

# POSITIVE AND OSCILLATORY RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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We prove that the nonlinear partial differential equation

$$\Delta u + f(u) + g(|x|, u) = 0, \text{ in } \mathbb{R}^n, n \geq 3,$$

with  $u(0) > 0$ , where  $f$  and  $g$  are continuous,  $f(u) > 0$  and  $g(|x|, u) > 0$  for  $u > 0$ , and

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^q} = B > 0, \text{ for } 1 < q < n/(n-2),$$

has no positive or eventually positive radial solutions. For  $g(|x|, u) \equiv 0$ , when  $n/(n-2) \leq q < (n+2)/(n-2)$  the same conclusion holds provided  $2F(u) \geq (1-2/n)uf(u)$ , where  $F(u) = \int_0^u f(s)ds$ . We also discuss the behavior of the radial solutions for  $f(u) = u^3 + u^5$  and  $f(u) = u^4 + u^5$  in  $\mathbb{R}^3$  when  $g(|x|, u) \equiv 0$ .

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## 1. Introduction

In recent years, numerous authors have given substantial attention to the existence of positive solutions of semilinear elliptic equations involving critical exponents (see [2], [5], [9], [10], [12], [13], [14], [15]). We shall consider the solutions of the nonlinear partial differential equation

$$\Delta u + f(u) + g(|x|, u) = 0, \text{ in } \mathbb{R}^n, n \geq 3, \tag{1.1}$$

where  $f$  and  $g$  are continuous functions, with  $f(u) > 0$  and  $g(|x|, u) > 0$  whenever  $u > 0$ . Such equations arise in many areas of applied mathematics (see [7], [12]); solutions that exist in  $\mathbb{R}^n$  and satisfy  $u(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  are called ground states.

By a positive solution we mean a solution  $u$  satisfying  $u(x) > 0$  for all  $x$  in  $\mathbb{R}^n$ . Equation (1.1) is said to involve critical exponents if  $f(u) = u^p + f_0(u)$ , where  $f_0(u)$  is an algebraic rational function in  $u$  with order of growth  $o(u^p)$  at infinity, where  $p = (n + 2)/(n - 2)$  is the critical Sobolev exponent. Examples of such  $f_0(u)$  in  $\mathbb{R}^3$  include  $f_0(u) = u^3\sqrt{1 + u^2}$  and  $f_0(u) = u^4/(1 + u)$ . When  $g \equiv 0$  and  $f(u) = u^p$ , Equation (1.1) is known to have a one parameter family of positive solutions. For  $f(u) = u^q$ ,  $q > p$  and  $g \equiv 0$  the authors of [1] have shown that (1.1) has a positive solution provided  $u(0) > 0$ .

In this paper we will deal only with radially symmetric solutions of (1.1). Hence, using polar coordinates, we need only consider the singular initial-value problem

$$u'' + \frac{n-1}{r}u' + f(u) + g(r, u) = 0, \text{ for } r > 0, \quad u(0) = u_0 \neq 0 = u'(0), \tag{1.2}$$

where  $f$  and  $g$  are continuous, with  $f(u) > 0$  and  $g(r, u) > 0$  whenever  $u > 0$ . Observe that the function  $u(r) = A/(B + r^2)^{1/(q-1)}$ ,  $A, B > 0$ , is a positive solution of the problem

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad u(0) = A/B^{q-1}, \quad u'(0) = 0,$$

with

$$f(u) = \frac{4qB}{(q-1)^2 A^{2(q-1)}} u^{2q-1} + \frac{2(q(n-2)-n)}{(q-1)^2 A^{q-1}} u^q.$$

For both terms of  $f(u)$  to have nonnegative coefficients, we must have  $q \geq n/(n-2)$ . In addition, the authors of [1] have shown, for  $g \equiv 0$ , that positive solutions to (1.2) exist whenever  $u_0 > 0$ ,  $f$  is Lipschitz,  $f(0) = 0$ ,  $f(u) > 0$  for  $u > 0$  and

$$2F(u) \leq (1 - \frac{2}{n})uf(u), \text{ for } u \geq 0, \tag{1.3}$$

where

$$F(u) = \int_0^u f(s)ds.$$

Thus, the existence of positive solutions to (1.2) depends on the order of growth of  $f(u)$  for small  $u > 0$  and on properties of the antiderivative of  $f(u)$ .

In this paper we show that the reversal of the inequality in (1.3) together with the assumption that  $uf(u) > 0$  and  $ug(r, u) > 0$  for all  $u \neq 0$ , and mild conditions on the order of the growth of  $f(u)$  to zero as  $u \rightarrow 0$ , lead to the *nonexistence* of positive solutions to (1.2) with  $u_0 > 0$ . Moreover, the solutions to (1.2) cannot be eventually positive or eventually negative, but must oscillate about 0 infinitely often. Results of this type can be inferred from careful reading of the literature (see [4], [8], [11]). However, proofs in the literature are often limited to functions of the form  $K(|x|)u^p$ , with assertions that they carry over to more general expressions, and frequently involve deep results about elliptic partial differential equations. Our results are completely elementary and give precise statements of the conditions required for nonexistence of positive and eventually positive or negative solutions. Our paper is organized as follows.

In Section 2 we present some elementary results that will be used in our proofs. The main tools consist of an “energy function” that was developed in [9] and a modi-

fication of an identity due to [14].

In Section 3 we prove that if  $f$  and  $g$  are continuous,  $f(u) > 0$ ,  $g(r, u) > 0$  for  $u > 0$ , and  $\lim_{u \rightarrow 0^+} f(u)/u^q = B > 0$  for  $1 < q < n/(n-2)$ , then the initial-value problem (1.2) with  $u_0 > 0$ , has no positive solutions. In Section 4 we shall assume that  $g(r, u) \equiv 0$ , so our results apply to solutions of

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \text{ for } r > 0, u(0) = u_0 \neq 0 = u'(0). \tag{1.5}$$

If  $f$  is continuous,  $f(u) > 0$  for  $u > 0$ ,  $\lim_{u \rightarrow 0^+} f(u)/u^q = B > 0$  for some  $q$  in  $1 < q < p$ , and

$$2F(u) \geq (1 - \frac{2}{n})u f(u), \tag{1.6}$$

for  $u > 0$ , then equation (1.5) has no positive solutions or eventually positive solutions.

Finally, in Section 5 we show that with an additional condition, solutions to (1.5) must oscillate infinitely and converge to 0 as  $r \rightarrow \infty$ . We also discuss the behavior of two such solutions: one with  $f(u) = u^3 + u^5$ , and one with  $f(u) = u^4 + u^5$ . Problems of these types have been studied in [5] and [6]. The first of these oscillates to 0 while the second becomes eventually negative and oscillates to  $-1$ , as  $r \rightarrow \infty$ .

## 2. Elementary Results

In what follows, we shall need some elementary facts concerning solutions of the initial value problems (1.2) and (1.5). Suppose  $f$  and  $g$  are continuous.

**(a)  $g(r, u) \neq 0$ .** Equation (1.2) can be rewritten as

$$(r^{n-1}u')' = -r^{n-1}[f(u) + g(r, u)].$$

Integration of this equality from 0 to  $r$  yields

$$r^{n-1}u'(r) = - \int_0^r [f(u(s)) + g(s, u(s))]s^{n-1}ds, \tag{2.1}$$

so that

$$u'(r) = - \frac{1}{r^{n-1}} \int_0^r [f(u(s)) + g(s, u(s))]s^{n-1}ds. \tag{2.2}$$

**Lemma 2.1:** *If  $f(u) > 0$  and  $g(r, u) > 0$  for  $u > 0$ , and  $u$  is a positive solution of equation (1.2), then  $u$  is strictly decreasing and tends to 0 as  $r \rightarrow \infty$ .*

**Proof:** Since the integrand in (2.2) is positive,  $u' < 0$  so the solution is strictly decreasing. Hence, there is a number  $c \geq 0$  such that  $u(r)$  decreases to  $c$  as  $r \rightarrow \infty$  and  $u'(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Suppose  $c > 0$ , then since  $f$  is continuous on the interval  $[c, u_0]$  it has a minimum  $f_{\min} > 0$  on this interval. Hence,

$$-r^{n-1}u'(r) = \int_0^r [f(u(s)) + g(s, u(s))]s^{n-1}ds \geq f_{\min} \frac{r^n}{n},$$

implying that  $u'(r) \leq -f_{\min}(r/n) \rightarrow -\infty$ . This is a contradiction and hence  $u$  tends

to zero as  $r \rightarrow \infty$ .

Suppose the solution  $u$  of (1.2) oscillates about zero a finite number of times and has a local maximum at  $r_0$  for which  $u(r) > 0$  for all  $r \geq r_0$ . We call such solutions *eventually positive solutions*. Then, because  $u'(r_0) = 0$ ,

$$u'(r) = -\frac{1}{r^{n-1}} \int_{r_0}^r [f(u(s)) + g(s, u(s))]s^{n-1} ds, \tag{2.3}$$

so that again  $u$  is strictly decreasing for  $r \geq r_0$ , and the same proof as above shows that  $u$  and  $u'$  converge to 0 as  $r \rightarrow \infty$ .

**(b)  $g(r, u) \equiv 0$ .** By the uniqueness theorem for solutions of initial value problems, a solution of (1.5) cannot satisfy both  $u'(r) = 0$  and  $f(u(r)) = 0$ , unless  $u$  is constant. Thus, except for such cases, the critical points of any solution of (1.5) are isolated, and are minima whenever  $f(u(r)) < 0$  and maxima whenever  $f(u(r)) > 0$ . Let  $u(r)$  be a solution of (1.5) and define the “energy function” of [9]:

$$Q(u(r)) = \frac{(u'(r))^2}{2} + F(u(r)). \tag{2.4}$$

If (1.5) is multiplied by  $u'$ , one obtains

$$\frac{dQ}{dr} = \left( \frac{(u')^2}{2} + F(u) \right)' = -\frac{n-1}{r}(u')^2 \leq 0. \tag{2.5}$$

This implies that the “energy” function  $Q$  is strictly decreasing because the critical points of  $u$  are isolated.

**Lemma 2.2:** *Suppose that  $u$  has a critical point at  $r_0$ . If  $u(r_0)$  is a local maximum, then  $u(r) < u(r_0)$  for all  $r > r_0$ , and if  $u(r_0)$  is a local minimum, then  $u(r) > u(r_0)$  for all  $r > r_0$ .*

**Proof:** Suppose  $u(r_1) = u(r_0)$  for  $r_1 > r_0$ . Then

$$Q(u(r_1)) = \frac{u'(r_1)^2}{2} + F(u(r_1)) \geq F(u(r_0)) = Q(u(r_0)),$$

contradicting the fact that  $Q$  is strictly decreasing.

We also need the following “energy” version of Pokhozhaev’s second identity valid for continuous  $f$  and functions  $u$  that are  $C^2(\mathbb{R}^n)$  and radial (see [14]):

$$\begin{aligned} & \int_0^r (\Delta u + f(u))(su' + \alpha u)s^k ds \\ &= r^{k+1}Q(u(r)) + \alpha r^k u(r)u'(r) + \frac{\alpha}{2}(n-1-k)r^{k-1}u^2(r) \\ &+ (2n-3-k-2\alpha) \int_0^r Q(u(s))s^k ds - \alpha \frac{(n-1-k)(k-1)}{2} \int_0^r u^2(s)s^{k-2} ds \\ &+ \int_0^r [\alpha u f(u) - 2(n-1-\alpha)F(u)]s^k ds, \text{ for integers } k > 1, \text{ and } \alpha \text{ real.} \end{aligned} \tag{2.6}$$

Here  $u = u(s)$  inside the integrals. If  $\Delta u + f(u) = 0$ , then the left side of (2.6) is zero. Verification of this identity is a routine task by using (1.5) instead of  $\Delta u + f(u)$ .

**Lemma 2.3:** *Let  $u$  be a solution of (1.5) and let*

$$J(r, u) = r^n u'^2(r) + (n - 2)r^{n-1}u(r)u'(r) + 2r^n F(u). \tag{2.7}$$

If  $u'(r_0) = 0$  for some  $r_0 \geq 0$ , then for all  $r \geq r_0$ ,

$$J(r, u) = \int_{r_0}^r [2nF(u(s)) - (n - 2)u(s)f(u(s))]s^{n-1}ds + 2r_0^n F(u(r_0)). \tag{2.8}$$

**Proof:** Differentiate (2.7) with respect to  $r$  and substitute (1.5) into the resulting equation to obtain

$$\frac{dJ}{dr} = [2nF(u) - (n - 2)uf(u)]r^{n-1}.$$

An integration yields the desired result.

**Lemma 2.4:** *Let  $u$  be a positive solution of the initial value problem (1.5) with  $u_0 > 0$ , and suppose that  $f$  is continuous,  $f(u) > 0$  for  $u > 0$ , and*

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^q} = B > 0, \text{ for } 1 < q < p = (n + 2)/(n - 2). \tag{2.9}$$

Then, for sufficiently large  $r$ , there is a constant  $c > 0$  such that

$$u(r) \leq cr^{-2/(q-1)}. \tag{2.10}$$

The result also holds for eventually positive solutions.

**Proof:** First assume that  $u$  is a positive solution. By Lemma 2.1,  $u$  decreases to 0, so some  $r_0$  exists for which  $f(u)/u^q > B/2$  for  $r \geq r_0$ . Then, by equation (2.2),

$$\begin{aligned} \frac{du}{dr} &= \frac{-1}{r^{n-1}} \int_0^{r_0} f(u(s))s^{n-1}ds - \frac{1}{r^{n-1}} \int_{r_0}^r f(u(s))s^{n-1}ds \\ &\leq -\frac{B}{2r^{n-1}} \int_{r_0}^r u^q(s)s^{n-1}ds \leq -\frac{Bu^q(r)}{2r^{n-1}} \int_{r_0}^r s^{n-1}ds. \end{aligned}$$

Integrating the resulting inequality:

$$\int_{u(r_0)}^{u(r)} u^{-q}du \leq \frac{-c}{n} \int_{r_0}^r [r - r_0^n r^{-n+1}]dr$$

yields an inequality from which the result follows for  $r \geq 2r_0$ .

If  $u$  is eventually positive, there is an  $r_1 > 0$  such that  $u(r) > 0$  for  $r > r_1$ . Then, there is a first maximum of  $u$  at  $r_2 > r_1$  for which (2.3) applies (with  $g \equiv 0$ )

$$u'(r) = -\frac{1}{r^{n-1}} \int_{r_1}^r f(u(s))s^{n-1}ds.$$

By the comments following the proof of Lemma 2.1,  $u \rightarrow 0$  as  $r \rightarrow \infty$ , so some  $r_0 > r_1$  exists for which  $f(u)/u^q > B/2$  for  $r \geq r_0$ . The proof then follows by the same argument as the positive solution case.

An identical proof yields:

**Corollary 2.5:** *Let  $u$  be a solution of (1.5) with  $u_0 < 0$ . Suppose  $f$  is continuous,  $f(u) < 0$  for  $u < 0$ , and*

$$\lim_{u \rightarrow 0^-} \frac{f(u)}{|u|^{q-1}} = B > 0, \text{ for } 1 < q < p. \tag{2.11}$$

*If there is an  $r_0$  such that  $u(r) < 0$  for all  $r \geq r_0$  ( $u$  is eventually negative), then for sufficiently large  $r$*

$$|u(r)| \leq cr^{-2/(q-1)}. \tag{2.12}$$

### 3. Nonexistence of Positive Solutions in the Range $1 < q < n/(n - 2)$

We now show that problem (1.2) does not have positive or eventually positive radial solutions if the order of growth  $q$  of  $f(u)$  to zero as  $u \rightarrow 0^+$  is in the interval  $1 < q < n/(n - 2)$ .

**Theorem 3.1:** *If  $f$  and  $g$  are continuous,  $f(u) > 0$ ,  $g(r, u) > 0$  for  $u > 0$ , and*

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^q} = B > 0, \text{ for } 1 < q < \frac{n}{n-2}, \tag{3.1}$$

*then the initial-value problem*

$$u'' + \frac{n-1}{r}u' + f(u) + g(r, u) = 0, \text{ for } r > 0, u(0) = u_0 > 0 = u'(0), \tag{3.2}$$

*has no positive solutions. Moreover, if  $u_0 < 0$  or  $u(r)$  becomes negative there is no  $r_0 \geq 0$  for which  $u(r) > 0$  for all  $r \geq r_0$ , that is (3.2) has no eventually positive solutions.*

**Proof:** Suppose a local maximum of the solution exists at  $r_0$  such that  $u(r) > 0$  for all  $r \geq r_0$ . By Lemma 2.1 and the remark following its proof, it follows that  $u \rightarrow 0$  as  $r \rightarrow \infty$ . Since  $n > (n - 2)q$  select  $0 < \epsilon < 1/q$  and an integer  $k > 0$  such that both of the following inequalities hold:

$$\eta = n - (n - 2)q - \frac{(n - 2 - \epsilon)(q - 1)}{q^k - 1} > 0$$

and

$$n - 2 - \frac{(n - 2 - \epsilon)(q - 1)}{q^k - 1} > \frac{1}{q} > \epsilon = \frac{2 - \eta - [n - (n - 2)q - \eta]q^k}{q - 1}.$$

By L'Hopital's rule, provided the right side's limit exists, for any  $\beta > 0$

$$\lim_{r \rightarrow \infty} \frac{u(r)}{r - \beta} = \lim_{r \rightarrow \infty} \frac{u'(r)}{-\beta r - \beta - 1} = \lim_{r \rightarrow \infty} \frac{\int_0^r [f(u(s)) + g(s, u(s))]s^{n-1} ds}{\beta r^{n-2-\beta}}. \tag{3.3}$$

In particular, let  $\beta = n - 2$ . Then, since the integral in (3.3) is positive, some constant  $c_0 > 0$  exists so that

$$u(r) \geq c_0 r^{-(n-2)}, \text{ for all } r \geq r_0. \tag{3.4}$$

Define  $\beta_{j+1} = \beta_j q - 2 + \eta$  with  $\beta_0 = n - 2$ . It is easy to prove by induction that

$$\beta_j = (n - 2 - \epsilon) \left( 1 - \frac{q^j - 1}{q^k - 1} \right) + \epsilon, \quad j \leq k. \tag{3.5}$$

By hypothesis,  $f(u)/u^q \geq \frac{1}{2}B$  for all  $u \leq u_*$ , and since  $u \rightarrow 0$  as  $r \rightarrow \infty$ , we can assume  $u(r) \leq u_*$  for all  $r \geq r_1 \geq r_0$ . Using (3.3) with  $\beta = \beta_1$ , observe that

$$\begin{aligned} \frac{\int_{r_0}^r [f(u(s)) + g(s, u(s))] s^{n-1} ds}{\beta r^{n-2-\beta}} &\geq \frac{\frac{1}{2}B \int_{r_1}^r u^q(s) s^{n-1} ds}{\beta r^{n-2-\beta}} \geq \frac{\frac{1}{2}B c_0^q \int_{r_1}^r s^{n-1-(n-2)q} ds}{\beta r^{n-2-\beta}} \\ &= \frac{\frac{1}{2}B c_0^q}{\beta r^{n-2-\beta}} \left( \frac{r^{n-(n-2)q} - r_1^{n-(n-2)q}}{n-(n-2)q} \right) = Cr^\eta \left( 1 - \left( \frac{r_1}{r} \right)^{n-(n-2)q} \right) \rightarrow \infty, \text{ as } r \rightarrow \infty. \end{aligned} \tag{3.6}$$

Thus, by (3.3), there exist constants  $c_1 > 0$  and  $r_1 > r_0$  such that

$$u(r) \geq c_1 r^{-\beta_1}, \text{ for all } r \geq r_1. \tag{3.7}$$

We can repeat the process in (3.6) with  $\beta = \beta_2$ , obtaining  $u(r) \geq c_2 r^{-\beta_2}$ , for  $r \geq r_2 > r_1$ , and in general,

$$u(r) \geq c_j r^{-\beta_j}, \text{ for all } r \geq r_j, \quad j \leq k. \tag{3.8}$$

Since  $\beta_k = \epsilon$ , we have proved that  $u(r) \geq c_k r^{-\epsilon}$  for all  $r \geq r_k$ . However, by (2.3) and (3.6)

$$\begin{aligned} |u'(r)| &= \frac{1}{r^{n-1}} \int_{r_0}^r [f(u(s)) + g(s, u(s))] s^{n-1} ds \geq \frac{\frac{1}{2}B c_k^q}{r^{n-1}} \int_{r_k}^r s^{n-1-\epsilon q} ds \\ &\geq Cr^{1-\epsilon q} \left( 1 - \left( \frac{r_k}{r} \right)^{n-\epsilon q} \right) \rightarrow \infty, \text{ as } r \rightarrow \infty, \end{aligned}$$

which is a contradiction to Lemma 2.1 and the remark following it. Thus, no local maximum can exist for which the solution is positive thereafter, and consequently no positive solution of (3.2) exists.

**Corollary 3.2:** *Let  $f$  and  $g$  be continuous,  $uf(u) > 0$  and  $ug(r, u) > 0$  for all  $r$  and  $u \neq 0$ , and assume that*

$$\lim_{u \rightarrow 0} \frac{f(u)}{|u|^{q-1}} = B > 0, \text{ for } 1 < q < \frac{n}{n-2}. \tag{3.9}$$

*Then the initial value problem (1.2) with  $u_0 \neq 0$  has no positive or negative solutions, nor eventually positive or negative solutions.*

**Proof:** Theorem 3.1 yields the case for positive or eventually positive solution. The case for nonnegative or essentially negative solutions follows trivially by setting  $v = -u$ .

**4. Nonexistence of Positive Solutions in the Range  $n/(n - 2) \leq q < p = (n + 2)/(n - 2)$**

In this section we extend Theorem 3.1, for  $g(r, u) \equiv 0$ , to the range  $n/(n - 2) \leq q < p$ , and show that with an additional condition, we again have nonexistence of positive or eventually positive solutions.

**Theorem 4.1:** *Let  $f$  be continuous,  $f(u) > 0$  for  $u > 0$ , and assume*

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^q} = B > 0, \text{ for } 1 < q < \frac{n+2}{n-2}. \tag{4.1}$$

Further, assume that

$$2F(u) \geq (1 - \frac{2}{n})uf(u) > 0 \text{ for } u > 0. \tag{4.2}$$

Then the initial-value problem

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \text{ for } r > 0, u(0) = u_0 > 0 = u'(0), \tag{4.3}$$

has no positive solutions. Moreover, if  $u_0 < 0$  or  $u(r)$  becomes negative, there is no  $r_* \geq 0$  such that  $u(r) > 0$  for all  $r > r_* \geq 0$ , that is, there is no eventually positive solution.

**Proof:** As a consequence of Theorem 3.1, we only need to prove the conclusions for  $n/(n - 2) \leq q < p$ , or, equivalently  $2q/(q - 1) \leq n < 2(q + 1)/(q - 1)$ . Suppose that the conclusions are not true. Then there exists a point  $r_* \geq 0$  such that  $u(r) > 0$  for all  $r > r_* \geq 0$ . By Lemma 2.4, we have

$$u(r) \leq cr^{-\frac{2}{q-1}}, \text{ for all } r \geq r_0 \geq r_*. \tag{4.4}$$

Using (2.3) with  $g(r, u) \equiv 0$  and (4.4), we get

$$\begin{aligned} |u'(r)| &= \left| \frac{1}{r^{n-1}} \int_0^r f(u(s))s^{n-1}ds \right| \\ &\leq \frac{C}{r^{n-1}} + \frac{2Bc^q}{r^{n-1}} \int_{r_0}^r s^{n-1-2q/(q-1)}ds \leq \frac{C}{r^{n-1}} + C_1r^{-(q+1)/(q-1)}, \end{aligned} \tag{4.5}$$

where we choose  $r_0 \geq r_*$  such that  $0 < f(u(r))u/u^q(r) \leq 2B$  for  $r \geq r_0$ . Since  $q \geq n/(n - 2)$ , it follows that  $n - 1 \geq (q + 1)/(q - 1)$ . Thus, from (4.5),

$$|u'(r)| \leq c_2r^{-(q+1)/(q-1)}, \text{ for large } r. \tag{4.6}$$

By L'Hopital's rule, we have from (4.1)

$$\lim_{r \rightarrow \infty} \frac{F(u(r))}{u^{q+1}(r)} = \lim_{r \rightarrow \infty} \frac{\int_0^{u(r)} f(u)du}{u^{q+1}(r)} = \lim_{r \rightarrow \infty} \frac{f(u(r))}{(q+1)u^q(r)} = \frac{B}{q+1}. \tag{4.7}$$

Now, using (4.4), (4.6), and (4.7) and  $\alpha = 2(q + 1)/(q - 1) - n > 0$ , we get



$$r^n F(u(r)) \leq r^n \frac{F(u(r))}{u^{q+1}(r)} [cr^{-2/(q-1)}]^{q+1} \leq cr^{-\alpha}, \tag{4.8}$$

and both

$$r^{n-1}u(r) |u'(r)| \leq cr^{-\alpha}, \tag{4.9}$$

$$r^n |u'(r)|^2 \leq cr^{-\alpha}, \tag{4.10}$$

for large  $r$ . Let  $r_1$  be the first maximum point such that  $u(r) > 0$  for all  $r \geq r_1$ . Then  $u'(r_1) = 0$ . Using Lemma 2.3 and (4.2) we get

$$|J(r, u)| \leq r^n |u'(r)|^2 + (n-2)r^{n-1} |u(r)| |u'(r)| + 2r^n F(u) \leq (n+1)cr^{-\alpha}, \tag{4.11}$$

and

$$J(r, u) = \int_{r_1}^r [2nF(u(s)) - (n-2)u(s)f(u(s))]s^{n-1}ds + 2r_1^n F(u(r_1)) \tag{4.12}$$

$$\geq 2r_1^n F(u(r_1)) > 0.$$

Letting  $r \rightarrow \infty$ , we see that (4.11) contradicts (4.12), so the theorem is proved.

**Corollary 4.2:** *Let  $f$  be continuous,  $uf(u) > 0$  for  $u \neq 0$ , and assume that*

$$\lim_{u \rightarrow 0} \frac{f(u)}{|u|^{q-1}} = B > 0, \quad 1 < q < \frac{n+2}{n-2},$$

and

$$2F(u) \geq (1 - \frac{2}{n})uf(u), \text{ for all } u.$$

Then,

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \text{ for } r > 0, u(0) = u_0 \neq 0 = u'(0), \tag{4.13}$$

has no eventually positive or eventually negative solutions.

**Proof:** The proof is almost identical with that of Theorem 4.1.

**Example 4.3:** The function  $f(u) = u^5 + u^3\sqrt{1+u^2}$  in  $\mathbb{R}^3$  satisfies the hypotheses of Corollary 4.2. Thus, (4.13) has no eventually positive or eventually negative solutions for this function. The function  $f(u) = u^5 + u^4/(1+u)$  in  $\mathbb{R}^3$  satisfies the hypotheses of Theorem 4.1, but  $f(u) > 0$  in  $-1 < u < 0$ . Thus, (4.13) has no eventually positive solution, but may have an eventually negative solution.

## 5. Oscillatory Behavior

Example 4.3 motivates the discussion and results in this section. By the existence theory for initial value problems, we know that (4.13) has a local solution. It is not difficult to show that this solution can be extended to  $\mathbb{R}^+$ . Under the conditions listed in Corollaries 3.2 or 4.2, equations (1.2) and (1.5) with  $u_0 \neq 0$ , respectively, do not have eventually positive or eventually negative solutions. So what is the behavior of the solutions as  $r \rightarrow \infty$ ? Do the solutions converge to some value  $c$ , or do they oscillate but have no limit as  $r \rightarrow \infty$ ? These are some of the questions we address in this section. For simplicity, we will assume that  $g(r, u) \equiv 0$ .

First we show that with a stronger condition than that in Corollary 4.2, we can prove that the solution converges to zero as  $r \rightarrow \infty$ .

**Lemma 5.1:** *Let  $f(u)$  be continuous, and satisfy*

$$\lim_{u \rightarrow 0} \frac{|f(u)|}{|u|^q} = B > 0, \text{ for } 1 < q < \frac{n+2}{n-2}. \tag{5.1}$$

Further, assume that

$$(n-2)uf(u) \geq 2F(u) \geq (1 - \frac{2}{n})uf(u) > 0 \text{ for } u \neq 0. \tag{5.2}$$

Suppose  $u$  is a solution of

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \text{ for } r > 0, u(0) = u_0 \neq 0, u'(0) = 0. \tag{5.3}$$

Then

$$|u(r)| \leq cr^{-\frac{2}{q+1}}, \text{ for large } r. \tag{5.4}$$

**Proof:** Pokhozhaev's first identity [14] with  $\alpha = n - 2$  can be rewritten as

$$r^2F(u(r)) + \frac{1}{2}[ru'(r) + (n-2)u(r)]^2 + \int_0^r [(n-2)u(s)f(u(s)) - 2F(u(s))]s ds = \frac{(n-2)^2u^2(0)}{2}.$$

Since the second and third terms on the left side of this equation are nonnegative, we have by (5.2)

$$0 < \frac{(n-2)u(r)f(u(r))}{n} \leq 2F(u(r)) \leq \frac{(n-2)^2u^2(0)}{r^2}.$$

Thus,  $F(u(r)) \rightarrow 0$  as  $r \rightarrow \infty$ , and by the first inequality,  $F$  is only zero at  $u = 0$ . Hence  $u \rightarrow 0$  as  $r \rightarrow \infty$ . Then

$$\frac{|u|^{q+1}|f(u)|}{|u|^q} \leq \frac{n(n-2)u^2(0)}{r^2},$$

and using (5.1) the conclusion in (5.4) follows.

**Example 5.2:** The function  $f(u) = u^5 + u^3$  in  $\mathbb{R}^3$  satisfies the hypotheses of Lemma 5.1 with  $q = 3$ . Hence, the solution to (5.3) with this  $f$  will oscillate infinitely and decay as  $|u(r)| \leq cr^{-1/2}$  as  $r \rightarrow \infty$ . On the other hand,  $f(u) = (u/(1+u^2))^3$  satisfies only the second inequality in (5.2) for  $u \neq 0$ , so the conclusion (5.4) cannot be assumed.

The situation for functions  $f(u)$  that are not negative for  $u < 0$  is more complicated. The following lemma can be extended to more general functions, and shows that they too oscillate, but now about a negative number.

**Lemma 5.3:** *Let  $u(r)$  be a solution of (5.3) with  $f(u) = u^4 + u^5$ . Then  $u(r)$  oscillates infinitely about  $-1$ , and tends to  $-1$ , as  $r \rightarrow \infty$ .*

**Proof:** Suppose otherwise, and let  $r_1$  be the last extreme point of  $u$ . Then  $u(r) + 1$  does not change sign for  $r > r_1$ . Since  $u'(r_1) = 0$ , by (2.1) and L'Hopital's rule with  $\beta \leq n - 2$ ,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{u(r) + 1}{r - \beta} &= \lim_{r \rightarrow \infty} \frac{u'(r)}{-\beta r - \beta - 1} \\ &= \lim_{r \rightarrow \infty} \frac{1}{\beta r - \beta - 2 + n} \int_{r_1}^r [u^4(s) + u^5(s)]s^{n-1} ds. \end{aligned} \tag{5.5}$$

If  $\beta = n - 2$ , we have

$$\lim_{r \rightarrow \infty} \frac{u(r) + 1}{r - (n - 2)} = \lim_{r \rightarrow \infty} \int_{r_1}^r [1 + u(s)] |u(s)|^4 s^{n-1} ds \neq 0,$$

and the limit exists or is infinite. Hence, there is a constant  $c_1 > 0$ , and an  $r_2 > r_1$  such that

$$|u(s) + 1| \geq c_1 s^{-(n-2)}, u(s) < 0, \text{ and } |u(s)|^4 \geq \frac{1}{2}, \text{ for } s \geq r_2. \tag{5.6}$$

Observe that

$$\begin{aligned} \left| \int_{r_1}^r [1 + u(s)] u^4(s) s^{n-1} ds \right| &= \int_{r_2}^r |u(s) + 1| |u(s)|^4 s^{n-1} ds \\ &\geq \frac{c_1}{2} \int_{r_2}^r s ds \rightarrow \infty, \text{ as } r \rightarrow \infty. \end{aligned} \tag{5.7}$$

If  $n - 2 > 1$ , take  $\beta = n - 3$  in (5.5), and we have from (5.7)

$$\left| \lim_{r \rightarrow \infty} \frac{u(r) + 1}{r - (n - 3)} \right| \geq \lim_{r \rightarrow \infty} \frac{\frac{c_1}{2} \int_{r_2}^r s ds}{(n - 3)r} = \infty.$$

So there exists a constant  $c_2 > 0$  such that  $|u(r) + 1| \geq c_2 r^{-(n-3)}$ , for  $r \geq r_2$ . If  $n - 3 > 1$ , repeat these steps until  $n - k \leq 1$ . Then,  $|u(r) + 1| \geq c_{n-2} r^{-1}$ , for  $r \geq r_2$ . On the other hand,

$$\begin{aligned} |u'(r)| &= \frac{1}{r^{n-1}} \int_{r_1}^r |u(s)|^4 |1 + u(s)| s^{n-1} ds \\ &\geq \frac{1}{r^{n-1}} \int_{r_2}^r |u(s)|^4 |1 + u(s)| s^{n-1} ds \geq \frac{c_{n-2}}{2r^{n-1}} \int_{r_2}^r s^{n-2} ds \geq c_0 > 0, \end{aligned}$$

for  $r > r_2 + 1$ , which is a contradiction because  $u'(r) \rightarrow 0$ , as  $r \rightarrow \infty$ . Thus, the solution  $u$  must oscillate infinitely about  $-1$ . Since  $u'(r) \rightarrow 0$ , as  $r \rightarrow \infty$ , it follows that  $u(r) \rightarrow -1$ .

**Remark 5.4:** We now study the solutions of the initial-value problem (5.3) numerically. We can rewrite (5.3) as the nonautonomous system

$$\begin{aligned} u' &= v, & u(0) &= u_0, \\ v' &= -\frac{(n-1)v}{r} - f(u), & v(0) &= 0. \end{aligned} \tag{5.8}$$

The numerical solution is very sensitive to the singularity  $r = 0$ . To decide what is appropriate there, we apply L'Hopital's rule:

$$\lim_{r \rightarrow 0} v' = \lim_{r \rightarrow 0} \left( -\frac{(n-1)v}{r} - f(u) \right) = -(n-1) \lim_{r \rightarrow 0} v' - f(u_0). \tag{5.9}$$

Hence

$$\lim_{r \rightarrow 0} v' = -\frac{f(u_0)}{n}. \tag{5.10}$$

Applying the Runge-Kutta method to (5.8) except at  $r = 0$ , where (5.10) is used, we obtain the graphs in Figures 1 and 2 for the solutions when  $f(u) = u^3 + u^5$  and

$f(u) = u^4 + u^5$  respectively.

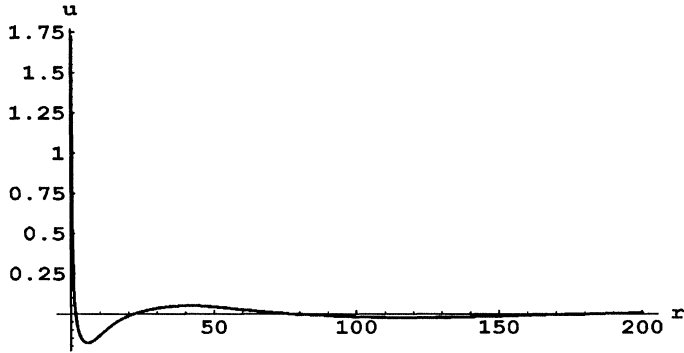


Figure 1

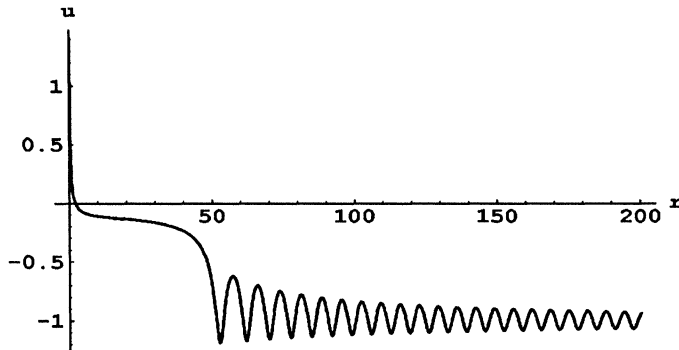


Figure 2

Observe that for the former, the solution oscillates and converges to zero. In a subsequent paper [3], we prove that the order of convergence to zero is  $|u| \leq cr^{-2/3}$ , somewhat faster than guaranteed by (5.4). Also, in that paper we prove that the zeros get further apart as  $r \rightarrow \infty$ .

For the latter, the solution becomes eventually negative and oscillates about  $u = -1$ . The numerics show that the oscillations damp to  $-1$  as  $r \rightarrow \infty$ , and we believe that the distance between zeros is bounded from below by a nonzero constant.

The main difference between the two functions  $f(u)$  is that the former is *restoring*, that is  $uf(u) > 0$  for  $u > 0$ , while the latter is not since it is positive in  $-1 < u < 0$ . For large  $r$  we can ignore the term  $(n-1)u'/r$  in (5.3), so the concavity of the solution changes with its sign for restoring functions, but does not change until  $u < -1$  for the latter function.

It is interesting to speculate what behavior might be expected from the solutions to (5.4) for more general  $f(u)$ . Numerical computations show when  $f(u) = u^3(u+1)(u+2)$  that for small  $u_0$  the solution behaves just as if  $f(u)$  were restoring, very similar to the behavior of  $u^3 + u^5$ . But for sufficiently large  $u_0$ , the solution oscillates about  $u = -2$ , with a behavior similar to that of  $u^4 + u^5$ . Thus,

some value of  $u_0$  is a bifurcation point for these two behaviors.

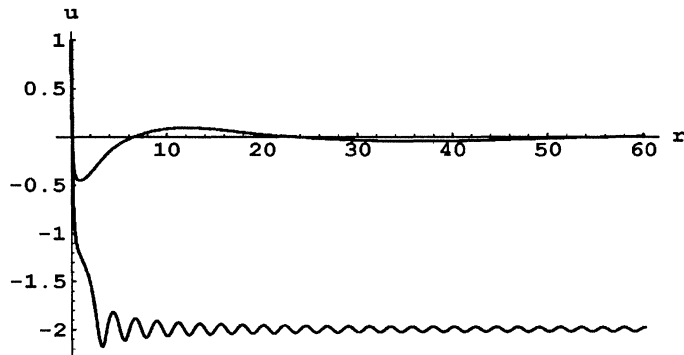


Figure 3

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