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NEW GENERALIZATIONS OF THE POISSON KERNEL

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The purpose of this paper is to give new generalizations of the Poisson Kernel in two dimensions and discuss integral formulas for them. This paper concludes with an open problem.

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1. Introduction

The Poisson Kernel in two dimensions is defined by

$$P(\theta, r) \overset{\text{def}}{=} \frac{1-r^2}{1-2r\cos\theta + r^2} = \frac{1-r^2}{(1-re^{i\theta})(1-re^{-i\theta})}. \tag{1}$$

Then, as is well-known, the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P(\theta, r)d\theta = 1 \tag{2}$$

holds. Here r is a real parameter satisfying $|r| < 1$.

In [3] which is a motive of our present paper, a proof of (2) is given by using the functional equation

$$F(r^2) = F(r),$$

where

$$F(r) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} P(\theta; r)d\theta$$

in $|r| < 1$.

In this paper we shall treat generalizations of (1) and (2):

First, if we set

$$Q(\theta;a,b)\overset{\text{def}}{=}\frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})}, \tag{3}$$

where a, b are complex parameters satisfying $|a|<1$ and $|b|<1$.

By taking $a=r$ and $b=r$ in (3) we find that (3) is a generalization of (1). In Section 2 we shall prove the integral formula for $Q(\theta;a,b)$

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta;a,b)d\theta = 1, \tag{4}$$

where a, b are complex parameters satisfying $|a|<1$ and $|b|<1$.

By taking $a=r$ and $b=r$ in (4) we find that (4) is a generalization of (2). The method of proof of (4) in this paper is similar to the proof given for (2) in [3], i.e., the method is by applying a functional equation.

Second, if we set

$$R(\theta;a,b,c,d)=\frac{L(a,b,c,d)}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-de^{-i\theta})}, \tag{5}$$

where a,b,c,d are complex parameters satisfying $|a|<1$, $|b|<1$, $|c|<1$ and $|d|<1$ and

$$L(a,b,c,d)\overset{\text{def}}{=}\frac{(1-ab)(1-ad)(1-bc)(1-cd)}{1-abcd}. \tag{6}$$

By taking $c=0$ and $d=0$ in (5) we find that (5) is a generalization of (3).

In Section 3 we shall prove the integral formula for $R(\theta;a,b,c,d)$

$$\frac{1}{2\pi}\int_0^{2\pi} R(\theta;a,b,c,d)d\theta = 1, \tag{7}$$

where a,b,c,d are complex parameters satisfying $|a|<1$, $|b|<1$, $|c|<1$ and $|d|<1$. The method of proof of (7) is the calculus of residues (cf. [1, pp. 147-151]).

Remark 1: The purpose of this paper is to prove (4) and (7).

2. Proof of the Integral Formula (4)

Theorem 1:

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta;a,b)d\theta = 1,$$

where a,b are complex parameters satisfying $|a|<1$ and $|b|<1$.

Proof: If we set

$$G(a,b)\overset{\text{def}}{=}\frac{1}{2\pi}\int_0^{2\pi} Q(\theta;a,b)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-ab}{(1-ae^{i\theta})(1-be^{-\theta})}d\theta \text{ (by (3))}, \tag{8}$$

then $G(a,b)$ is a continuous function of a,b when $|a|<1$ and $|b|<1$. Also, it is clear that

$$G(0,0)=1. \tag{9}$$

By (8) let us write

$$G(a,b)=\frac{1}{2\pi}\int_0^{\pi}\frac{1-ab}{(1-ae^{i\theta})(1-be^{i\theta})}d\theta+\frac{1}{2\pi}\int_{\pi}^{2\pi}\frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})}d\theta$$

for all complex a,b satisfying $|a|<1$ and $|b|<1$.

Making the substitution $\theta=\varphi+\pi$ in the second integral and using the formulas $e^{i\pi}=e^{-i\pi}=-1$, one obtains

$$G(a,b)=\frac{1}{2\pi}\int_0^{\pi}\frac{1-ab}{(1-ae^{i\theta})(1-be^{-\theta})}d\theta+\frac{1}{2\pi}\int_0^{\pi}\frac{1-ab}{(1+ae^{i\theta})(1+be^{-i\theta})}d\theta$$

$$=\frac{1}{2\pi}\int_0^{\pi}\left(\frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})}+\frac{1-ab}{(1+ae^{i\theta})(1+be^{-i\theta})}\right)d\theta$$

for all complex values of a,b satisfying $|a|<1$ and $|b|<1$.

From the identity

$$(1+ae^{i\theta})(1+be^{-i\theta})+(1-ae^{i\theta})(1-be^{-i\theta})=2(1+ab),$$

we get

$$G(a,b)=\frac{1}{2\pi}\int_0^{\pi}\frac{2(1-a^2b^2)}{(1-a^2e^{2i\theta})(1-b^2e^{-2i\theta})}d\theta.$$

Making the substitution $\theta=\frac{1}{2}\psi$ in the above integral yields

$$G(a,b)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-a^2b^2}{(1=a^2e^{i\psi})(1-b^2e^{-i\psi})}d\psi=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-a^2b^2}{(1-a^2e^{i\theta})(1-b^2e^{-i\theta})}d\theta(10)$$

for all complex numbers a,b satisfying $|a|<1$ and $|b|<1$.

In view of (8), (10) we obtain

$$G(a,b)=G(a^2,b^2) \tag{11}$$

for all complex numbers a,b satisfying $|a|<1$ and $|b|<1$.

By repeated applications of (11) we have

$$G(a,b)+G(a^{2^n},b^{2^n})(n=1,2,3,\ldots)$$

for all complex values of a,b satisfying $|a|<1$ and $|b|<1$.

Letting $n\to+\infty$ in the above inequality, using $\lim\limits_{n\to\infty}a^{2^n}=\lim\limits_{n\to+\infty}b^{2^n}=0$ which follow from the hypothesis that $|a|<1$ and $|b|<1$ and applying the continuity of $G(a,b)$ at $(0,1)$ yields

$$G(a,b)=G(0,0) \tag{12}$$

for all a,b satisfying $|a|<1$ and $|b|<1$.

By (9), (12) we obtain

$$G(a,b)=1$$

for all complex numbers a, b satisfying $|a| < 1$ and $|b| < 1$. Hence, by (8) we get (4). *Q.E.D.*

Remark: Another proof of Theorem 1 is given as Corollary 1 to Theorem 2.

3. Proof of the Integral Formula (7)

Theorem 2:

$$\frac{1}{2\pi}\int_0^{2\pi} R(\theta; a, b, c, d)d\theta = 1,$$

where a, b, c, d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$ and $|d| < 1$.

Proof: We have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-de^{-i\theta})} \tag{13}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{i\theta}}{(1-ae^{i\theta})(e^{i\theta}-b)(1-ce^{i\theta})(e^{i\theta}-d)} ie^{i\theta}d\theta.$$

If we set $z = e^{i\theta}$, then we obtain

$$ie^{i\theta}d\theta = dz. \tag{14}$$

Furthermore, we set

$$f(z) \stackrel{\text{def}}{=} \frac{z}{(1-az)(z-b)(1-cz)(z-d)}. \tag{15}$$

Hence, by (13), (14) and (15) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-de^{-i\theta})} = \frac{1}{2\pi i}\int_{|z|=1} f(z)dz, \tag{16}$$

where the right-hand side means the complex integral of the function $f(z)$ along the unit circle $|z| = 1$ on the z-plane in the positive direction.

By (15) we note that $f(z)$ is an analytic function in $|z| \le 1$ except at $z = b$ and $z = d$ each of which is a simple pole of f.

We consider two cases.

Case 1: Let $b \ne d$.

Suppose that R_1 and R_2 denote the residues of $f(z)$ at $z = b$ and $z = d$, respectively. By the Residue Theorem (cf. [1, pp. 147-151]) we get

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)dz = R_1 + R_2. \tag{17}$$

Next, by a standard method (cf. [2, p. 242]), we shall calculate R_1 and R_2. By (15) we have

$$R_1 = \lim_{z\to b}((z-b)f(z)) = \lim_{z\to b}\frac{z}{(1-az)(1-cz)(z-d)} = \frac{b}{(1-ab)(1-bc)(b-d)} \tag{18}$$

and

$$R_2 = \lim_{z\to d}((z-d)f(z)) = \lim_{z\to d}\frac{z}{(1-az)(z-b)(1-cz)} = \frac{d}{(1-ad)(d-b)(1-cd)}. \quad (19)$$

By (17), (18) and (19) we can write

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)dz = \frac{b}{(1-ab)(1-bc)(b-d)} + \frac{d}{(1-ad)(d-b)(1-cd)}$$

$$= \frac{1-abcd}{(1-ab)(1-ad)(1-bc)(1-cd)} \quad (20)$$

$$= \frac{1}{L(a,b,c,d)} \text{ (by (6))}.$$

By (16), (20) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-de^{-i\theta})} = \frac{1}{L(a,b,c,d)}$$

and therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{L(a,b,c,d)}{(1-ae^{i\theta})(1-be^{-\theta})(1-ce^{i\theta})(1-de^{-i\theta})}d\theta = 1 \quad (21)$$

for all complex values of a, b, c, d satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$ and $|d| < 1$.

By (5), (21) we get (7).

Case 2: Let $b = d$.

In this case, by (15) we have

$$f(z) = \frac{z}{(1-az)(1-cz)(z-b)^2}. \quad (22)$$

By (22) we see that $f(z)$ is an analytic function in $|z| \leq 1$ except at $z = b$ which is a double pole of the function.

In this case, let R denote the residue of $f(z)$ at $z = b$.

By the Residue Theorem we get

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)dz = R. \quad (23)$$

In the following, we shall calculate R.

By Cauchy's Integral Formula for the derivative (cf. [2, pp. 178-179]) we obtain

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{z}{(1-az)(1-cz)} \,/\, (z-b)^2 dz$$

$$= \left(\frac{d}{dz}\left(\frac{z}{(1-az)(1-cz)}\right)\right)_{z=b} \quad (24)$$

$$= \frac{1-ab^2c}{(1-ab)^2(1-bc)^2}$$

$$= \frac{1}{L(a,b,c,b)} \text{ (by (6)).}$$

By (16), (24) we obtain (note that $b = d$)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-be^{-i\theta})} = \frac{1}{L(a,b,c,b)}$$

and thus

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{L(a,b,c,b)}{(1-ae^{i\theta})(1-be^{-i\theta})(1-ce^{i\theta})(1-be^{-i\theta})} d\theta = 1 \tag{25}$$

for all complex values of a, b, c satisfying $|a| < 1$, $|b| < 1$ and $|c| < 1$.

By (5), (25) we get

$$R(\theta; a, b, c, b) = 1$$

for all complex numbers a, b, c satisfying $|a| < 1$, $|b| < 1$ and $|c| < 1$.

From Case 1 and Case 2 we get the desired result (7). *Q.E.D.*

Corollary 1: (to Theorem 2) *If we set $c = 0$ and $d = 0$ in Theorem 2, we obtain Theorem 1. Therefore, Theorem 2 gives another proof of Theorem 1.*

Corollary 2: (to Theorem 2) *If we set $c = a$ and $d = b$ in Theorem 2 we obtain*

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)^2 d\theta = \frac{1+ab}{1-ab},$$

where

$$Q(\theta; a, b) = \frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})} \quad (\textit{see } (3))$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

4. Open Problem

Let

$$I_n \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)^{n+1} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1-ab}{(1-ae^{i\theta})(1-be^{-\theta})}\right)^{n+1} d\theta \ (n = 0, 1, \ldots), \tag{26}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

By Theorems 1 and 2 we obtain

$$I_0 = 1 \text{ and } I_1 = \frac{1+ab}{1-ab}.$$

Open Problem: Compute I_n for $n = 2, 3, 4, \ldots$.

References

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