x'(s))ds](t).$$

Using this control we will show that the operator defined by

$$(Fx)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x'(s))ds + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds, \quad t \in J,$$

has a fixed point. This fixed point is then a solution of Equation (2).

Clearly, $(Fx)(T) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time T , provided we obtain a fixed point of the nonlinear operator F . Consider the space $Z = C^1(J, X)$ with norm

$$\|x\|^* = \max\{\|x\|, \|x'\|\}.$$

In order to study the controllability problem for the system (1) we apply Lemma 3 to

the following system

$$x''(t) = \lambda Ax(t) + \lambda f(t, x(t), x'(t)) + \lambda Bu(t), \quad t \in J, \quad \lambda \in (0, 1). \tag{3}$$

Let x be a mild solution of system (3). Then from

$$\begin{aligned} x(t) = & \lambda(C(t)x_0 + S(t)y_0) + \lambda \int_0^t S(t-s)f(s, x(s), x'(s))ds \\ & + \lambda \int_0^t S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ & - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds \end{aligned}$$

we have

$$\begin{aligned} \|x(t)\| \leq & M \|x_0\| + MT \|y_0\| + MT \int_0^t m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \\ & + MT^2 \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\ & + MT \int_0^T m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds] \\ \equiv & K_1 + MT \int_0^t m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds. \end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$v(0) = K_1, \quad \|x(t)\| \leq v(t)$$

and

$$v'(t) + MTm(t)\Omega(\|x(t)\| + \|x'(t)\|), \quad t \in J.$$

On the other hand,

$$\begin{aligned} x'(t) = & \lambda[AS(t)x_0 + C(t)y_0] + \lambda \int_0^t C(t-s)f(s, x(s), x'(s))ds \\ & + \lambda \int_0^t C(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ & - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \|x'(t)\| &\leq M^* \|x_0\| + M \|y_0\| + M \int_0^t m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \\ &+ MT \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\ &+ MT \int_0^T m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds] \\ &\equiv K_2 + M \int_0^t m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds. \end{aligned}$$

Denoting by $r(t)$ the right-hand side of the above inequality we have

$$r(0) = K_2, \quad \|x'(t)\| \leq r(t)$$

and

$$r'(t) = Mm(t)\Omega(\|x(t)\| + \|x'(t)\|), \quad t \in J.$$

Let

$$w(t) = v(t) + r(t), \quad t \in J.$$

Then

$$w(0) = v(0) + r(0) = c,$$

and

$$\begin{aligned} w'(t) &= v'(t) + r'(t) \\ &\leq MTm(t)\Omega(w(t)) + Mm(t)\Omega(w(t)) \\ &= M(T+1)m(t)\Omega(w(t)), \quad t \in J. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s)} \leq M(T+1) \int_0^T m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad t \in J.$$

The last inequality implies that there is a constant K such that

$$w(t) = v(t) + r(t) \leq K, \quad t \in J.$$

Then

$$\|x(t)\| \leq v(t), \quad t \in J,$$

$$\|x'(t)\| \leq r(t), \quad t \in J,$$

and hence

$$\|x\|^* = \max\{\|x\|, \|x'\|\} \leq K,$$

where K depends only on T and on the functions m and Ω .

We shall now prove that the operator $F: Z \rightarrow Z$ defined by

$$\begin{aligned} (Fx)(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x'(s))ds \\ &\quad + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds, \quad t \in J, \end{aligned}$$

is a completely continuous operator.

Let $B_k = \{x \in Z, \|x\|^* \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned} \|(Fx)(t_1) - (Fx)(t_2)\| &\leq \|C(t_1) - C(t_2)\| \|x_0\| + \|S(t_1) - S(t_2)\| \|y_0\| \\ &\quad + \left\| \int_0^{t_1} [S(t_1-s) - S(t_2-s)]f(s, x(s), x'(s))ds \right\| \\ &\quad + \left\| \int_{t_1}^{t_2} S(t_2-s)f(s, x(s), x'(s))ds \right\| \\ &\quad + \left\| \int_0^{t_1} [S(t_1-s) - S(t_2-s)]BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \right. \\ &\quad \left. - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds \right\| \\ &\quad + \left\| \int_{t_1}^{t_2} S(t_2-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \right. \\ &\quad \left. - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s)ds \right\| \\ &\leq \|C(t_1) - C(t_2)\| \|x_0\| + \|S(t_1) - S(t_2)\| \|y_0\| \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \alpha_k(s) ds \\
 & + \int_{t_1}^{t_2} \|S(t_2 - s)\| \alpha_k(s) ds \\
 + & \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
 & + MT \int_0^T \alpha_k(\theta) d\theta] ds \\
 + & \int_{t_1}^{t_2} \|S(t_2 - s)\| \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
 & + MT \int_0^T \alpha_k(\theta) d\theta] ds, \tag{4}
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 \|(Fx)'(t_1) - (Fx)'(t_2)\| & \leq \|C'(t_1) - C'(t_2)\| \|x_0\| + \|S'(t_1) - S'(t_2)\| \|y_0\| \\
 & + \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)] f(s, x(s), x'(s)) ds \right\| \\
 & + \left\| \int_{t_1}^{t_2} C(t_2 - s) f(s, x(s), x'(s)) ds \right\| \\
 + & \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)] BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\
 & \left. - \int_0^T S(T - \theta) f(\theta, x(\theta), x'(\theta)) d\theta] (s) ds \right\| \\
 & + \left\| \int_{t_1}^{t_2} C(t_2 - s) BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\
 & \left. - \int_0^T S(T - \theta) f(\theta, x(\theta), x'(\theta)) d\theta] (s) ds \right\| \\
 \leq & \|A(S(t_1) - S(t_2))\| \|x_0\| + \|C(t_1) - C(t_2)\| \|y_0\| \\
 & + \int_0^{t_1} \|C(t_1 - s) - C(t_2 - s)\| \alpha_k(s) ds
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_1}^{t_2} \|C(t_2 - s)\| \alpha_k(s) ds \\
 + \int_0^{t_1} & \|C(t_1 - s) - C(t_2 - s)\| \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
 & + MT \int_0^T \alpha_k(\theta) d\theta] ds \\
 + \int_{t_1}^{t_2} & \|C(t_2 - s)\| \|B\| \|W^{-1}\| [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
 & + MT \int_0^T \alpha_k(\theta) d\theta] ds. \tag{5}
 \end{aligned}$$

The right-hand side of (4) and (5) are independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \rightarrow 0$, since $C(t)$ and $S(t)$ are uniformly continuous for $t \in J$. The compactness of $C(t)$ and $S(t)$ for $t > 0$ imply the continuity in the uniform operator topology (or norm topology, see [12, p. 182]). Furthermore, the compactness of $S(t)$ follows from that of $C(t)$ (see [15]).

Thus F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next we show that $\overline{FB_k}$ is compact. Since we have shown that FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X .

Let $0 < t \leq T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}
 (F_\epsilon x)(t) &= C(t)x_0 + S(t)y_0 + \int_0^{t-\epsilon} S(t-s)f(s, x(s), x'(s)) ds \\
 &+ \int_0^{t-\epsilon} S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\
 &- \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s) ds.
 \end{aligned}$$

Since $C(t)$ and $S(t)$ are compact operators, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t); x \in B_k\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have that

$$\begin{aligned}
 \|(Fx)(t) - (F_\epsilon x)(t)\| &\leq \int_{t-\epsilon}^t \|S(t-s)f(s, x(s), x'(s))\| ds \\
 &+ \int_{t-\epsilon}^t \|S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s) \| ds. \\
 & \leq \int_{t-\epsilon}^t \| S(t-s) \| \alpha_k(s)ds \\
 & + \int_{t-\epsilon}^t \| S(t-s) \| \| B \| \| W^{-1} \| [\| x_1 \| + M \| x_0 \| + MT \| y_0 \| \\
 & \quad + MT \int_0^T \alpha_k(\theta)d\theta]ds
 \end{aligned}$$

and

$$\begin{aligned}
 \| (Fx)'(t) - (F_\epsilon x)'(t) \| & \leq \int_{t-\epsilon}^t \| C(t-s)f(s, x(s), x'(s)) \| ds \\
 & + \int_{t-\epsilon}^t \| C(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\
 & \quad - \int_0^T S(T-\theta)f(\theta, x(\theta), x'(\theta))d\theta](s) \| ds. \\
 & \leq \int_{t-\epsilon}^t \| C(t-s) \| \alpha_k(s)ds \\
 & + \int_{t-\epsilon}^t \| C(t-s) \| \| B \| \| W^{-1} \| [\| x_1 \| + M \| x_0 \| + MT \| y_0 \| \\
 & \quad + MT \int_0^T \alpha_k(\theta)d\theta]ds.
 \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fx)(t): x \in B_k\}$. Hence the set $\{(Fx)(t): x \in B_k\}$ is precompact in X .

It remains to show that $F: Z \rightarrow Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \rightarrow x$ in Z . Then there is an integer q such that $\|x_n(t)\| \leq q$, $\|x'_n(t)\| \leq q$ for all n and $t \in J$, so $\|x(t)\| \leq q$, $\|x'(t)\| \leq q$ and $x, x' \in Z$. By (H_2)

$$f(t, x_n(t), x'_n(t)) \rightarrow f(t, x(t), x'(t))$$

for each $t \in J$ and since

$$\| f(t, x_n(t), x'_n(t)) - f(t, x(t), x'(t)) \| \leq 2\alpha_q(t),$$

we have by the dominated convergence theorem that

$$\| Fx_n - Fx \| = \sup_{t \in J} \left\| \int_0^t S(t-s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))]ds \right\|$$

$$\begin{aligned}
 & - \int_0^t S(t-s)BW^{-1} \\
 & \times \int_0^T S(T-\theta)[f(\theta, x_n(\theta), x'_n(\theta)) - f(\theta, x(\theta), x'(\theta))]d\theta ds \parallel \\
 & \leq \int_0^T \parallel S(t-s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))] \parallel ds \\
 & \quad + \int_0^T \parallel S(t-s)BW^{-1} \\
 & \times \int_0^T S(T-\theta)[f(\theta, x_n(\theta), x'_n(\theta)) - f(\theta, x(\theta), x'(\theta))]d\theta \parallel ds \rightarrow 0
 \end{aligned}$$

and

$$\begin{aligned}
 \parallel (Fx_n)' - (Fx)'\parallel & = \sup_{t \in J} \parallel \int_0^t C(t-s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))]ds \\
 & \quad - \int_0^t C(t-s)BW^{-1} \\
 & \times \int_0^T S(T-\theta)[f(\theta, x_n(\theta), x'_n(\theta)) - f(\theta, x(\theta), x'(\theta))]d\theta ds \parallel \\
 & \leq \int_0^T \parallel C(t-s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))] \parallel ds \\
 & \quad + \int_0^T \parallel C(t-s)BW^{-1} \\
 & \times \int_0^T S(T-\theta)[f(\theta, x_n(\theta), x'_n(\theta)) - f(\theta, x(\theta), x'(\theta))]d\theta \parallel ds \rightarrow 0.
 \end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\zeta(F) = \{x \in Z: x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded as we proved it in the first step. Consequently, by Schaefer's theorem, the operator F has a fixed point in Z . This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus the system (1) is controllable on J . \square

4. Example

Consider the partial differential equation

$$\begin{aligned}
 z_{tt}(y, t) & = z_{yy}(y, t) + \sigma(t, z(y, t), z_t(y, t)) + \mu(y, t), \\
 z(0, t) & = z(\pi, t) = 0,
 \end{aligned}$$

$$z(y, 0) = z_0(y), \quad z_t(y, 0) = z_1(y), \quad 0 < y < \pi, \quad t \in J = [0, T]. \tag{6}$$

Now we have to show that there exists a control μ which steers (6) from any specified initial state to the final state in a Banach space X .

Let $X = L^2[0, \pi]$ and let $A: X \rightarrow X$ be defined by

$$Aw = w'', \quad w \in D(A),$$

where $D(A) = \{w \in X: w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\}$.

Then,

$$Aw = \sum_{n=1}^{\infty} -n^2(w, w_n)w_n, \quad w \in D(A),$$

where $w_n(s) = \sqrt{2/\pi} \sin ns$, $n = 1, 2, 3, \dots$ is the orthogonal set of eigenvalues of A .

It can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, w_n)w_n, \quad w \in X,$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(w, w_n)w_n, \quad w \in X.$$

Let $f: J \times X \times X \rightarrow X$ be define by

$$f(t, v, w)(y) = \sigma(t, v(y), w(y)), \quad v, w \in X, \quad y \in [0, \pi],$$

where $\sigma: J \times [0, \pi] \times [0, \pi] \rightarrow [0, \pi]$ is continuous and strongly measurable.

Let $u: J \rightarrow U \subset X$ be defined by

$$(u(t))(y) = \mu(y, t), \quad y \in [0, \pi],$$

where $\mu: [0, \pi] \times J \rightarrow [0, \pi]$ is continuous.

Assume that there exists a bounded invertible operator W^{-1} (with range $L^2(J, U)/kerW$) such that

$$Wu = \int_0^T S(T-s)u(s)ds.$$

Further, the function σ satisfies the following condition:

(H) There exists a continuous function $p: J \rightarrow [0, \infty)$ such that

$$\|\sigma(t, u, v)\| \leq p(t)\Omega(\|u\| + \|v\|), \quad t \in J, \quad u, v \in X,$$

where $\Omega: (0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function and

$$M(T+1) \int_0^T p(s)ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

where c is a known constant.

With this choice of A , f and $B = I$, (1) is an abstract formulation of (6). Furthermore, all the conditions stated in the above theorem are satisfied. Hence, system (6) is controllable on J .

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