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AN APPLICATION OF A NONCOMPACTNESS TECHNIQUE TO AN INVESTIGATION OF THE EXISTENCE OF SOLUTIONS TO A NONLOCAL MULTIVALUED DARBOUX PROBLEM

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The aim of the paper is to prove two theorems on the existence of solutions to a nonlocal multivalued Darboux problem. The first theorem concerns the case when the orientor field is convex valued. The second theorem concerns the case when the orientor field is nonconvex valued. A compactness type condition involving the ball measure of noncompactness is applied.

Key words: Multivalued Darboux Problem, Nonlocal Conditions, Measure of Noncompactness, Measurable Multifunctions, Upper and Lower Semicontinuous Multifunctions, Fixed Point Principle.

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1. Introduction

In this paper we study the existence of solutions to a nonlocal multivalued Darboux problem in a separable Banach space. Applying a compactness type condition involving the ball measure of noncompactness, we obtain two theorems on the existence of the solutions of the nonlocal multivalued Darboux problem. The first theorem concerns the case when the orientor field is convex valued. The second theorem concerns the case when the orientor field is nonconvex valued.

The results obtained are generalizations of those given by Papageorgiou in [13] on the existence of solutions of a classical multivalued Darboux problem. They are also generalizations of those given by Byszewski in [3, 4, 6] and by Byszewski and Lakshmikantham in [5] on the existence and uniqueness of solutions of nonlocal Darboux problems.

The approach applied in the paper is based on results of Papageorgiou [11-13], Kandilakis and Papageorgiou [8, 9], and Byszewski [3, 4, 6].

The existence of a solution of a classical multivalued problem, where the orientor field has compact values in a separable Banach space, was also examined by Dawidowski and Kubiaczyk in [7] using a contraction principle for multifunctions.

2. Preliminaries

Let (Ω, Σ) be a measurable space and Y a separable Banach space. We will need the following sets:

$$P_{f(c)}(Y) := \{A \subset Y\colon A \text{ is nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(Y) := \{A \subset Y\colon A \text{ is nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(Y)$ is said to be *measurable* if there exists a sequence of measurable functions $f_n\colon \Omega \to Y$ $(n \in \mathbb{N})$ such that $F(\omega) = \overline{\{f_n(\omega)\}}_{n \in \mathbb{N}}$ for all $\omega \in \Omega$.

The multifunction F is said to be *weakly measurable* if for every $y^* \in Y^*$ the $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$-valued function $\omega \to \sigma(y^*, F(\omega)) = \sup\{(y^*, y)\colon y \in F(\omega)\}$ is measurable.

Let $\mathfrak{B}$ be the family of bounded subsets of Y. Then the ball measure of noncompactness $\beta\colon \mathfrak{B} \to [0, \infty)$ is defined by

$$\beta(B) := \inf\{\rho > 0\colon B \text{ can be covered by finitely many balls of the radii } \rho\}, \quad B \in \mathfrak{B}. \tag{2.1}$$

Let Y_i $(i = 1, 2)$ be Hausdorff topological spaces.

A multifunction $G\colon Y_1 \to 2^{Y_2} \backslash \{\emptyset\}$ is said to be *upper semicontinuous* [respectively, *lower semicontinuous*] if for every open set $A \subset Y_2$, $\{y \in Y_1\colon G(y) \subset A\}$ [respectively, $\{y \in Y_1\colon G(y) \cap A \neq \emptyset\}$] is open in Y_1.

We will need the sets $\mathbb{R}_+ := (0, \infty)$ and $\Delta := I \times I$, where $I := [0, c]$ and $c > 0$.

Let $X := C(I, E)$, where E is a separable Banach space with the norm $\|\cdot\|$, and let C be the Banach space defined by

$$C := \left\{(\tilde{\phi}, \tilde{\psi}) \in X \times X\colon \ \tilde{\phi}(0) = \tilde{\psi}(0)\right\}$$

and equipped with the norm $\|\cdot\|_C$ given by the formula

$$\|(\tilde{\phi}, \tilde{\psi})\|_C := \|\tilde{\phi}\|_X + \|\tilde{\psi}\|_X, \quad (\tilde{\phi}, \tilde{\psi}) \in C.$$

By E_w we denote the Banach space E with the weak topology.

Let p and r be arbitrary natural numbers.

For given a multifunction $F: \Delta \times E \to P_{fc}(E)$ [$F: \Delta \times E \to P_f(E)$, respectively] and, given a $(\phi, \psi) \in C$, satisfying some assumptions, we will study the existence of a solution of the following Darboux problem:

$$\left.\begin{aligned} &\frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u(x,y)) \text{ a.e. in } \Delta, \\ &u(x,0) + \sum_{j=1}^{r} h_j(x) u(x, b_j) = \phi(x), \quad x \in I, \\ &u(0,y) + \sum_{i=1}^{p} k_i(y) u(a_i, y) = \psi(y), \quad y \in I, \end{aligned}\right\} \tag{2.2}$$

where a_i $(i = 1, \ldots, p)$ and b_j $(j = 1, \ldots, r)$ are given numbers such that

$$0 < a_1 < \ldots < a_p \leq c$$

and

$$0 < b_1 < \ldots < b_r \leq c.$$

A function $u \in C(\Delta, E)$ is said to be a *solution* of problem (2.2) if there is $f \in L^1(\Delta, E)$ such that

$$f(\xi, \eta) \in F(\xi, \eta, u(\xi, \eta)) \text{ a.e. in } \Delta \tag{2.3}$$

and

$$u(x,y) = \alpha(x,y) - \sum_{j=1}^{r} h_j(x) u(x, b_j) - \sum_{i=1}^{p} k_i(y) u(a_i, y) + \int_0^x \int_0^y f(\xi, \eta) d\xi d\eta, \quad (x,y) \in \Delta, \tag{2.4}$$

where

$$\alpha(x,y) := \phi(x) + \psi(y) - \phi(0), \quad (x,y) \in \Delta. \tag{2.5}$$

Let Z be a fixed nonempty compact subset of E.

By $K(\Delta, E)$ we denote the set of the functions w belonging to $C(\Delta, E)$ such that

$$w(x,0) + w(0,y) \in Z \text{ for } (x,y) \in \Delta.$$

3. Theorem About the Existence of a Solution of the Nonlocal Multivalued Darboux Problem with the Convex Valued Orientor Field

Theorem 3.1: *Assume that $F: \Delta \times E \to P_{fc}(E)$ is a multifunction such that:*

(*i*) *$(x,y,z) \to F(x,y,z)$ is weakly measurable, $z \to F(x,y,z)$ is upper semicontinuous from E into E_w and for all $(x,y) \in \Delta$, $F(x,y,\cdot)$ maps the bounded sets into relatively weakly compact sets,*

(*ii*)
$$|F(x,y,z)| := \sup\{\|s\| : s \in F(x,y,z)\} \leq c_1(x,y) + c_2(x,y)\|z\| \text{ a.e.}, \tag{3.1}$$

where c_i $(i=1,2)$ are some functions belonging to $L^1(\Delta,\mathbb{R}_+)$, and

$$\beta(F(x,y,B)) \leq \mathfrak{S}(x,y)\beta(B) \ a.e. \tag{3.2}$$

for every nonempty and bounded set $B \subset E$, where $\mathfrak{S}$ is a function belonging to $L^\infty(\Delta,\mathbb{R}_+)$.

Moreover, assume that

(iii) $\phi,\psi \in C$,

(iv) $h_j \in C(I,\mathbb{R})$ $(j=1,\ldots,r)$, $k_i \in C(I,\mathbb{R})$ $(i=1,\ldots,p)$, $h_j(0)=0$ $(j=1,\ldots,r)$, $k_i(0)=0$ $(i=1,\ldots,p)$ *and*

$$\left(\sum_{j=1}^{r} \| h_j \|_{C(I,\mathbb{R})} + \sum_{i=1}^{p} \| k_i \|_{C(I,\mathbb{R})}\right) e^{\| c_2 \|_{L^1(\Delta,\mathbb{R}_+)}} < 1. \tag{3.3}$$

Then in a class of functions $w \in K(\Delta,E)$ problem (2.2) possesses a solution.

Proof: Firstly, we will obtain an a priori evaluation for the solutions of problem (2.2). For this purpose suppose that u is a solution of problem (2.2). Consequently, we have, from (2.3)-(2.5), that

$$\begin{aligned} \| u(x,y) \| \leq \| \alpha(x,y) \| + \sum_{j=1}^{r} | h_j(x) | \, \| u(x,b_j) \| + \sum_{i=1}^{p} | k_i(y) | \, \| u(a_i,y) \| \\ + \int_0^x \int_0^y \| f(\xi,\eta) \| d\xi d\eta, \quad (x,y) \in \Delta. \end{aligned} \tag{3.4}$$

Formulas (3.4), (2.3) and (3.1) imply that

$$\begin{aligned} \| u(x,y) \| &\leq \alpha_0 + c_{h,k} \| u \|_{C(\Delta,E)} + \int_0^x \int_0^y [c_1(\xi,\eta) + c_2(\xi,\eta) \| u(\xi,\eta) \|] d\xi d\eta \\ &\leq \alpha_0 + c_{h,k} \| u \|_{C(\Delta,E)} + \| c_1 \|_{L^1(\Delta,\mathbb{R}_+)} \\ &\quad + \int_0^x \int_0^y c_2(\xi,\eta) \| u(\xi,\eta) \| d\xi d\eta, \quad (x,y) \in \Delta, \end{aligned} \tag{3.5}$$

where

$$\alpha_0 := \| \phi \|_X + \| \psi \|_X + \| \phi(0) \|$$

and

$$c_{h,k} := \sum_{j=1}^{r} \| h_j \|_{C(I,\mathbb{R})} + \sum_{i=1}^{p} \| k_i \|_{C(I,\mathbb{R})}.$$

By (3.5) and Gronwall's inequality,

$$\| u(x,y) \| \leq \left(\alpha_0 + c_{h,k} \| u \|_{C(\Delta,E)} + \| c_1 \|_{L^1(\Delta,\mathbb{R}_+)}\right) e^{\| c_2 \|_{L^1(\Delta,\mathbb{R}_+)}}, \quad (x,y) \in \Delta. \tag{3.6}$$

From (3.6), we obtain

$$\left(1 - c_{h,k}e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}\right) \| u \|_{C(\Delta, E)}$$

$$\leq \left(\alpha_0 + \| c_1 \|_{L^1(\Delta, \mathbb{R}_+)}\right) e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}. \tag{3.7}$$

Consequently, by (3.7) and (3.3),

$$\| u(x,y) \| \leq \frac{\alpha_0 + \| c_1 \|_{L^1(\Delta, \mathbb{R}_+)}}{1 - c_{h,k}e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}} e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}, \ (x,y) \in \Delta. \tag{3.8}$$

Then, from (3.8),

$$\| u(x,y) \| \leq M, \ (x,y) \in \Delta, \tag{3.9}$$

where

$$M: = \frac{\alpha_0 + \| c_1 \|_{L^1(\Delta, \mathbb{R}_+)}}{1 - c_{h,k}e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}} e^{\| c_2 \|_{L^1(\Delta, \mathbb{R}_+)}}. \tag{3.10}$$

Define $\widehat{F}: \Delta \times E \to P_{fc}(E)$ by the formula

$$\widehat{F}(x,y,z): = \begin{cases} F(x,y,z) & \text{if } \| z \| \leq M, \\ F(x,y,\frac{Mz}{\| z \|}) & \text{if } \| z \| > M, \end{cases} \tag{3.11}$$

where M is given by (3.10).

It follows, from (3.11), that

$$\widehat{F}(x,y,z) = F(z,y,p_M(z)) \text{ in } \Delta \times E, \tag{3.12}$$

where p_M is the M-radial retraction in E.

Since p_M is Lipschitz continuous then (3.11) and the first part of assumption (i) imply that $(x,y,z) \to \widehat{F}(x,y,z)$ is weakly measurable. Moreover, from (3.11), the second part of assumption (i) and Theorem 7.3.11 from [10], $z \to \widehat{F}(x,y,z)$ is upper semicontinuous from E into E_w. By (2.1), (3.12), (3.2) and by the inclusion,

$$p_M(B) \subset \overline{\text{conv}}(B \cup \{0\}), \ \ B \subset E,$$

we have

$$\begin{aligned} \beta(\widehat{F}(x,y,B)) = \beta(F(x,y,p_M(B))) &\leq \mathfrak{S}(x,y)\beta(p_M(B)) \\ &\leq \mathfrak{S}(x,y)\beta(\overline{\text{conv}}(B \cup \{0\})) \leq \mathfrak{S}(x,y)\beta(B) \text{ a.e. in } \Delta, \ B \subset E. \end{aligned} \tag{3.13}$$

Observe that, from (3.11) and (3.1),

$$|\widehat{F}(x,y,z)| = \sup\{\|s\| : s \in \widehat{F}(x,y,z)\} \tag{3.14}$$

$$\leq c_1(x,y) + Mc_2(x,y) =: c_3(x,y) \text{ a.e. with } c_3 \in L^1(\Delta, \mathbb{R}_+).$$

Define a set $\mathcal{U}$ by

$$\mathcal{U} := \Big\{u \in C(\Delta, E): u(x,0) + u(0,y) \in Z, (x,y) \in \Delta,$$

$$u(x,y) = \alpha(x,y) - \sum_{j=1}^{r} h_j(x)u(x,b_j) - \sum_{i=1}^{p} k_i(y)u(a_i,y)$$

$$+ \int_0^x \int_0^y g(\xi,\eta)d\xi d\eta, \quad (x,y) \in \Delta,$$

$$\|g(\xi,\eta)\| \leq c_3(\xi,\eta) \text{ a.e. in } \Delta\Big\} \tag{3.15}$$

and a multifunction

$$T: \mathcal{U} \to 2^{\mathcal{U}}$$

by

$$T(u) := \Bigg\{v \in C(\Delta, E): v(x,y) = \alpha(x,y) - \sum_{j=1}^{r} h_j(x)u(x,b_j) - \sum_{i=1}^{p} k_i(y)u(a_i,y)$$

$$+ \int_0^x \int_0^y f(\xi,\eta)d\xi d\eta, \quad (x,y) \in \Delta,$$

$$f \in L^1(\Delta, E), f(\xi,\eta) \in \widehat{F}(\xi,\eta,u(\xi,\eta)) \text{ a.e. in } \Delta\Bigg\}, \quad u \in \mathcal{U}. \tag{3.16}$$

Since $(\xi,\eta,z) \to \widehat{F}(\xi,\eta,z)$ is weakly measurable, $(\xi,\eta) \to \widehat{F}(\xi,\eta,u(\xi,\eta))$ is weakly measurable on Δ with the Lebesgue σ-field which is complete with respect to the Lebesgue measure on Δ. So, $(\xi,\eta) \to \widehat{F}(\xi,\eta,u(\xi,\eta))$ is measurable and, therefore, by Aumann's selection theorem (see Theorem 5.10 from [15]), there is $f \in L^1(\Delta, E)$ such that $f(\xi,\eta) \in \widehat{F}(\xi,\eta,u(\xi,\eta))$ a.e. in Δ. Consequently, T has nonempty values.

Moreover, since from Proposition 3.1 given by Papageorgiou in [11],

$$\mathcal{S}^1_{\widehat{F}(\cdot,\cdot,u(\cdot,\cdot))} = \Big\{g \in L^1(\Delta, E): g(\xi,\eta) \in \widehat{F}(\xi,\eta,u(\xi,\eta))\Big\}$$

$$\in P_{wkc}(L^1(\Delta, E)), \tag{3.17}$$

T has values in $P_{fc}(C(\Delta, E))$.

Additionally, from (3.15), (3.16), and the first part of assumption (iv),

$$v(x,0)+v(0,y)=\alpha(x,0)+\alpha(0,y)-\sum_{j=1}^{r}h_j(x)u(x,b_j)-\sum_{i=1}^{p}k_i(y)u(a_i,y)$$
$$=u(x,0)+u(0,y)\in Z,\ \ (x,y)\in\Delta, \tag{3.18}$$

if $v\in T(u)$, where $u\in\mathcal{U}$.

Let $B\subset\mathcal{U}$ be a nonempty set. We have, by (2.1), (3.16), (3.15), (3.18) and (3.17), that

$$\beta(T(B)(x,y))\le\beta\left(\int_0^x\int_0^y f(\xi,\eta)d\xi d\eta:\ f\in S^1_{\widehat{F}(\cdot,\cdot,u(\cdot,\cdot))},u\in B\right)$$
$$\le\beta\left(\int_0^x\int_0^y \widehat{F}(\xi,\eta,\overline{B(\xi,\eta)})d\xi d\eta\right),\ \ (x,y)\in\Delta, \tag{3.19}$$

where

$$\overline{B(\xi,\eta)}:=\overline{\{u(\xi,\eta):u\in B\}},\ \ (\xi,\eta)\in\Delta, \tag{3.20}$$

and

$$\int_0^x\int_0^y \widehat{F}(\xi,\eta,\overline{B(\xi,\eta)})d\xi d\eta:=\left\{\int_0^x\int_0^y h(\xi,\eta)d\xi d\eta:\ h\in L^1(\Delta,E),\right.$$
$$\left.h(\xi,\eta)\in\widehat{F}(\xi,\eta,\overline{B(\xi,\eta)})\text{ a.e. in }\Delta\right\},\ (x,y)\in\Delta. \tag{3.21}$$

Observe that, for every $x^*\in E^*$, $(\xi,\eta,z)\to\sigma(x^*,\widehat{F}(\xi,\eta,z))$ is measurable and $(\xi,\eta)\to\overline{B(\xi,\eta)}$ is graph measurable. Therefore, from the Kandilakis-Papageorgiou theorem (see [9], Theorem 6.1), $(\xi,\eta)\to\sup[\sigma(x^*,\widehat{F}(\xi,\eta,u)):u\in\overline{B(\xi,\eta)}]$ is Lebesgue measurable on Δ. Hence, by the third part of assumption (i), $(\xi,\eta)\to\overline{\text{conv}}\,\widehat{F}(\xi,\eta,B(\xi,\eta))\in P_{wkc}(E)$. Consequently, applying an argument given by Papageorgiou in [13] and by Kandilakis-Papageorgiou in [8], we obtain that

$$\beta\left(\int_0^x\int_0^y \widehat{F}(\xi,\eta,\overline{B(\xi,\eta)})d\xi d\eta\right)\le\int_0^x\int_0^y \mathfrak{H}(\xi,\eta)\beta(\overline{B(\xi,\eta)})d\xi d\eta$$
$$=\int_0^x\int_0^y \mathfrak{H}(\xi,\eta)\beta(B(\xi,\eta))d\xi d\eta,\ \ (x,y)\in\Delta. \tag{3.22}$$

Formulae (3.19)-(3.22) imply that

$$\beta(T(B)(x,y))\le\int_0^x\int_0^y \mathfrak{H}(\xi,\eta)\beta(B(\xi,\eta))d\xi d\eta \tag{3.23}$$

$$\leq \int_0^x \int_0^y \| \mathfrak{G} \|_\infty \beta(B(\xi,\eta))d\xi d\eta, \quad (x,y) \in \Delta.$$

Let

$$\Psi(B) := \sup_{(\xi,\eta)\in\Delta} \left[e^{-\lambda \| \mathfrak{G} \|_\infty^{1/2}(\xi+\eta)} \beta(B(\xi,\eta)) \right] \tag{3.24}$$

for $B \subset \mathfrak{U}$ and with $\lambda > 0$.

Since $\mathfrak{U} \subset C(\Delta, E)$ is bounded and equicontinuous, by properties of β and by Arzela-Ascoli theorem, Ψ is a sublinear measure of noncompactness in the sense of Banas-Goebel [1]. Therefore, from (3.23) and (3.24),

$$\beta(T(B)(x,y)) \leq \int_0^x \int_0^y \| \mathfrak{G} \|_\infty e^{\lambda \| \mathfrak{G} \|_\infty^{1/2}(\xi+\eta)} e^{-\lambda \| \mathfrak{G} \|_\infty^{1/2}(\xi+\eta)} \beta(B(\xi,\eta))d\xi d\eta$$

$$\leq \int_0^x \int_0^y \| \mathfrak{G} \|_\infty e^{\lambda \| \mathfrak{G} \|_\infty^{1/2}(\xi+\eta)} \Psi(B) d\xi d\eta \leq \frac{\Psi(B)}{\lambda^2} e^{\lambda \| \mathfrak{G} \|_\infty^{1/2}(x+y)}, \; (x,y) \in \Delta. \tag{3.25}$$

By (3.25) and (3.24),

$$\Psi(T(B)) \leq \frac{1}{\lambda^2} \Psi(B). \tag{3.26}$$

The above inequality implies that T is a Ψ contraction if $\lambda > 1$.

Next, we will show that the graph of T

$$\mathrm{Gr}T := \{(u,v) \in \mathfrak{U} \times \mathfrak{U} : \; v \in T(u)\}$$

is closed in $C(\Delta, E) \times C(\Delta, E)$.

To this end, let $(u_n, v_n) \in \mathrm{Gr}T$ $(n = 1,2,\ldots)$ and assume that

$$(u_n, v_n) \underset{n\to\infty}{\to} (u,v) \text{ in } \mathfrak{U} \times \mathfrak{U} \subset C(\Delta, E) \times C(\Delta, E). \tag{3.27}$$

We know that functions u_n and v_n $(n = 1,2,\ldots)$ satisfy the following conditions:

$$\begin{aligned} &u_n(x,0) + u_n(0,y) \in Z, \quad (x,y) \in \Delta \quad (n = 1,2,\ldots), \\ &v_n(x,0) + v_n(0,y) \in Z, \quad (x,y) \in \Delta \quad (n = 1,2,\ldots), \end{aligned} \tag{3.28}$$

$$\begin{aligned} v_n(x,y) = {} & \alpha(x,y) - \sum_{j=1}^r h_j(x) u_n(x, b_j) - \sum_{i=1}^p k_i(y) u_n(a_i, y) \\ & + \int_0^x \int_0^y g_n(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Delta \quad (n = 1,2,\ldots), \end{aligned} \tag{3.29}$$

with $g_n \in L^1(\Delta, E)$, $g_n(\xi, \eta) \in \widehat{F}(\xi, \eta, u_n(\xi, \eta))$ $(n = 1, 2, \ldots)$ a.e. in Δ.

Since $\widehat{F}(\xi, \eta, \cdot)$ is upper semicontinuous from E into E_w with values in $P_{kc}(E)$, then, applying Theorem 7.4.2 from [10],

$$(\xi, \eta) \rightarrow \overline{\text{conv}} \bigcup_{n \in \mathbb{N}} \widehat{F}(\xi, \eta, u_n(\xi, \eta)) = : G(\xi, \eta)$$

is a measurable, $P_{wkc}(E)$-valued multifunction such that

$$|\, G(\xi, \eta)\,| : = \sup\{ \|\, s \,\| : s \in G(\xi, \eta)\} \leq c_3(\xi, \eta) \;\text{ a.e. in } \Delta.$$

Consequently, from [11], we have that

$$\mathcal{S}^1_G : = \{ g \in L^1(\Delta, E) : g(\xi, \eta) \in G(\xi, \eta) \text{ a.e. in } \Delta\}$$

is weakly compact in $L^1(\Delta, E)$. Thus, passing to a subsequence if necessary, we may assume that

$$g_n \underset{n \to \infty}{\overset{w}{\rightarrow}} g \text{ in } L^1(\Delta, E).$$

By the fact that $z \rightarrow \widehat{F}(x, y, z)$ is upper semicontinuous from E into E_w, by the convergence of u_n to u in $C(\Delta, E)$ and by an argument given by Papageorgiou in [12, 13],

$$g(\xi, \eta) \in \widehat{F}(\xi, \eta, u(\xi, \eta)) \text{ a.e. in } \Delta.$$

Therefore, from (3.27) and (3.29),

$$v(x, y) = \alpha(x, y) - \sum_{j=1}^{r} h_j(x) u(x, b_j) - \sum_{i=1}^{p} k_i(y) u(a_i, y)$$
$$+ \int_0^x \int_0^y g(\xi, \eta) d\xi d\eta, \qquad (x, y) \in \Delta,$$

with $g \in L^1(\Delta, E)$ and $g(\xi, \eta) \in \widehat{F}(\xi, \eta, u(\xi, \eta))$ a.e. in Δ.

Moreover, by (3.27) and (3.28),

$$u(x, 0) + u(0, y) \in Z \text{ for } (x, y) \in \Delta$$

and

$$v(x, 0) + v(0, y) \in Z \text{ for } (x, y) \in \Delta.$$

So $(u, v) \in \text{Gr}T$ and, consequently, T has a closed graph in $\mathcal{U} \times \mathcal{U} \subset C(\Delta, E) \times C(\Delta, E)$.

Inequality (3.26) and the fact that $\text{Gr}T$ is closed imply, by the Tarafdar-Vyborny theorem (see [14], Theorem 4.1), that $u \in T(u)$.

As in the beginning of the proof of Theorem 3.1, applying the definition of $\widehat{F}$ and Gronwall's inequality, we obtain that

$$\|\, u \,\|_{C(\Delta, E)} \leq M, \tag{3.30}$$

where M is given by formula (3.10).

Therefore,

$$\widehat{F}(x,y,u(x,y)) = F(x,y,u(x,y)), \quad (x,y) \in \Delta. \tag{3.31}$$

Consequently, $u \in K(\Delta, E)$ is a solution of problem (2.2).
The proof of Theorem 3.1 is complete.

4. Theorem About the Existence of a Solution of the Nonlocal Multivalued Darboux Problem with the Nonconvex Valued Orientor Field

Theorem 4.1: *Suppose that $F: \Delta \times E \to P_f(E)$ is a multifunction such that:*

(a) *$(x,y,z) \to F(x,y,z)$ is measurable, $z \to F(x,y,z)$ is lower semicontinuous and for all $(x,y) \in \Delta$, $F(x,y,\cdot)$ maps the bounded sets into relatively weakly compact sets,*

(b) *assumptions (ii)-(iv) of Theorem 3.1 are satisfied.*

Then in a class of functions $w \in K(\Delta, E)$ problem (2.2) possesses a solution.

Proof: As in the proof of Theorem 3.1, if $u \in C(\Delta, E)$ is a solution of problem (2.2) then

$$\| u(x,y) \| \leq M, \quad (x,y) \in \Delta,$$

where M is given by (3.10).

Define $\widehat{F}: \Delta \times E \to P_f(E)$ by formula (3.11). Consequently,

$$\widehat{F}(x,y,z) = F(x,y,p_M(z)) \text{ in } \Delta \times E,$$

where p_M in the M-radial retraction in E, and Theorem 7.3.11 from [10] implies that $z \to \widehat{F}(x,y,z)$ is lower semicontinuous.

Introduce a set $\mathcal{U}$ by the formula

$$\mathcal{U}: = \{u \in C(\Delta, E): u(x,0) + u(0,y) \in Z, \quad (x,y) \in \Delta,$$

$$u(x,y) = \alpha(x,y) - \sum_{j=1}^{r} h_j(x)u(x,b_j) - \sum_{i=1}^{p} k_i(y)u(a_i,y)$$

$$+ \int_0^x \int_0^y g(\xi,\eta)d\xi d\eta, \quad (x,y) \in \Delta,$$

$$\| g(\xi,\eta) \| \leq c_3(x,y) \text{ a.e. in } \Delta\}. \tag{4.1}$$

The above set is a nonempty, bounded, closed an equicontinuous subset of $C(\Delta, E)$.

Let $\Gamma: \mathcal{U} \to P_f(L^1(\Delta, E))$ be the multifunction defined by

$$\Gamma(u): = S^1_{\widehat{F}(\cdot,\cdot,u(\cdot,\cdot))} = \left\{g \in L^1(\Delta,E): g(\xi,\eta) \in \widehat{F}(\xi,\eta,u(\xi,\eta)), (\xi,\eta) \in \Delta\right\},$$

$$u \in \mathfrak{U}.$$

The Papageorgiou theorem (see [12], Theorem 4.1) implies that Γ is lower semicontinuous. Therefore, from the Bressan-Colombo theorem (see [2], Theorem 3), there is a continuous mapping $\gamma: \mathfrak{U} \to L^1(\Delta, E)$ such that $\gamma(u) \in \Gamma(u)$ for all $u \in \mathfrak{U}$.

Let

$$\mu(u)(x,y) := \alpha(x,y) - \sum_{j=1}^{r} h_j(x)u(x,b_j) - \sum_{i=1}^{p} k_i(y)u(a_i,y) + \int_0^x \int_0^y \gamma(u)(\xi,\eta)d\xi d\eta, \quad u \in \mathfrak{U}, \ (x,y) \in \Delta. \tag{4.2}$$

Then $\mu: \mathfrak{U} \to \mathfrak{U}$ and, by the continuity of γ, μ is continuous.

Let B be a nonempty, bounded and closed subset of $\mathfrak{U}$. Moreover, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $\{u_n\}_{n \in \mathbb{N}} \subset B$ and $\overline{\{u_n\}_{n \in \mathbb{N}}}^{C(\Delta, E)} = B$. We have, from (2.1), (4.2), (4.1), (3.18) and (3.2), that

$$\beta(\mu(B)(x,y)) \leq \beta\left(\int_0^x \int_0^y \gamma(B)(\xi,\eta)d\xi d\eta\right)$$
$$\leq \beta\left(\int_0^x \int_0^y \gamma(\{u_n\}_{n \in \mathbb{N}})(\xi,\eta)d\xi d\eta\right) \leq \int_0^x \int_0^y \| \mathfrak{G} \|_\infty \beta(\{u_n(\xi,\eta)\}_{n \in \mathbb{N}})d\xi d\eta$$
$$= \int_0^x \int_0^y \| \mathfrak{G} \|_\infty \beta(B(\xi,\eta))d\xi d\eta, \quad (x,y) \in \Delta. \tag{4.3}$$

Define the sublinear measure of noncompactness Ψ by formula (3.24). Arguing as in the proof of Theorem 3.1, we obtain

$$\Psi(\mu(B)) \leq \frac{1}{\lambda^2}\Psi(B). \tag{4.4}$$

The above inequality implies that μ is a Ψ contraction if $\lambda > 1$.

Inequality (4.4) and the Tarafdar-Vyborny theorem (see [14], Theorem 4.1) imply that $u = \mu(u)$ for some $u \in \mathfrak{U}$.

As in the beginning of the proof of Theorem 3.1, we obtain inequality (3.30). Therefore, (3.31) holds and $u \in K(\Delta, E)$ is a solution of problem (2.2).

The proof of Theorem 4.1 is complete.

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