Journal of Applied Mathematics and Stochastic Analysis, **10**:2 (1997), 127-130.

RANDOM FIXED POINTS OF NON-SELF MAPS AND RANDOM APPROXIMATIONS[1]

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(Received June, 1996; Revised December, 1996)

In this paper we prove random fixed point theorems in reflexive Banach spaces for nonexpansive random operators satisfying inward or Leray-Schauder condition and establish a random approximation theorem.

Key words: Random Fixed Point, Nonexpansive Random Operator, Weak Inward Condition, Leray-Schauder Condition.

AMS subject classifications: 47H10, 60H25, 41A50.

1. Introduction

Lin [6] proved a random version of an approximation theorem of Fan [3] and obtained several random fixed point theorems. Recently Xu [12] and Lin [7] obtained some more random fixed point theorems for self and non-self nonexpansive or condensing random operators. For other related work we refer the reader to [1, 2, 8, 9, 10, 11, 13]. In this paper we prove random fixed point theorems in reflexive Banach spaces for nonexpansive random operators, and generalize the results obtained by Lin [6, 7] and Xu [11]. A random version of best approximation theorem of Fan [3] is also derived.

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space with Σ a sigma algebra of subsets of Ω. Let (X,d) be a metric space, 2^X be family of all subsets of X, and $WK(X)$ be family of all nonempty weakly compact subsets of X. A mapping $F:\Omega \rightarrow 2^X$ is called *measurable* if for any open subset C of X, $F^{-1}(C) = \{w \in \Omega: F(w) \cap C \neq \emptyset\} \in \Sigma$. A mapping $\xi:\Omega \rightarrow X$ is said to be a *measurable selector* of a measurable mapping $F:\Omega \rightarrow 2^X$ if ξ is measurable and for any $w \in \Omega$, $\xi(w) \in F(w)$. Let M be a subset of X. A mapping $T:\Omega \times M \rightarrow X$ is called a *random operator* if for any

[1]This research partially supported by the Kuwait University research grant No. SM 119.

$x \in M$, $T(\cdot, x)$ is measurable. A measurable mapping $\xi: \Omega \to M$ is called a *random fixed point* of a random operator $T: \Omega \times M \to X$ if for every $w \in \Omega$, $\xi(w) = T(w, \xi(w))$.

A mapping $T: M \to X$ is called *k-set-Lipschitz* ($k \geq 0$) if T is continuous and for any bounded subset B of M, $\alpha(T(B)) \leq k\ \alpha(B)$, where $\alpha(B) = \inf\{e > 0: B$ can be covered by a finite number of sets of diameter $\leq e\}$. The number $\alpha(B)$ is called the *(set)-measure of noncompactness* of B. A k-set-Lipschitz mapping T is a *k-set-contraction* if $k < 1$. A mapping $T: M \to X$ is called *(set-) condensing* if T is continuous and for each bounded subset C of M with $\alpha(C) > 0$, $\alpha(T(C)) < \alpha(C)$. Clearly a k-set-contraction mapping is condensing. A mapping $T: M \to X$ is called *nonexpansive* if $\| T(x) - T(y) \| \leq \| x - y \|$ for all $x, y \in M$. A random operator $T: \Omega \times M \to X$ is *continuous (condensing, nonexpansive*, etc.) if for each $w \in \Omega$, $T(w, \cdot)$ is continuous (condensing, nonexpansive, etc.) A random operator $T: \Omega \times M \to X$ is said to be *weakly inward* if for each $w \in \Omega$, $T(w, x) \in \text{cl } I_M(x)$ for $x \in M$, where cl denotes closure and $I_M(x) = \{z \in X: z = x + a(y - x)$ for some $y \in M$ and $a \geq 0\}$. When M has a nonempty interior, a random operator $T: \Omega \times M \to X$ is said to *satisfy the Leray-Schauder condition* if for each $w \in \Omega$, there exists an element $z \in \text{int}(M)$ (depending on w) such that

$$T(w, y) - z \neq a(y - z) \tag{1}$$

for all y in the boundary of M and $a > 1$.

A mapping $T: M \to X$ is said to be *demiclosed at* $y \in X$ if, for any sequence $\{x_n\}$ in M, the conditions $x_n \to x \in M$ weakly and $T(x_n) \to y$ strongly imply $T(x) = y$.

Theorem 2.1: [Xu, 12]. *Let C be a nonempty closed convex subset of a separable Banach space $X, T: \Omega \times C \to X$ be a condensing random operator that is either (i) weakly inward or (ii) satisfies the Leray-Schauder condition. Suppose, for each $w \in \Omega$, $T(w, C)$ is bounded. Then T has a random fixed point.*

Remark 2.2: Theorem 2.1 remains true if C is separable instead of X being separable.

3. The Main Results

Theorem 3.1: *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach space X and let $T: \Omega \times C \to X$ be a weakly inward nonexpansive random operator. Suppose for each $w \in \Omega$, $I - T(w, \cdot)$ is demiclosed at zero. Then T has a random fixed point.*

Proof: Take an element $v \in C$ and a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. For each n, define a mapping $f_n: \Omega \times C \to X$ by $f_n(w, x) = k_n v + (1 - k_n) T(w, x)$. Then, f_n is a weakly inward $(1 - k_n)$-set-contraction random operator. Hence by Theorem 2.1 (*i*) and Remark 2.2, there is a random fixed point ξ_n of f_n. Since X is a reflexive Banach space, $w - \text{cl}\{\xi_i(w)\}$ is weakly compact.

Let C be a weakly closed and bounded subset of X containing $w - \text{cl}\{\xi_i(w)\}$. For each n, define $F_n: \Omega \to WK(C)$ by $F_n(w) = w - \text{cl}\{\xi_i(w): i \geq n\}$. Let $F: \Omega \to WK(C)$ be a mapping defined by $F(w) = \bigcap_{n=1}^{\infty} F_n(w)$. Then, as in Itoh [5, proof of Theorem 2.5], F is w-measurable and has a measurable selector ξ. This ξ is the desired random fixed point of T. Indeed, fix any $w \in \Omega$, then some subsequence $\{\xi_m(w)\}$ of

$\{\xi_n(w)\}$ converges weakly to $\xi(w)$. On the other hand, we have $\xi_m(w) - T(w,\xi_m(w)) = k_m\{v - T(w,\xi_m(w))\}$. Thus $\{\xi_m(w) - T(w,\xi_m(w))\}$ converges to 0. Since $I - T(w,\cdot)$ is demiclosed at zero, it follows that $\xi(w) = T(w,\xi(w))$. □

If $T:\Omega\times C\to C$ then we have the following:

Theorem 3.2: *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach space and let $T:\Omega\times C\to C$ be a nonexpansive random operator. Suppose for each $w\in\Omega$, $I - T(w,\cdot)$ is demiclosed at zero. Then T has a random fixed point.*

Theorem 3.3: *Let C be a nonempty closed bounded convex separable subset of a reflexive Banach space X and has a nonempty interior. Let $T:\Omega\times C\to X$ be a nonexpansive random operator that satisfies the Leray-Schauder condition. Suppose for each $w\in\Omega$, $I - T(w,\cdot)$ is demiclosed at zero. Then T has a random fixed point.*

Proof: Let $z = z(w)\in \mathrm{int}(C)$ satisfy inequality (1). Take a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n\to 0$ as $n\to\infty$. For each n, define a mapping f_n: $\Omega\times C\to X$ by $f_n(w,x) = k_n z + (1-k_n)T(w,x)$. Then f_n is a random $(1-k_n)$-set-contraction operator that satisfies the Leray-Schauder condition. Then, by Theorem 2.1 (*ii*) and Remark 2.2, f_n has a random fixed point ξ_n. Define a sequence of mappings $F_n:\Omega\to WK(C)$ and a mapping $F:\Omega\to WK(C)$ as in the proof of Theorem 3.1. Then F is measurable and has a measurable selector ξ. This ξ is the desired random fixed point of T.

The following is a special case of Theorem 3.2, which extends the results of Lin [6, Theorem 3] and Lin [7, Corollary 3.2].

Theorem 3.4: *Let C be a nonempty closed bounded convex separable subset of a Hilbert space X and let $T:\Omega\times C\to X$ be a nonexpansive random operator. Then there exists a measurable map $\xi:\Omega\to C$ such that*

$$\|\xi(w) - T(w,\xi(w))\| = d(T(w,\xi(w)),C),$$

for each $w\in\Omega$.

Proof: Let P be the proximity map on C, that is, P is a continuous map from X into C such that for each $y\in X$ we have

$$\|P(y) - y\| = d(y,C).$$

Since both P and T are nonexpansive, the random operator $P\circ T:\Omega\times C\to C$ is also nonexpansive. By Theorem 3.2 there exists a random fixed point of $P\circ T$, that is, there exists a measurable map $\xi:\Omega\to C$ such that $P\circ T(w,\xi(w)) = \xi(w)$, for each $w\in\Omega$. Therefore,

$$\begin{aligned}\|\xi(w) - T(w,\xi(w))\| &= \|P\circ T(w,\xi(w)) - T(w,\xi(w))\| \\ &= d(T(w,\xi(w)),C),\end{aligned}$$

for each $w\in\Omega$. □

Remark 3.5:

(*i*) Immediate corollaries to Theorems 3.1 are Lin [6, Theorem 6′(*ii*)] and Lin [7, Corollary 4.2 (*iii*)].

(*ii*) Theorem 3.2 generalizes Lin [6, Lemma 1] and Xu [12, Theorem 1].

(*iii*) The fixed point property of C and strict convexity of X in Xu [12, Theorem 1] are not needed.

(*iv*) Theorem 3.3 extends Xu [12, Theorem 4].

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