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LYAPUNOV EXPONENTS FOR HIGHER DIMENSIONAL RANDOM MAPS[1]

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A random map is a discrete time dynamical system in which one of a number of transformations is selected randomly and implemented. Random maps have been used recently to model interference effects in quantum physics. The main results of this paper deal with the Lyapunov exponents for higher dimensional random maps, where the individual maps are Jabłoński maps on the n-dimensional cube.

Key words: Random Maps, Higher Dimensional, Dynamical System, Lyapunov Exponent.

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1. Introduction

Ergodic theory of dynamical systems deals with the qualitative analysis of iterations of a single transformation. Ulam and von Neuman [12] suggested the study of more general systems where one applies at each iteration a different transformation chosen at random from a set of transformations. In this setting one could consider a single transformation, where parameters defining the transformation are allowed to vary discretely or even continuously.

The importance of studying higher dimensional random maps is, in part, inspired by fractals that are fixed points of iterated functions systems [1]. Iterated function systems can be viewed as random maps, where the individual transformations are contractions. Recently, random maps were used in modeling interference effects such as those that occur in the two-slit experiment of quantum physics [2]. For a general study of ergodic theory of random maps, the reader is referred to the text by Kifer [8]. Additional ergodic properties of random maps can be found in [4, 5, 10] and [11].

One of the most important ways of quantifying the complexity of a dynamical system is by means of the Lyapunov exponent. This quantifier of chaos can be defined for random maps. In this paper, we develop formulas for the individual Lyapunov exponents for higher dimensional maps, where the basic maps are Jabłoński maps on the n-dimensional cube [7].

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2. Lyapunov Exponents

Our considerations are based on Oseledec's Multiplicative Ergodic Theorem [9]. Let $(X, \mathfrak{B}, m)$ be a probability space and let τ be a measurable transformation $\tau: X \to X$ preserving an invariant measure μ, absolutely continuous with respect to m. Let A: $X \to GL(n, \mathbb{R})$ be a measurable map with $\int_X \log^+ \| A(\cdot) \| d\mu < +\infty$.

Then, in particular, the limit

$$\lambda_{\mathbf{v}} = \lim_{k \to +\infty} \frac{1}{k} \log \| A(\tau^{k-1}x)A(\tau^{k-2}x)\ldots(A(\tau x))A(x)\mathbf{v} \|$$

exists for any $\mathbf{v} \in \mathbb{R}^n$ and μ almost any $x \in X$. The number $\lambda_{\mathbf{v}}$ can have one of at most n values $\lambda_1, \ldots, \lambda_n$.

In this note, $X = I^n = [0,1]^n$ and m is the Lebesgue measure on I^n. τ is a piecewise expanding C^2 transformation and $A(x)$ is the derivative matrix of τ, where it is well defined (it is not defined on a set of measure 0). In this case, the numbers $\lambda_1, \ldots, \lambda_n$ are called Lyapunov exponents.

More precisely, let $\mathcal{P} = \{D_1, \ldots, D_q\}$ be a partition of I^n into subsets with piecewise C^2 boundaries. Let

$$\tau(x) = \psi_j(x), \quad x \in D_j,$$

where ψ_j is C^2, 1-1 and onto its image, $j = 1, \ldots, q$. Then,

$$A(x) = A_j(x), \quad x \in D_j,$$

where A_j is the derivative matrix of ψ_j. If

$$\psi_j(x) = (\psi_{1j}(x), \ldots, \psi_{nj}(x)),$$

then

$$A_j(x) = \begin{pmatrix} \frac{\partial \psi_{1j}}{\partial x_1} & \frac{\partial \psi_{1j}}{\partial x_2} & \cdots & \frac{\partial \psi_{1j}}{\partial x_n} \\ \frac{\partial \psi_{2j}}{\partial x_1} & \frac{\partial \psi_{2j}}{\partial x_2} & \cdots & \frac{\partial \psi_{2j}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \psi_{nj}}{\partial x_1} & \frac{\partial \psi_{nj}}{\partial x_2} & \cdots & \frac{\partial \psi_{nj}}{\partial x_n} \end{pmatrix}.$$

3. Jabłoński Transformations

We say that $\tau: I^n \to I^n$ is a Jabłoński transformation if

$$\tau(x) = (\psi_1(x), \ldots, \psi_n(x)) = (\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n)), \; x \in D_j, j = 1, 2, \ldots, q,$$

where $\mathcal{P} = \{D_1, \ldots, D_q\}$ is a rectangular partition of I^n, i.e., $D_j = \Pi_{i=1}^n [a_{ij}, b_{ij})$, where $[a, b) = [a, b]$ if $b = 1$.

These transformations are among the simplest non-trivial higher dimensional transformations that have absolutely continuous invariant measures. They are useful in approximating the ergodic behavior of more complex higher dimensional transformations [3]. Jabłoński transformations have also proven to be useful in modeling cellular automata problems [6].

For the Jabłoński transformation τ, the derivative matrix

$$A(x) = A_j(x) = \begin{pmatrix} \psi'_{1j}(x_1) & & & \\ & \psi'_{2j}(x_2) & & 0 \\ & & \ddots & \\ 0 & & & \psi'_{nj}(x_n) \end{pmatrix}, x \in D_j.$$

If τ is piecewise C^2 and there exists a constant $s > 1$ such that

$$\inf_{i,j} \inf_{[a_{ij}, b_{ij}]} | \psi'_{ij} | \geq s,$$

then ([7]) there exists a measure μ invariant under τ (with density f) with respect to Lebesgue measure.

All measures considered in this paper are assumed to be probability measures. The transformations τ we consider have a finite number of ergodic absolutely continuous measures. To simplify our considerations, we assume that the absolutely continuous invariant measure μ is unique. Maps, which are piecewise onto, will satisfy this condition as will maps which are Markov and for which the matrix A is irreducible. In the general case we would consider each ergodic absolutely continuous measure μ_i, $i = 1, \ldots, k$, separately and our formulas would hold for each of them μ_i-a.e.

For any $i = 1, 2, \ldots, n$, the basic vector $\mathbf{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the only nonzero term is in the i^{th} position), we have the i^{th} Lyapunov exponent of τ:

$$\begin{aligned}
\lambda_i &= \lim_{k \to \infty} \frac{1}{k} \log \| A(\tau^{k-1}x) \ldots A(\tau x) A(x) \mathbf{v}_i \| \\
&= \lim_{k \to \infty} \frac{1}{k} \log \| \frac{\partial \psi_i}{\partial x_i}(\tau^{k-1}x) \ldots \frac{\partial \psi_i}{\partial x_i}(\tau x) \frac{\partial \psi_i}{\partial x_i}(x) \| \\
&= \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \log | \frac{\partial \psi_i}{\partial x_i}(\tau^l x) | = \int_{I^n} \log | \frac{\partial \psi_i}{\partial x_i}(x) | \, d\mu \\
&= \int_{I^n} \log | \frac{\partial \psi_i}{\partial x_i}(x) | f(x) dx = \sum_{j=1}^{p} \int_{D_j} \log | \frac{\partial \psi_i}{\partial x_i}(x) | f(x) dx \\
&= \sum_{j=1}^{p} \int_{D_j} \log | \psi'_{ij}(x_i) | f(x) dx.
\end{aligned}$$

We do not assume that the λ_i's are numbered in increasing order nor that they are all different.

4. Quasi-Jabłoński Transformations

We now restrict ourselves to dimension 2. We say that $\tau: I^2 \to I^2$ is a quasi-Jabłoński transformation if $\tau(x) = (\psi_1(x), \psi_2(x)) = (\psi_{1j}(x_2), \psi_{2j}(x_1))$, $x \in D_j$, $j = 1, 2, \ldots, q$, where $\mathcal{P} = \{D_1, \ldots, D_q\}$ is a rectangular partition of I^2, and $D_j = [a_{1j}, b_{1j}) \times [a_{2j}, b_{2j})$. Note that ψ_{1j} is a function of x_2 and ψ_{2j} is a function of x_1.

For the quasi-Jabłoński transformation τ, the derivative matrix is given by

$$A = A_j = \begin{pmatrix} 0 & \psi'_{1j}(x_2) \\ \psi'_{2j}(x_1) & 0 \end{pmatrix}, \quad x \in D_j.$$

If there exists a constant $s > 1$ such that

$$\inf_{i,j} \inf_{[a_{ij}, b_{ij}]} |\psi'_{ij}| \geq s,$$

then τ^2 is an expanding Jabłoński transformation and there exists a measure μ invariant under τ^2 with density f with respect to Lebesgue measure.

If we take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$A(x)\mathbf{v}_1 = \begin{pmatrix} 0 \\ \frac{\partial \psi_2}{\partial x_1}(x) \end{pmatrix},$$

$$A(\tau x)A(x)\mathbf{v}_1 = \left(\frac{\partial \psi_1}{\partial x_2}(\tau x)\frac{\partial \psi_2}{\partial x_1}(x), 0\right),$$

$$\vdots$$

It follows that

$$\begin{aligned}
\lambda_1 &= \lim_{k\to\infty} \frac{1}{k} \log \| A(\tau^{k-1}x)\ldots A(\tau x)A(x)\mathbf{v}_1 \| \\
&= \lim_{k\to\infty} \frac{1}{2k} \log \| A(\tau^{2k-1}x\ldots A(\tau x)A(x)\mathbf{v}_1 \| \\
&= \lim_{k\to\infty} \frac{1}{2k} \log \left| \frac{\partial \psi_1}{\partial x_2}(\tau^{2k-1}x)\frac{\partial \psi_2}{\partial x_1}(\tau^{2k-2}x)\ldots\frac{\partial \psi_2}{\partial x_1}(\tau^2 x)\frac{\partial \psi_1}{\partial x_2}(\tau x)\frac{\partial \psi_2}{\partial x_1}(x) \right| \\
&= \frac{1}{2} \lim_{k\to\infty} \frac{1}{k} \left[\sum_{l=0}^{k-1} \log \left| \frac{\partial \psi_2}{\partial x_1}(\tau^{2l}x) \right| + \sum_{l=0}^{k-1} \log \left| \frac{\partial \psi_1}{\partial x_2}(\tau^{2l+1}x) \right| \right] \\
&= \frac{1}{2} \int_{I^2} \log \left| \frac{\partial \psi_2}{\partial x_1} \right| d\mu + \frac{1}{2} \int_{I^2} \log \left| \frac{\partial \psi_1}{\partial x_2} \right| d\mu \\
&= \frac{1}{2} \sum_{j=1}^{p} \int_{D_j} (\log | \psi'_{1j}(x_2) | + \log | \psi'_{2j}(x_1) |) f(x) dx.
\end{aligned}$$

If we take $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then, as above, we have

$$\begin{aligned}\lambda_2 &= \lim_{k\to\infty} \frac{1}{k} \| A(\tau^{k-1}x)\ldots A(\tau x)A(x)\mathbf{v}_2 \| \\ &= \frac{1}{2}\sum_{j=1}^{p} \int_{D_j} (\log |\psi'_{1j}(x_2)| + \log |\psi'_{2j}(x_1)|)f(x)dx = \lambda_1.\end{aligned}$$

It is easy to see that no other value is possible as a limit $\lambda_{\mathbf{v}}$.

5. Random Maps

Let $\{\tau_t\}_{t=1}^{\infty}$ be a sequence of transformations from X into X. A random map $\mathfrak{I} = \{\{\tau_t\}_{t=1}^{\infty}, \{p_t\}_{t=1}^{\infty}\}$ is a discrete dynamical system, where at each iteration, τ_t is chosen with probability p_t, $p_t > 0$, $\sum_{t=1}^{\infty} p_t = 1$.

Formally, we define $\mathfrak{I}^{(1)}(x) = \tau_{t_1}(x)$, and

$$\mathfrak{I}^{(k)}(x) = \tau_{t_k}(\mathfrak{I}^{(k-1)}(x)) = \tau_{t_k} \circ \tau_{t_{k-1}} \circ \ldots \circ \tau_{t_1}(x),$$

where each t_s is chosen with probability p_{t_s}, $s = 1, \ldots, k$, $t_s \in \{1, 2, \ldots\}$. We also define:

$$\begin{aligned}A^{(k)}(x) &= (\mathfrak{I}^{(k)}(x))' \\ &= A_{t_k}(\mathfrak{I}^{(k-1)}(x)) \cdot A_{t_{k-1}}(\mathfrak{I}^{(k-2)}(x))\ldots A_{t_2}(\mathfrak{I}^{(1)}(x)) \cdot A_{t_1}(x),\end{aligned}$$

where $A_{t_s} = (\tau_{t_s})'$ is the matrix derivative of τ_{t_s}.

The above definition represents $\mathfrak{I}$ as a Markov stochastic process with transition probability function

$$P(x, A) = \sum_{t=1}^{\infty} p_t \chi_A(\tau_t x).$$

There is an alternative way [10] of defining the random map $\mathfrak{I}$. Let $\Omega = \{\omega = (\omega_0, \omega_1, \ldots): \omega_i = 1, 2, \ldots;\ i = 0, 1, 2, \ldots\}$. Let $\sigma: \Omega \to \Omega$ be the left shift, i.e.,

$$\sigma(\omega_0, \omega_1, \omega_2, \ldots) = (\omega_1, \omega_2, \ldots).$$

Then the random map $\mathfrak{I}$ can be represented as the skew-product transformation $T: X \times \Omega \to X \times \Omega$ defined by $T(x, \omega) = (\tau_{\omega_0}(x), \sigma(\omega)), (x, \omega) \in X \times \Omega$.

6. Random Jabłoński Transformation

Let $X = I^n$ and $\mathfrak{I} = \{\{\tau_t\}_{t=1}^{\infty}, \{p_t\}_{t=1}^{\infty}\}: I^n \to I^n$ be a random Jabłoński transformation, i.e., τ_t is a Jabłoński transformation for $t = 1, 2, \ldots$. Assume that there exists an absolutely continuous measure $\overline{\mu}$ with density $\overline{f}$ invariant under the random map $\mathfrak{I}$. Sufficient conditions for existence are given in Theorem 2 of [4]. For any measurable subset B of I^n, we have $\sum_{t=1}^{\infty} p_t \bar{\mu}(\tau_t^{-1}B) = \bar{\mu}(B)$. Now consider the representation of the random map by a skew-product transformation $T: I^n \times \Omega \rightarrow I^n \times \Omega$.

Let P be the product measure on Ω, $P((\omega_0,\ldots)) = p_{\omega_0}$, for any $\omega_0 \in \{1,2,\ldots\}$. We will show that the measure μ given by $\mu(B \times (\omega_0,\ldots)) = \bar{\mu}(B)P(\omega_0)$, where $P(\omega_0) = P((\omega_0,\ldots))$, is T invariant. We have:

$$\begin{aligned}\mu(T^{-1}(B \times (\omega_0,\ldots)) &= \mu(\bigcup_{t=0}^{\infty}[\tau_t^{-1}B \times (t,\omega_0,\ldots)]) = \sum_{t=1}^{\infty}\mu(\tau_t^{-1}B \times (t,\omega_0,\ldots))\\ &= \sum_{t=1}^{\infty}\bar{\mu}(\tau_t^{-1}B)P(t)P(\omega_0) = P(\omega_0)\sum_{t=1}^{\infty}P(t)\bar{\mu}(\tau_t^{-1}B)\\ &= P(\omega_0)\bar{\mu}(b) = \mu(B \times (\omega_0,\ldots)).\end{aligned}$$

Theorem 1: *Let $\mathfrak{J} = \{\{\tau_t\}_{t=1}^{\infty},\{p_t\}_{t=1}^{\infty}\}$ be a random Jabłoński transformation for which there exists a unique absolutely continuous invariant measure. Then, the Lyapunov exponents of $\mathfrak{J}$ are*

$$\lambda_i = \sum_{t=1}^{\infty} p_t \int_{I^n} \log \left| \frac{\partial \psi_i^t}{\partial x_i} \right| d\bar{\mu}, \tag{6.1}$$

$i = 1,2,\ldots,n$, where

$$\begin{aligned}\tau_t(x) &= (\psi_1^t(x),\ldots,\psi_n^t(x))\\ &= (\psi_{1j}^t(x_1),\ldots,\psi_{nj}^t(x_n)),\ x \in D_j^t,\ j = 1,2,\ldots,q_t.\end{aligned}$$

If $n = 1$, then (6.1) reduces to

$$\lambda = \sum_{t=1}^{\infty} p_t \int_0^1 \log|\tau_t'(x)|\bar{f}(x)dx.$$

Proof: Let $f_i(x,\omega) = \log\left|\frac{\partial \psi_i^{\omega_0}}{\partial x_i}(x)\right|$. Since the shift (σ, P) is ergodic (even exact), and we assume uniqueness of the absolutely continuous τ_t-invariant measures, there exists a unique $\mathfrak{J}$ invariant absolutely continuous measures $\bar{\mu}$ ([10]), and it gives a T-ergodic measure $\mu = \bar{\mu}P$. We have:

$$\begin{aligned}\lambda_i &= \lim_{k\rightarrow+\infty} \frac{1}{k}\log \| A^{(k)}(x)\mathbf{v}_i \|\\ &= \log_{k\rightarrow\infty} \frac{1}{k}\sum_{s=0}^{k-1}\log\left|\frac{\partial \psi_i^{t_s}}{\partial x_i}(\mathfrak{J}^{(s)}(x))\right|\\ &= \lim_{k\rightarrow\infty} \frac{1}{k}\sum_{s=0}^{k-1} f_i(T^s(x,\omega)) = \int_{I^n\times\Omega} f_i(x,\omega)d\mu\\ &= \int_{I^n}\int_{\Omega} f_i(x,\omega)d\bar{\mu}\,dP\end{aligned}$$

$$= \int_{I^n} \sum_{\omega_0 = 1}^{\infty} p_{\omega_0} \log \left| \frac{\partial \psi_i^{\omega_0}}{\partial x_i} \right| d\overline{\mu}$$

$$= \sum_{t=1}^{\infty} p_t \int_I^n \log \left| \frac{\partial \psi_i^t}{\partial x_i} \right| d\overline{\mu}.$$

In the general case, we can have a finite number of such measures, not more then minimal number of absolutely continuous invariant measures for τ_t, $t = 1, 2, \ldots$. In particular, if a least one of τ_t has a unique absolutely continuous measure, then the $\mathfrak{I}$-invariant measure μ is unique ([5]). In the case of more then one invariant measure, the above formula holds for x in the support of any fixed $\overline{\mu}$, $\overline{\mu}$-a.e.

7. Random Quasi-Jabłoński Transformation

Let $\mathfrak{I} = \{\{\tau_t\}_{t=1}^{\infty}; \{p_t\}_{t=1}^{\infty}\}: I^2 \to I^2$ be a random quasi-Jabłoński transformation. If for $t = 1, 2, \ldots, \tau_t = (\psi_1^t(x), \psi_2^t(x))$ satisfies all the conditions in Section 4, with a partition $\mathcal{P}_t = \{D_1^t, \ldots, D_{q_t}^t\}$, then $\mathfrak{I}^2 = \{\tau_t \cdot \tau_s; p_t p_s; t, s = 1, 2, \ldots\}: I^2 \to I^2$ is a random Jabłoński transformation which satisfies all the conditions in Section 6 and there exists an absolutely continuous measure $\overline{\mu}$ with density $\overline{f}$ invariant under $\mathfrak{I}^2$. For any measurable subset B of I^2, we have

$$\sum_{t=1}^{\infty} \sum_{s=1}^{\infty} p_t p_s \overline{\mu}(\tau_t^{-1}(\tau_s^{-1} B)) = \overline{\mu}(B).$$

Theorem 2: *Let $\mathfrak{I} = \{\{\tau_t\}_{t=1}^{\infty}, \{p_t\}_{t=1}^{\infty}\}$ be a random quasi-Jabłoński transformation. Then*

$$\lambda_1 = \frac{1}{2} \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} p_t p_s \int_{I^2} \left(\log \left| \frac{\partial \psi_1^t}{\partial x_2} \circ \tau_s \right| + \log \left| \frac{\partial \psi_2^s}{\partial x_1} \right| \right) d\overline{\mu}$$

$$\lambda_2 = \frac{1}{2} \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} p_t p_s \int_{I^2} \left(\log \left| \frac{\partial \psi_2^t}{\partial x_1} \circ \tau_s \right| + \log \left| \frac{\partial \psi_1^s}{\partial x_2} \right| \right) d\overline{\mu},$$

where $\overline{\mu}$ is $\mathfrak{I}^2$-invariant and absolutely continuous. If $\mathfrak{I}^2$ has a unique absolutely continuous invariant measure $\overline{\mu}$ (e.g., if at least one τ_t, $t \in \{1, 2, \ldots\}$ has a unique absolutely continuous invariant measure) then $\overline{\mu}$ is also $\mathfrak{I}$-invariant and

$$\lambda_1 = \lambda_2 = \frac{1}{2} \sum_{t=1}^{\infty} p_t \int_{I^2} \left(\log \left| \frac{\partial \psi_1^t}{\partial x_2} \right| + \log \left| \frac{\partial \psi_2^t}{\partial x_1} \right| \right) d\overline{\mu}.$$

Proof: The first part of the theorem follows from Theorem 1 and the observation that $\lambda(\mathfrak{I}) = \frac{1}{2}\lambda(\mathfrak{I}^2)$. The last equality follows from the definition of Lyapunov exponent and the fact that there are two iterations of $\mathfrak{I}$ for each iterate of $\mathfrak{I}^2$.

If $\overline{\mu}$ is $\mathfrak{I}^2$ invariant, then the measure

$$\mu_1 = \frac{1}{2}\left(\overline{\mu} + \sum_{t=1}^{\infty} p_t \tau_{t*} \overline{\mu} \right),$$

where $\tau_{t*}\bar{\mu} = \bar{\mu} \circ \tau_t^{-1}$, is $\mathfrak{I}$-invariant. Since any $\mathfrak{I}$-invariant measure is $\mathfrak{I}^2$-invariant, if $\bar{\mu}$ is unique, then $\mu_1 = \bar{\mu}$ and $\bar{\mu}$ is also $\mathfrak{I}$-invariant. This implies that for any integrable function $f \in L^1(X, \mathfrak{B}, m)$,

$$\sum_{s=1}^{\infty} p_s \int_{I^2} f \circ \tau_s d\bar{\mu} = \int_{I^2} f d\bar{\mu},$$

which simplifies formulas for λ_1 and λ_2. □

8. Random Composition of Jabłoński and Quasi-Jabłoński Transformations

Let τ_1 be a Jabłoński transformation and τ_0 a quasi-Jabłoński transformation on I^2. In this section we will find Lyapunov exponents for a random map $\mathfrak{I} = \{\tau_1, \tau_0, p, q\}$, where $p, q \geq 0$, $p + q = 1$.

Consider a sequence of Jabłoński transformations:

$$\begin{aligned}
\tau_1 &= \tau_1 \\
\tau_2 &= \tau_0^2 \\
\tau_3 &= \tau_0 \circ \tau_1 \circ \tau_0 \\
&\vdots \\
\tau_t &= \tau_0 \circ \tau_1^{t-2} \circ \tau_0,
\end{aligned}$$

and let $p_1 = p$, $p_t = p^{t-2}q^2$, $t = 2, 3, \ldots$. Instead of $\mathfrak{I}$, we will consider the random Jabłoński transformation $\mathfrak{I}_1 = \{\{\tau_t\}_{t=1}^{\infty}, \{p_t\}_{t=1}^{\infty}\}$.

Lemma 1: $\lambda_i(\mathfrak{I}) = \frac{1}{2}\lambda_i(\mathfrak{I}_1)$, $i = 1, 2$.

Proof:

$$\begin{aligned}
\lambda_i(\mathfrak{I}) &= \lim_{k \to +\infty} \frac{1}{k} \log \| A^{(k)}(x)\mathbf{v}_i \| \\
&= \lim_{k \to +\infty} \frac{1}{k} \log \| A_{t_k}(\mathfrak{I}^{(k-1)}(x)) \ldots A_{t_2}(\tau_{t_1}(x)) A_{t_1}(x)\mathbf{v}_i \|
\end{aligned}$$

For $\lambda_i(\mathfrak{I}_1)$ we have the same expression under the norm sign, but the averaging factor k is different. It is enough to prove that, on average, there are two iterations of $\mathfrak{I}$ for each interaction of $\mathfrak{I}_1$.

The average number of iterations of $\mathfrak{I}$ in each iterate of $\mathfrak{I}_1$ is:

$$\sum_{t=1}^{\infty} p_t \,\{\text{number of iterates of } \mathfrak{I} \text{ in } \tau_t\} = p + \sum_{t=2}^{\infty} p^{t-2}q^2 t = 2.$$

Lemma 2: *If μ is $\mathfrak{I}$-invariant, then μ is $\mathfrak{I}_1$-invariant.*

Proof: If μ is $\mathfrak{I}$-invariant then $(p\tau_{1*} + q\tau_{0*})\mu = \mu$. Then $q\tau_{0*}\mu = \mu - p\tau_{1*}\mu$. To prove that μ is $\mathfrak{I}_1$-invariant, we have to show that

$$\overline{\mu} = p\tau_{1*}\mu + \sum_{t=2}^{\infty} p^{t-2}q^2\tau_{0*}\tau_{1*}^{t-2}\tau_{0*}\mu = \mu.$$

We have

$$\overline{\mu} = p\tau_{1*}\mu + \sum_{t=2}^{\infty} p^{t-2}q\tau_{0*}\tau_{1*}^{t-2}\mu - \sum_{t=2}^{\infty} p^{t-1}q\tau_{0*}\tau_{1*}^{t-1}\mu = p\tau_{1*}\mu + q\tau_{0*}\mu = \mu.$$

□

Theorem 3: *Let $\Im = \{\tau_1, \tau_0, p, q\}$, where τ_1 is Jabłoński and τ_0 is quasi-Jabłoński on I^2. Then the Lyapunov exponents of $\Im$ are*

$$\lambda_1 = \tfrac{1}{2}p \int_{I^2} \log \left| \frac{\partial \psi_1^1}{\partial x_1} \right| d\overline{\mu} + \tfrac{1}{2}\sum_{t=2}^{\infty} p^{p-2}q^2$$

$$\times \int_{I^2} \left(\log \left| \frac{\partial \psi_2^0}{\partial x_1} \circ \tau_1^{t-2} \circ \tau_0 \right| + \log \left| \frac{\partial \psi_2^1}{\partial x_2} \circ \tau_1^{t-3} \circ \tau_0 \right| \right.$$

$$\left. + \ldots + \log \left| \frac{\partial \psi_2^1}{\partial x_2} \circ \tau_0 \right| + \log \left| \frac{\partial \psi_1^0}{\partial x_2} \right| \right) d\overline{\mu}$$

and

$$\lambda_2 = \tfrac{1}{2}p \int_{I^2} \log \left| \frac{\partial \psi_2^1}{\partial x_2} \right| d\overline{\mu} + \tfrac{1}{2}\sum_{t=2}^{\infty} p^{t-2}q^2$$

$$\times \int_{I^2} \left(\log \left| \frac{\partial \psi_1^0}{\partial x_2} \circ \tau_1^{t-2} \circ \tau_0 \right| + \log \left| \frac{\partial \psi_1^1}{\partial x_1} \circ \tau_1^{t-3} \circ \tau_0 \right| \right.$$

$$\left. + \ldots + \log \left| \frac{\partial \psi_1^1}{\partial x_1} \circ \tau_0 \right| + \log \left| \frac{\partial \psi_2^0}{\partial x_1} \right| \right) d\overline{\mu},$$

where $\overline{\mu}$ is the $\Im_1$-invariant absolutely continuous measure. If $\overline{\mu}$ is unique (e.g., if τ_1 or τ_0 admits a unique acim), then $\overline{\mu}$ is also $\Im$ invariant and then:

$$\lambda_1 = \lambda_2 = \tfrac{1}{2}p \left(\int_{I^2} (\log \left| \frac{\partial \psi_1^1}{\partial x_1} \right| + \log \left| \frac{\partial \psi_2^1}{\partial x_2} \right|) d\overline{\mu} \right)$$

$$+ \tfrac{1}{2}q \left(\int_{I^2} (\log \left| \frac{\partial \psi_1^0}{\partial x_2} \right| + \log \left| \frac{\partial \psi_2^0}{\partial x_1} \right| d\overline{\mu} \right).$$

Proof: The first part of the theorem follows from Theorem 1 and Lemma 1. If $\overline{\mu}$ is unique, then from Lemma 2 it follows that $\overline{\mu}$ is also $\Im$ invariant. This means that for any $f \in L^1(X, \mathfrak{B}, m)$, we have

$$\int (pf \circ \tau_1 + qf \circ \tau_0) d\overline{\mu} = \int f d\overline{\mu}.$$

Using this equality we can prove, for $s = 0, 1, 2, \ldots$

$$\int_{I^n} \left(\sum_{t=2+s}^{\infty} p^{t-2} q^2 f \circ \tau^{t-2-s} \circ \tau_0 \right) d\overline{\mu}$$

$$= \int_{I^n} \left(\sum_{t=2+s}^{\infty} p^{t-2} q f \circ \tau_1^{t-2-s} - \sum_{t=2+s}^{\infty} p^{t-2+1} q f \circ \tau_1^{t-2-s+1} \right) d\overline{\mu}$$

$$= p^s q \int f d\overline{\mu}.$$

This in turn, applied to the general formulas for λ_1 and λ_2, reduces them to the last statement of the theorem. □

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