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QUASI-RETRACTIVE REPRESENTATION OF SOLUTION SETS TO STOCHASTIC INCLUSIONS

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Continuous dependence and retraction properties of solution sets to stochastic inclusions $x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau)\widetilde{\nu}\,(d\tau, dr)$ are considered.

Key words: Stochastic Inclusions, Lower Semicontinuous Dependence of Solution Set, Retraction Theorem.

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1. Introduction

Properties of solution sets to stochastic inclusions play a crucial role in stochastic optimal control theory. The first results dealing with this topic are given in the author's paper [4], in which, by rather strong assumptions the weak compactness of the set of all solutions to stochastic inclusions

$$x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau)\widetilde{\nu}\,(d\tau, dr)$$

has been obtained. In the present paper, we show that for a given random variable λ, the solution set $\mathcal{C}_\lambda$ to an initial value problem

$$x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau)\widetilde{\nu}\,(d\tau, dr), x_0 = \lambda,$$

has quasi-retractive representation. As a result, we obtain lower semicontinuous dependence of solution set $\mathcal{C}_\lambda$ on an initial date.

We begin with basic notations dealing with set-valued stochastic integrals. Some properties of fixed point sets to subtrajectory integral mappings are investigated. Hence, the main results of this paper readily follow.

2. Basic Definitions and Notations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq}, P)$ be a complete, filtered probability space. Given $T > 0$, let $I = [0, T]$ and let $\mathcal{B}(I)$ denote the Borel σ-algebra on I. We consider set-valued stochastic processes $(F_t)_{t \in I}$, $(\mathcal{G}_t)_{t \in I}$, and $(\mathcal{R}_{t,r})_{t \in I, r \in \mathbb{R}^n}$, taking on values from the space $\text{Conv}(\mathbb{R}^n)$ of all nonempty, compact convex subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. These processes are assumed to be nonanticipative such that $\int_0^T \| F_t \|^2 dt < \infty$; $\int_0^T \| \mathcal{G}_t \|^s dt < \infty$; and $\int_0^T \int_{\mathbb{R}^n} \| \mathcal{R}_{t,z} \|^2 dtq(dz) < \infty$, a.s., where q is a measure on a Borel σ-algebra $\mathcal{B}^n$ of $\mathbb{R}^n$, $A \in \text{Conv}(\mathbb{R}^n)$, and $\| A \| := \sup\{ |a| : a \in A\}$. The space $\text{Conv}(\mathbb{R}^n)$ is endowed with the Hausdorff metric h defined in the usual way (i.e., $h(A,B) = \max\{\bar{h}(A,B), \bar{h}(B,A)\}$, for $A, B \in \text{Conv}(\mathbb{R}^n)$, where $\bar{h}(A,B) = \{\text{dist}(a,B): a \in A\}$ and $\bar{h}(B,A) = \{\text{dist}(b,A): b \in B\}$). $\text{Cl}(X)$ denotes the family of all nonempty closed subsets of a metric space (X, ρ).

Filtered, complete probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ are assumed to satisfy the usual hypotheses: (i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$; and (ii) $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$, all $t, 0 \leq t < \infty$. As usual, we shall consider a set $I \times \Omega$ as a measurable space with the product σ-algebra $\mathcal{B}(I) \otimes \mathcal{F}$.

$(X_t)_{t \in I}$ denotes an n-dimensional stochastic process x, understood as a function $x: I \times \Omega \to \mathbb{R}^n$ with $\mathcal{F}$-measurable sections x_t, each $t \in I$. This process is measurable if x is $\mathcal{B}(I) \otimes \mathcal{F}$-measurable. The process $(x_t)_{t \in I}$ is $\mathcal{F}_t$-*adapted* or *adapted* if x_t is $\mathcal{F}_t$-measurable for $t \in I$. Every measurable and adapted process is called *nonanticipative.*

The Banach spaces $L^2(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ and $L^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$, with the usual norm $\| \cdot \|_{L^2_n}$, are denoted by $L^2_n(\mathcal{F}_t)$ and $L^2_n(\mathcal{F})$, respectively. $\mathcal{M}^2(\mathcal{F}_t)$ denotes the family (i.e., equivalence classes) of all n-dimensional nonanticipative processes $(f_t)_{t \in I}$ such that $\int_0^T | f_t |^2 dt < \infty$, a.s. We shall also consider a subspace $\mathcal{L}^2$ of $\mathcal{M}^2(\mathcal{F}_t)$ defined by $\mathcal{L}^2 = \{(f_t)_{t \in 0} \in \mathcal{M}^2(\mathcal{F}_t): | f |_{\mathcal{L}^2} < \infty\}$, with $| f |^2_{\mathcal{L}^2} = E \int_0^T | f_t |^2 dt$. Finally, $M_n(\mathcal{F}_t)$ we denote the space (i.e., equivalence classes) of all n-dimensional $\mathcal{F}_t$-measurable mappings.

$(w_t)_{t \in I}$ defines a one-dimensional $\mathcal{F}_t$-Brownian motion starting at 0. $\nu(t, A)$ denotes a $\mathcal{F}_t$-Poisson measure on $I \times \mathcal{B}^n$. We define a $\mathcal{F}_t$-centered Poisson measure $\tilde{\nu}(t, A)$, $t \in I$, $A \in \mathcal{B}^n$ by taking $\tilde{\nu}(t, A) = \nu(t, A) - tq(A)$, $t \in I$, $A \in \mathcal{B}^n$, where q is a measure on $\mathcal{B}^n$ such that $E\nu(t, B) = tq(B)$ and $q(B) < \infty$ for $B \in \mathcal{B}^n_0 := \{A \in \mathcal{B}^n: 0 \notin \bar{A}\}$.

$\mathcal{M}^2(\mathcal{F}_t, q)$ denotes the family (i.e., equivalence classes) of all $\mathcal{B}(I) \otimes \mathcal{F} \otimes \mathcal{B}^n$-measurable and $\mathcal{F}_t$-adapted functions $h: I \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\int_0^T \int_{\mathbb{R}^n} | h_{t,r} |^2 dtq(dr) < \infty$, a.s. Recall, a function $h: I \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mathcal{F}_t$-adapted or adapted if $h(t, \cdot, r)$ is $\mathcal{F}_t$-measurable for every $r \in \mathbb{R}^n$ and $t \in I$. Elements of $\mathcal{M}^2(\mathcal{F}_t, q)$ will be denoted by $h = (h_{t,r})_{t \in I, r \in \mathbb{R}^n}$. Finally, we let $\mathcal{W}^2_n = \{h \in \mathcal{M}^2(\mathcal{F}_t, q): | h |^2_{\mathcal{W}^2_n} < \infty\}$, where $| h |^2_{\mathcal{W}^2_n} = E \int_0^T \int_{\mathbb{R}^n} | h_{t,r} |^2 dtq(dr)$.

Given $f, g \in \mathcal{M}^2(\mathcal{F}_t)$ and $h \in \mathcal{M}^2(\mathcal{F}_t, q)$, $(\int_0^t f_\tau d\tau)_{t \in I}$, $(\int_0^t g_\tau dw_\tau)t \in I$, and

$(\int_0^t \int_{\mathbb{R}^n} h_{\tau,r}\tilde{\nu}(d\tau, dr))_{t\in I}$ denote their stochastic integrals with respect to Lebesgue measure on $\mathbb{R}^+$, the $\mathcal{F}_t$-Brownian motion $(w_t)_{t\in I}$, and the $\mathcal{F}_t$-centered Poisson measure $\tilde{\nu}(t, A), t \in I, A \in \mathcal{B}^n$, respectively. For fixed $t \in I$ and $(f, g, h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$, we equate $\mathfrak{I}_t(f) = \int_0^t f_\tau d\tau$, $\mathfrak{J}_t(g) = \int_0^t g_\tau dw_\tau$, and $\mathcal{T}_t(h) = \int_0^t \int_{\mathbb{R}^n} h_{\tau,z}\tilde{\nu}(d\tau, dz)$. $\mathfrak{I}, \mathfrak{J}$, and $\mathcal{T}$ denote linear mappings defined by $\mathcal{L}^2 \ni f \to (\mathfrak{I}_t(f))_{t\in I} \in D$, $\mathcal{L}^2 \ni g \to (\mathfrak{J}_t(g))_{t\in I} \in D$, and $\mathcal{W}^2 \ni h \to (\mathcal{T}_t(h))_{t\in I} \in D$, respectively. Here, D denotes the family of all n-dimensional $\mathcal{F}_t$-adapted cádlág (see [7]) processes $(x_t)_{t\in I}$ such that $E\sup_{t\in I}|x_t|^2 < \infty$. The space D is considered a normed space with norm $\|\xi\|_\ell = \|\sup_{t\in I}|\xi_t|\|_{L_1^2}$ for $\xi = (\xi_t)_{t\in I} \in D$. It can be verified that $(D, \|\cdot\|_\ell)$ is a Banach space.

Given a measure space $(X, \mathcal{B}, m)$, a set-valued function $\mathfrak{R}: X \to \mathrm{Cl}(\mathbb{R}^n)$ is said to be $\mathcal{B}$-measurable if $\{x \in X: \mathfrak{R}(x) \cap C \neq \emptyset\} \in \mathcal{B}$ for every closed set $C \subset \mathbb{R}^n$. For such a multifunction, we define subtrajectory integrals as a set $\mathcal{S}(\mathfrak{R}) = \{g \in L^2(X, \mathcal{B}, m, \mathbb{R}^n): g(x) \in \mathfrak{R}(x) \text{ a.e.}\}$. We shall assume that the $\mathcal{B}$-measurable, set-valued function $\mathfrak{R}: X \to \mathrm{Cl}(\mathbb{R}^n)$ is square integrable bounded (i.e., a real-valued mapping $X \ni x \to \|\mathfrak{R}(x)\| \in \mathbb{R}_+$ belongs to $L^2(X, \mathcal{B}, m, \mathbb{R})$).

Let $\mathcal{G} = (\mathcal{G}_t)_{t\in I}$ be *a set-valued stochastic process* with values in $\mathrm{Cl}\mathbb{R}^n$), (i.e., a family of $\mathcal{F}$-measurable set-valued mappings $\mathcal{G}_t: \Omega \to \mathrm{Cl}(\mathbb{R}^n)$, $t \in I$). We call $\mathcal{G}$ *measurable* if it is $\mathcal{B}(I) \otimes \mathcal{F}$-measurable. Similarly, $\mathcal{G}$ is said to be $\mathcal{F}_t$*-adapted* or *adapted* if $\mathcal{G}_t$ is $\mathcal{F}_t$-measurable for each $t \in I$. A measurable and adapted set-valued stochastic process is called *nonanticipative.*

We shall also consider $\mathcal{B}(I) \otimes \mathcal{F} \otimes \mathcal{B}^n$-measurable set-valued mappings $\mathcal{R}: I \times \Omega \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$. These mappings will be denoted by $(\mathcal{R}_{t,r})_{t\in I, r\in\mathbb{R}^n}$, and called measurable *set-valued stochastic processes depending on a parameter* $r \in \mathbb{R}^n$. The process $\mathcal{R} = (\mathcal{R}_{t,zr})_{t\in I, r\in\mathbb{R}^n}$ is said to be $\mathcal{F}_t$*-adapted* or *adapted* if $\mathcal{R}_{t,r}$ is $\mathcal{F}_t$-measurable for each $t \in I$ and $z \in \mathbb{R}^n$. We call this process *nonanticipative* if it is measurable and adapted.

$\mathcal{M}^2_{s-v}(\mathcal{F}_t)$ and $\mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$ denote families of all nonanticipative set-valued processes $\mathcal{G} = (\mathcal{G}_t)_{t\in I}$ and $\mathcal{R} = (\mathcal{R}_{t,r})_{t\in I, r\in\mathbb{R}^n}$, respectively, such that $\int_0^T \|\mathcal{G}_t\|^2 dt < \infty$ and $\int_0^T \int_{\mathbb{R}^n} \|\mathcal{R}_{t,r}\|^2 dt q(dr) < \infty$, a.s. From Kuratowski and Ryll-Nardzewski measurable selection theorem (see [3]) it immediately follows that for every $F, \mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$ and $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{F}_{t,q})$, their subtrajectory integrals $\mathcal{S}(F) := \{f \in \mathcal{M}^2(\mathcal{F}_t): f_t(\omega) \in F_t(\omega), dt \times P\text{-a.e.}\}$, $\mathcal{S}(\mathcal{G}) := \{g \in \mathcal{M}^2(\mathcal{F}_t): g_t(\omega) \in \mathcal{G}_t(\omega),\ dt \times P\text{-a.e.}\}$, and $\mathcal{S}_q(\mathcal{R}) := \{h \in \mathcal{M}^2(\mathcal{F}_t, q): h_{t,r}(\omega) \in \mathcal{R}_{t,r}(\omega), dt \times P \times q\text{-a.e.}\}$ are nonempty. Indeed, we let $\Sigma = \{Z \in \mathcal{B}(I) \otimes \mathcal{F}: Z_t \in \mathcal{F}_t, \text{ each } t \in I\}$, where Z_t denotes a section of Z determined by $t \in I$. Σ is a σ-algebra on $I \times \Omega$, and a function $f: I \times \Omega \to \mathbb{R}^n$ (a multifunction $F: I \times \Omega \to \mathrm{Cl}(\mathbb{R}^n)$) is nonanticipative if and only if it is Σ-measurable. Therefore, by Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$, such selectors belong to $\mathcal{M}^2(\mathcal{F}_t)$. Similarly, we define on $I \times \Omega \times \mathbb{R}^n$ a σ-algebra $\Sigma = \{Z \in \mathcal{B}(I) \otimes \mathcal{F} \otimes \mathcal{B}^n: Z^u_t \in \mathcal{F}_t, \text{ each } t \in I \text{ and } u \in \mathbb{R}^n\}$, where $Z^u_t = (Z^u)_t$, and Z^u denotes a section of Z determined by $u \in \mathbb{R}^n$. The foregoing arguments can be repeated to obtain the above result for nonanticipative, set-valued processes depending on a parameter $r \in \mathbb{R}^n$.

It can be verified (see [2, 3]) that for given $F = (\mathcal{F}_t)_{t \in I} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$, $\mathcal{G} = (\mathcal{G}_t)_{t \in I} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$, and $\mathcal{R} = (\mathcal{R}_{t,r})_{t \in t, r \in \mathbb{R}^n} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$, their stochastic integrals are defined as families $(\int_0^t F_\tau d\tau)_{t \in I}, (\int_0^t \mathcal{G}_\tau dw_\tau)_{t \in I}$, and $(\int_0^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,z} \widetilde{\nu}(d\tau, dz))_{t \in I}$ of the subsets of $M(\mathcal{F}_t)$, of the form $\int_0^t \mathcal{F}_\tau d\tau = \{\int_0^t f_\tau d\tau: f \in \mathcal{S}(F)\}$, $\int_0^t \mathcal{G}_\tau dw_\tau = \{\int_0^t g_\tau dw_\tau : g \in \mathcal{S}^2(\mathcal{G})\}$ and $\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \widetilde{\nu}(d\tau, dz) : h \in \mathcal{S}_q(\mathcal{R})\}$. Given $0 \leq \alpha < \beta < \infty$, we also define $\int_\alpha^\beta F_s ds := \{\int_\alpha^\beta f_s ds : f \in \mathcal{S}^p(F)\}$, $\int_\alpha^\beta \mathcal{G}_s dw_s := \{\int_\alpha^\beta g_s dw_s : g \in \mathcal{S}^2(\mathcal{G})\}$, and $\int_\alpha^\beta \int_{\mathbb{R}^n} \mathcal{R}_{s,r} \widetilde{\nu}(ds, dz) := \{\int_\alpha^\beta \int_{\mathbb{R}^n} h_{s,r} \widetilde{\nu}(ds, dz) : h \in \mathcal{S}_q(\mathcal{R})\}$.

The following selection property of set-valued stochastic integrals has been obtained in [5]:

Proposition 1. *Let* $F, \mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t), \mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$, *and* $(x_t)_{t \in I} \in D$. *Then:*

$$x_t - x_s \in \int_s^t F_\tau d\tau + \int_s^t G_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,r} \widetilde{\nu}(d\tau, dr)$$

for $0 \leq s \leq t \leq T$ *if and only if there exists* $(f, g, h) \in \mathcal{S}(\mathcal{F}) \times \mathcal{S}(\mathcal{G}) \times \mathcal{S}_q(\mathcal{R})$ *such that*

$$x_t = \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,r} \widetilde{\nu}(d\tau, dr)$$

for $t \in I$.

3. Stochastic Inclusions and Subtrajectory Integrals Depending on Parameters

Let

$$F = \{(F_t(x))_{t \in I} : x \in \mathbb{R}^n\},\ G = \{(G_t(x))_{t \in I} : x \in \mathbb{R}^n\},$$
$$\text{and } H = \{(H_{t,r}(x))_{t \in I, r \in \mathbb{R}^n} : x \in \mathbb{R}^n\}.$$

Assume F, G, and H are such that $(F_t(x))_{t \in I} \in \mathcal{M}^p_{s-v}(\mathcal{F}_t)$, $(G_t(x))_{t \in I} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$, and $(H_{t,r}(x))_{t \in I, r \subset n\mathbb{R}^n} \in \mathcal{M}^2_{s-v}(\mathcal{F}^n_t, q)$, $x \in \mathbb{R}^n$. A stochastic inclusion denoted by $SI(F, G, H)$, corresponding to the aforementioned F, G, and H is the relation:

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau) \widetilde{\nu}(d\tau, dz),$$

which is satisfied for every $0 \leq s < t < T$ by a stochastic process $x = (x_t)_{t \in I} \in D$ such that $F \circ x \in \mathcal{M}^2_{s-v}(\mathfrak{F}_t), G \circ x \in \mathcal{M}^2_{s-v}(\mathfrak{F}_t)$, and $H \circ x \in \mathcal{M}^2_{s-v}(\mathfrak{F}_t, q)$, where $F \circ x = (F_t(x_t))_{t \in I}, G \circ x = (G_t(x_t))_{t \in I}$, and $H \circ x = (H_{t,r}(x_t))_{t \in I, r \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \in I} \in D$, satisfying conditions mentioned above, is said to be a global solution to $SI(F,G,H)$. Given $\lambda \in L^2_n(\mathfrak{F}_0)$ we shall consider $SI(F,G,H)$ together with an initial value condition $x_0 = \lambda$. This type of initial value problem will be denoted by $SI_\lambda(F,G,H)$.

We shall assume that F, G, and H satisfy the following condition:

$(\mathcal{A}_1)$: (*i*) $F = \{(F_t(x))_{t \in I}: x \in \mathbb{R}^n\}, G = \{(G_t(x))_{t \in I}: x \in \mathbb{R}^n$, and $H = \{(H_{t,r}(x))_{t \in I, r \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$, such that mappings $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \to F_t(x)(\omega) \in \text{Conv}(\mathbb{R}^n)$, $I \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \to G_t(x)(\omega) \in \text{conv}(\mathbb{R}^n)$, and $I \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, \omega, r, x) \to H_{t,r}(x)(\omega) \in \text{Conv}(\mathbb{R}^n)$ are $\Sigma \otimes \mathfrak{B}^n$ and $\widetilde{\Sigma} \otimes \mathfrak{B}^n$-measurable, respectively;

(*ii*) $(F_t(x))_{t \in I}, (G_t(x))_{t \in I}$, and $(H_{x,r}(x))_{t \in I, r \in \mathbb{R}^n}$ are uniformly square integrable bounded (i.e., functions $(t, \omega) \to \sup_{x \in \mathbb{R}^n} \| F_t(x)(\omega) \| \in \mathbb{R}^+$, $(t, \omega) \to \sup_{x \in \mathbb{R}^n} \| G^t(x)(\omega) \| \in \mathbb{R}^+$, and $(t, \omega, r) \to \sup_{x \in \mathbb{R}^n} \| H_{t,r}(x)(\omega) \| \in \mathbb{R}^+$) are square integrable on $\mathbb{R}^+ \times \Omega$ and $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n$, respectively.

We denote $B_F = \{u \in \mathcal{L}^2: | u_t | \leq \sup_{x \in \mathbb{R}^n} \| F_t(x) \|$ a.e. on $I \times \Omega\}$, $B_G = \{u \in \mathcal{L}^2: | v_t | \leq \sup_{x \in \mathbb{R}^n} \| G_t(x) \|$ a.e. on $I \times \Omega\}$, and $B_H = \{z \in \mathcal{W}^2: \; | z_{t,r} | \leq \sup_{x \in \mathbb{R}^n} \| H_{t,r}(x) \|$ a.e. on $I \times \Omega \times \mathbb{R}^n\}$. Then we define $B = B_F \times B_G \times B_H$.

Corollary 1. *If F, G, H satisfy $(\mathcal{A}_1)$, then B is a nonempty convex and weakly compact subset of $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. Moreover, for every $(x_t)_{t \in I} \in D$, one has $F \circ x$, $G \circ x \in \mathcal{M}^2_{s-v}(\mathfrak{F}_t)$ and $H \circ x \in \mathcal{M}^2_{s-v}(\mathfrak{F}^n_t, q)$.*

Let Φ be a linear mapping on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ defined by $\Phi = \mathfrak{I} + \mathfrak{J} + \mathfrak{T}$, (i.e., $\Phi(f,g,h) = (\mathfrak{I}_t f + \mathfrak{J}_t g + \mathfrak{T}_g h)_{t \in I}$ for $(f,g,h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$). For fixed $\lambda \in L^2_n(\mathfrak{F}_0)$, Φ^λ denotes an affine mapping on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ defined by $\Phi^\lambda(u,v,r) = \lambda + \Phi(f,g,h)$ for $(f,g,h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. Given F, G, and H and $\lambda \in L^2_n(\mathfrak{F}_0)$, we set

$$\mathcal{H}_\lambda(x) = \Phi^\lambda(\mathcal{S}(F \circ x) \times \mathcal{S}(G \circ x) \times \mathcal{S}_q(H \circ x))$$

for $x = (x_t)_{t \in I} \in D$. It can be verified (see [4, 5]) that for every $x \in D$, $\mathcal{H}_\lambda(x)$ is a convex, weakly compact subset of D. $\mathcal{H}_\lambda$ denotes a set-valued mapping $D \ni x \to \mathcal{H}_\lambda(x) \subset D$. From Proposition 1, it immediately follows that for every $\lambda \in L^2_n(\mathfrak{F}_0)$, and F, G, and H satisfying condition $(\mathcal{A}_1)$, $x \in D$ is a solution to $SI_\lambda(F,G,h)$ if and only if x is a fixed point to $\mathcal{H}_\lambda$.

Suppose $F = \{(F_t(x))_{t \in I}: x \in \mathbb{R}^n\}, G = \{(G_t(x))_{t \in I}: x \in \mathbb{R}^n\}$, and $H\{(H_{t,r}(x))_{t \in I, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$ satisfy condition $(\mathcal{A}_1)$ and the following condition

$(\mathcal{A}_2)$ There are $k_F, k_G \in L^2_1(\mathfrak{B}(I))$ and $m \in L^2_1(\mathfrak{B}(I) \times \mathfrak{B}^n)$ such that $h(F_t(x_2), F_t(x_1)) \leq k_F(t) \, | x_2 - x_1 |$, $h(G_t(x_2), G_t(x_1)) \leq k_G(t) \, | x_2 - x_1 |$, and $h(H_{t,r}(x_2), H_{t,r}(x_1)) \leq m(t,r) \, | x_2 - x_1 |$ a.s., each $t \in I$ and $x_1, x_2 \in \mathbb{R}^n$.

Consider for fixed $\lambda \in L^2_n(\mathfrak{F}_0)$ a subtrajectory integrals mapping S_λ defined by:

$$S_\lambda(u,v,r) = \mathcal{S}(F\circ\Phi^\lambda(u,v,r)) \times \mathcal{S}(G\circ\Phi^\lambda(u,v,r)) \times \mathcal{S}_q(H\circ\Phi^\lambda u,v,r))$$

for $(u,v,r) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. It is clear that for $\lambda \in L^2_n(\mathcal{F}_0)$, $(u,v,r) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$; and F, G, and H satisfying condition $(\mathcal{A}_1)$, one has $S_\lambda(u,v,r) \in \mathrm{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \mathcal{W}^2)$. We shall show that if condition $(\mathcal{A}_2)$ is satisfied, then it is possible to renorm a space $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ by an equivalent norm $\|\cdot\|$ such that $S_\lambda(\cdot)$ is a contraction from $(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2, \|\cdot\|)$ into $(\mathrm{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2), \ell)$, where ℓ is the Hausdorff metric induced by $\|\cdot\|$. A similar result is also true for $S_{.}(u,v,r)\colon L^2_n(\mathcal{F}_0) \ni \lambda \to S_\lambda(u,v,z) \in \mathrm{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2)$. Observe that a norm $\|\cdot\|$ is defined by $\|(u,v,r)\| = \max(\|u\|_{\mathcal{L}^2}, \|v\|_{\mathcal{L}^2}, \|z\|_{\mathcal{W}^2})$, where $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ are appropriate norms on $\mathcal{L}^2$ and $\mathcal{W}^2$ equivalent to $|\cdot|_{\mathcal{L}^2}$ and $|\cdot|_{\mathcal{W}^2}$ defined above.

Finally, observe that for every $A, \widetilde{A}, B, \widetilde{B} \in \mathrm{Cl}(\mathcal{L}^2)$ and $C, \widetilde{C} \in \mathrm{Cl}(\mathcal{W}^2)$ one has:

$$\ell(A\times B\times C, \widetilde{A}\times\widetilde{B}\times\widetilde{C}) \le \max\{\ell_{\mathcal{L}^2}(A,\widetilde{A}), \ell_{\mathcal{L}^2}(B,\widetilde{B}), \ell_{\mathcal{W}^2}(C,\widetilde{C})\}$$

where $\ell_{\mathcal{L}^2}$ and $\ell_{\mathcal{W}^2}$ are Hausdorff metrics on $\mathrm{Cl}(\mathcal{L}^2)$ and $\mathrm{Cl}(\mathcal{W}^2)$ induced by the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$, respectively.

Proposition 2. *Suppose F, G, and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. For every $L > 0$, there are norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ on $\mathcal{L}^2$ and $\mathcal{W}^2$ equivalent to $|\cdot|_{\mathcal{L}^2}$ and $|\cdot|_{\mathcal{W}^2}$, respectively, such that:*

$$\ell_{\mathcal{L}_2}(S(F\circ\Phi^\lambda(u,v,r)), S(F\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$$
$$\le L\,max(\|u-\widetilde{u}\|_{\mathcal{L}^2}, \|v-\widetilde{v}\|_{\mathcal{L}^2}, \|z=\widetilde{z}\|_{\mathcal{W}^2}),$$
$$\ell_{\mathcal{L}^2}(S(G\circ\Phi^\lambda(u,v,r)), S(G\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$$
$$\le L\,max(\|u-\widetilde{u}\|_{\mathcal{L}^2}, \|v-\widetilde{v}\|_{\mathcal{L}^2}, \|z-\widetilde{z}\|_{\mathcal{W}^2}),$$

and

$$\ell_{\mathcal{L}^2}(S_q(H\circ\Phi^\lambda(u,v,r)), S_q(H\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$$
$$\le L\,max(\|u-\widetilde{u}\|_{\mathcal{L}^2}, \|v-\widetilde{v}\|_{\mathcal{L}^2}, \|z-\widetilde{z}\|_{\mathcal{W}^2}),$$

for (u,v,r), $(\widetilde{u},\widetilde{v},\widetilde{z}) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$.

Proof. Let $L > 0$ be given and fix (u,v,z), $(\widetilde{u},\widetilde{v},\widetilde{z}) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. For every $f \in S(F\circ\Phi^\lambda_t(u,v,z))$, there is $\widetilde{f} \in S(F\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$ such that:

$$|f_t - \widetilde{f}_t) \le h(F_t(\Phi^\lambda_t(u,v,z)), F_t(\Phi^\lambda_t(\widetilde{u},\widetilde{v},\widetilde{z})) \le k_F(t)\,|\Phi^\lambda_t(u,v,z) - \Phi^\lambda_t(\widetilde{u},\widetilde{v},\widetilde{z})|$$
$$\le k_F(t)\left\{\int_0^t |u_r - \widetilde{u}_r|\,d\tau + \left|\int_0^t [v_r - \widetilde{v}_r]d\omega_\tau\right| + \left|\int_0^t\int_{\mathbb{R}^m}[z_{\tau,r} - \widetilde{z}_{\tau,r}]\widetilde{v}\,(d\tau,dr)\right|\right\}$$

a.s., each $t \in I$. Similarly, for every $g \in S(G\circ\Phi^\lambda(u,v,z))$ and $h \in S_q(H\circ\Phi^\lambda(u,v,z))$, there are $\widetilde{g} \in S(G\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$ and $\widetilde{h} \in S_q(H\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$ such that:

$$| g_t - \widetilde{g}_t |$$

$$\leq k_G(t)\left\{ \int_0^t | u_\tau - \widetilde{u}_\tau | d\tau + \left| \int_0^t [v_\tau - \widetilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}\,(d\tau, dr) \right| \right\}$$

and

$$| h_{t,r} - \widetilde{h}_{t,r} |$$

$$\leq m(t,r)\left\{ \int_0^t | u_r - \widetilde{u}_r | d\tau + \left| \int_0^t [v_r - \widetilde{v}_r] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}\,(d\tau, dr) \right| \right\}$$

a.s., each $t \in I$. Let:

$$\gamma = \max\{(3/L)^2 T, (3/L)^2\}, k^2(t) = \max\{k_F^2(t), k_G^2(t), \int_{\mathbb{R}^m} m^2(t,r) q(dr)\},$$

and $\mathcal{K}(t) = \int_0^t k^2(\tau) d\tau$ for $t \in I$. Let us renorm $\mathcal{L}^2$ and $\mathcal{W}^2$ with equivalent norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ defined by:

$$\| u \|_{\mathcal{L}^2} = \left(E \int_0^T e^{-\gamma \mathcal{K}(t)} | u_t |^2 dt \right)^{1/2}$$

and

$$\| z \|_{\mathcal{W}^2} = \left(E \int_0^T \int_{\mathbb{R}^m} e^{-\gamma \mathcal{K}(t)} | z_{t,r} |^2 q(dr) dt \right)^{1/2}$$

for $u \in \mathcal{L}^2$ and $z \in \mathcal{W}^2$. We obtain:

$$\| f - \widetilde{f} \|_{\mathcal{L}^2} \leq \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left[\int_0^t | u_r - \widetilde{u}_r | d\tau \right]^2 dt \right)^{1/2}$$

$$+ \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left| \int_0^t [v_\tau - \widetilde{v}_\tau] d\omega_\tau \right|^2 dt \right)^{1/2}$$

$$+ \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}\,(d\tau, dr) \right|^2 dt \right)^{1/2}.$$

We have:

$$E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left[\int_0^t | u_\tau - \widetilde{u}_\tau | d\tau \right]^2 dt$$

$$| g_t - \widetilde{g}_t | \leq k_G(t)\left\{ \int_0^t | u_\tau - \widetilde{u}_\tau | d\tau + \left| \int_0^t [v_\tau - \widetilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}(d\tau, dr) \right| \right\}$$

and

$$| h_{t,r} - \widetilde{h}_{t,r} | \leq m(t,r)\left\{ \int_0^t | u_r - \widetilde{u}_r | d\tau + \left| \int_0^t [v_r - \widetilde{v}_r] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}(d\tau, dr) \right| \right\}$$

a.s., each $t \in I$. Let:

$$\gamma = \max\{(3/L)^2 T, (3/L)^2\}, k^2(t) = \max\{k_F^2(t), k_G^2(t), \int_{\mathbb{R}^m} m^2(t,r) q(dr)\},$$

and $\mathcal{K}(t) = \int_0^t k^2(\tau) d\tau$ for $t \in I$. Let us renorm $\mathcal{L}^2$ and $\mathcal{W}^2$ with equivalent norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ defined by:

$$\| u \|_{\mathcal{L}^2} = \left(E \int_0^T e^{-\gamma \mathcal{K}(t)} | u_t |^2 dt \right)^{1/2}$$

and

$$\| z \|_{\mathcal{W}^2} = \left(E \int_0^T \int_{\mathbb{R}^m} e^{-\gamma \mathcal{K}(t)} | z_{t,r} |^2 q(dr) dt \right)^{1/2}$$

for $u \in \mathcal{L}^2$ and $z \in \mathcal{W}^2$. We obtain:

$$\| f - \widetilde{f} \|_{\mathcal{L}^2} \leq \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left[\int_0^t | u_r - \widetilde{u}_r | d\tau \right]^2 dt \right)^{1/2}$$

$$+ \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left| \int_0^t [v_\tau - \widetilde{v}_\tau] d\omega_\tau \right|^2 dt \right)^{1/2}$$

$$+ \left(E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \widetilde{z}_{\tau,r}] \widetilde{v}(d\tau, dr) \right|^2 dt \right)^{1/2}.$$

We have:

$$E \int_0^T k^2(t) e^{-\gamma \mathcal{K}(t)} \left[\int_0^t | u_\tau - \widetilde{u}_\tau | d\tau \right]^2 dt$$

Therefore,

$$\ell_{\mathcal{L}^2}(S(F\circ\Phi^\lambda(u,v,z)),S(F\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z})),$$

$$\leq L\max(\|u-\widetilde{u}\|_{\mathcal{L}^2},\|v-\widetilde{v}\|_{\mathcal{L}^2},\|z-\widetilde{z}\|_{\mathcal{W}^2}.$$

Similarly,

$$\ell_{\mathcal{L}^2}(S(G\circ\Phi^\lambda(u,v,z)),S(G\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z})),$$

$$\leq L\max(\|u-\widetilde{u}\|_{\mathcal{L}^2},\|v-\widetilde{v}\|_{\mathcal{L}^2},\|z-\widetilde{z}\|_{\mathcal{W}^2}.$$

Finally,

$$\|h-\widetilde{h}\|_{\mathcal{W}^2}$$

$$\leq\left(E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t[u_\tau-\widetilde{u}_\tau]d\tau\right|^2 q(d\tau)dt\right)^{1/2}$$

$$+\left(E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t[v_\tau-\widetilde{v}_\tau]d\omega_\tau\right|^2 q(d\tau)dt\right)^{1/2}$$

$$+\left(E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t\int_{\mathbb{R}^m}[z_{\tau,r}-\widetilde{z}_{\tau,r}]\nu(d\tau,dr)\right|^2 q(d\tau)dt\right)^{1/2}.$$

Similarly, as above,

$$E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t[u_\tau-\widetilde{u}_\tau]d\tau\right|^2 q(d\tau)dt$$

$$\leq TE\int_0^T e^{-\gamma\mathcal{K}(t)}\int_0^t|u_\tau-\widetilde{u}_\tau|^2 d\tau dt\leq(L/3)^2\|u-\widetilde{u}\|^2_{\mathcal{L}^2},$$

$$E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t[v_\tau-\widetilde{v}_\tau]d\omega\right|^2 q(d\tau)dt$$

$$\leq E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_0^t|v_\tau-\widetilde{v}_\tau|^2 d\tau dt\leq(L/3)^2\|v-\widetilde{v}\|^2_{\mathcal{L}^2},$$

and

$$E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_{\mathbb{R}^m} m^2(t,r)\left|\int_0^t\int_{\mathbb{R}^m}[z_{\tau,r}-\widetilde{z}_{\tau,r}]\widetilde{\nu}(d\tau,dr)\right|^2 q(d\tau)dt$$

$$\leq E\int_0^T e^{-\gamma\mathcal{K}(t)}\int_0^t\int_{\mathbb{R}^m}|z_{\tau,r}-\widetilde{z}_{\tau,r}|^2q(dr)d\tau dt\leq(L/3)^2\|z-\widetilde{z}\|^2_{\mathcal{W}^2}.$$

Therefore,

$$\ell_{\mathcal{W}^2}(S_q(H\circ\Phi^\lambda(u,v,r)),S_q(H\circ\Phi^\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))$$

$$\leq L\max(\|u-\widetilde{u}\|_{\mathcal{L}^2},\|v-\widetilde{v}\|_{\mathcal{L}^2},\|z-\widetilde{z}\|_{\mathcal{W}^2}.\qquad\square$$

Now we can prove the following basic lemma.

Lemma 1. *Suppose F,G, and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. There is a norm $\|\cdot\|$ on $\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2$ equivalent to the norm defined on $\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2$ by $|\cdot|_{\mathcal{L}^2}$ and $|\cdot|_{\mathcal{W}^2}$ such that $S_\lambda(\cdot)$ and $S_{\cdot}(u,v,z)$ are contractions from $(\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2,\|\cdot\|)$ and $(L^2_n(\mathcal{F}_0),\|\cdot\|_{L^2_n})$, respectively, into $(\mathrm{Cl}(\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2),\ell)$, where ℓ is the Hausdorff metric induced by the norm $\|\cdot\|$.*

Proof. Let $L\in[0,1)$ and $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ be such as in Proposition 2, corresponding to the given L. Set $\|(u,v,z)\|=\max(\|u\|_{\mathcal{L}^2},\|v\|_{\mathcal{L}^2},\|z\|_{\mathcal{W}^2})$ and let ℓ be the Hausdorff metric on $\mathrm{Cl}(\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2)$ induced by the norm $\|\cdot\|$. By Proposition 2, we obtain

$$\ell(S_\lambda(u,v,z),S_\lambda(\widetilde{u},\widetilde{v},\widetilde{z}))\leq L\|(u,v,z)-(\widetilde{u},\widetilde{v},\widetilde{z})\|$$

for $\lambda\in L^2_n(\mathcal{F}_0)$ and (u,v,z), $(\widetilde{u},\widetilde{v},\widetilde{z})\in\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2$. Quite similarly,

$$\ell(S_\lambda(u,v,z),S_{\widetilde{\lambda}}(u,v,z))\leq L\|\lambda-\widetilde{\lambda}\|_{L^2_n}$$

for $\lambda,\widetilde{\lambda}\in L^2_n(\mathcal{F}_0)$ and $(u,v,z)\in\mathcal{L}^2\times\mathcal{L}^2\times\mathcal{W}^2$. $\square$

4. Quasi-Retractive Representation of Solution Set

We shall show that if conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied, then the solution set mapping $\lambda\to\mathcal{C}_\lambda$, where $\mathcal{C}_\lambda$ denotes a set of all solutions to an initial value problem $SI_\lambda(F,G,H)$, has quasi-retractive representation. In particular, it will follow that this mapping is lower semicontinuous. Moreover, it will follow that in some special cases the solution set $\mathcal{C}_\lambda$ is weakly compact in $(D,\|\cdot\|_\ell)$. These results are consequences of Lemma 1 and a general retractive representation theorem presented in [1].

Let Λ be a topological space and $(X,|\cdot|)$ be a Banach space. Denote $\mathcal{N}(X)=\{A\subset X: A\neq\emptyset\}$. Given $S:\Lambda\to\mathcal{N}(X)$ and $C\subset X$, let $S^-(C)=\{\lambda\in\Lambda: S(\lambda)\cap C\neq\emptyset\}$. We say that $S:\Lambda\to\mathcal{N}(X)$ is lower semicontinuous (l.s.c.) [upper semicontinuous (u.s.c.)] if $S^-(C)$ is open [closed] for every open [closed] set $C\subset X$. A set-valued mapping $S:\Lambda\to\mathcal{N}$ is said to be W-upper semicontinuous (W-u.s.c.) if for every $x\in X$ the function $\lambda\to\mathrm{dist}(x,S(\lambda))$ is lower semicontinuous in the usual sense. Finally, S is said to be W-continuous if it is l.s.c. and W-u.s.c.

We say that $S: \Lambda \rightarrow \mathcal{N}(X)$ has a retractive representation if there exists a set $B \in \mathcal{N}(X)$ and a continuous mapping $p: \Lambda \times B \rightarrow B$ such that $p(\lambda, x) \in S(\lambda)$ for every $(\lambda, x) \in \Lambda \times B$ and $p(\lambda, x) = x$ if and only if $x \in S(\lambda)$.

We say that the solution set mapping $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \mathcal{C}_\lambda \subset D$ has quasi-retractive representation if there is a set-valued mapping $S: L_n^2(\mathcal{F}_0) \rightarrow \mathcal{N}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2)$ having a retractive representation $p: \Lambda \times B \rightarrow B$ such that $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$, each $\lambda \in L_n^2(\mathcal{F}_0)$.

We present the following general results (see [1, 8]) dealing with retractive representation of set-valued mappings.

Theorem 2. ([8], Th. 1) *Let Λ be a paracompact and perfectly normal topological space, $(X, |\cdot|)$ be a Banach space, and $B \in \mathrm{Cl}(X)$. Suppose $\mathcal{P}: \Lambda \times B \rightarrow \mathrm{Cl}(X)$ takes on convex values and is such that:*

- (*i*) *for every $x \in B$ the set-valued mapping $\mathcal{P}(\cdot, x)$ is W-continuous,*
- (*ii*) *there is $L \in [0,1)$ such that $h(\mathcal{P}(\lambda, x), \mathcal{P}(\lambda, \tilde{x})) \leq L|x - \tilde{x}|$ for fixed $\lambda \in \Lambda$ and $x, \tilde{x} \in B$, where h is Hausdorff metric on $\mathrm{Cl}(X)$ induced by the norm $|\cdot|$.*

Let $S_{\mathcal{P}}(\lambda) := \{x \in B: x \in \mathcal{P}(\lambda, x)\}$, each $\lambda \in \Lambda$. A set-valued mapping $S_{\mathcal{P}}: \Lambda \ni \lambda \rightarrow S_{\mathcal{P}}(\lambda) \in \mathcal{N}(B)$ has a retractive representation $p: \Lambda \times B \rightarrow B$.

We now apply Theorem 2 and Lemma 1 to the subtrajectory integrals mapping $S_{\cdot}$ defined above. Recall that for given F, G, and H satisfying condition $(\mathcal{A}_1)$, we can define a convex, weakly compact set B (see Corollary 1), where B is a subset of a Banach space $(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2, \|\cdot\|)$ with a norm $\|\cdot\|$ defined in Lemma 1 corresponding to any $L \in [0,1)$, containing the set $C(\lambda)$ of all fixed points to subtrajectory integrals mapping $S_\lambda(\cdot)$. From Theorem 2 and Lemma 1 we immediately obtain the following result.

Lemma 3. *Suppose F, G, and H satisfy conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. A set-valued mapping $C: L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow C(\lambda) \in \mathcal{N}(B)$ has a retractive representation $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$.*

Corollary 2. *Let F, G, and H satisfy conditions (A_1) and $(\mathcal{A}_2)$, and $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ be a retractive representation for S. Then $C(\lambda) = p(\lambda, B)$, each $\lambda \in L_n^2(\mathcal{F}_0)$.*

Corollary 3. *The set-valued mapping $\lambda \rightarrow C(\lambda)$ is continuous as a mapping from $L^2(\mathcal{F}_0)$ into a metric space $(\mathrm{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2, \ell)$.*

Given F, G, and H satisfying conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, $\mathcal{C}_\lambda$ denotes a set of all solutions to the initial value problem $SI_\lambda(F, G, H)$. As an immediate consequence of Proposition 1, we obtain $\mathcal{C}_\lambda = \Phi^\lambda(C(\lambda))$, where $C(\lambda)$ is defined as above. $\mathcal{C}$ denotes a set-valued mapping $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \mathcal{C}_\lambda \subset D$. From the above definitions, Lemma 3, and properties of Φ^λ, we immediately obtain the following main result of this paper.

Theorem 4. *If F, G, and H satisfy conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then $\mathcal{C}$ has a quasi-retractive representation and is l.s.c. on $L^2(\mathcal{F}_0)$.*

Proof. Let $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ be a retractive representation for the set-valued mapping C defined in Lemma 3. We have $\mathcal{C}_\lambda = \Phi^\lambda(C(\lambda))$ and $C(\lambda) = p(\lambda, B)$, each $\lambda \in L_n^2(\mathcal{F}_0)$. Therefore, $\mathcal{C}$ has a quasi-retractive representation. Moreover, a function $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \Phi^\lambda(p(\lambda, x)) \in D$ is continuous for fixed $x \in B$. Therefore, a set-valued mapping $\mathcal{C}$ (see [3], Proposition II 2.5) is l.s.c. on $L^2(\mathcal{F}_0)$. □

Corollary 4. *If F, G, and H satisfy conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ and are such that a set-valued mapping C has a retractive representation $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ that is weakly-weakly continuous, then $\mathcal{C}_\lambda$ is a weakly compact subset of D for every $\lambda \in L_n^2(\mathcal{F}_0)$ and a set-valued mapping $\mathcal{C}$ is weak-weak continuous on $L_n^2(\mathcal{F}_0)$.*

Proof. Indeed, if p has properties mentioned above, then (see [3], Th. II 2.6)

$p(\lambda, B)$ is a weakly compact subset of B for each $\lambda \in L_n^2(\mathcal{F}_0)$. $\mathcal{C}_\lambda$ is also a weakly compact subset of B for each $\lambda \in L_n^2(\mathcal{F}_0)$ because $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$. Finally, by weak-weak continuity of the linear mapping $L_n^2(\mathcal{F}_0) \times B \ni (\lambda, x) \rightarrow \Phi^\lambda(x)$, weak compactness of B, and an equality $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$, each $\lambda \in L_n^2(\mathcal{F}_0)$, it follows (see again [3], Proposition II 2.5), that $\mathcal{C}$ is weak-weak continuous on $L_n^2(\mathcal{F}_0)$. □

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