

ON A QUEUE WITH CORRELATED ARRIVALS

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Abstract. In this note we analyze the performance measures of a one server queue when arrivals are not independent. The analysis is based on the correlated Poisson distribution for customers arrival. Service may have any distribution. This type of queue is defined as $M^C/G/1$. The formulas for the performance measures of the queue are derived without any approximation. Surprisingly, these formulas are very simple.

Keywords: Queuing, Correlated arrivals.

1. Introduction

The standard $M/G/1$ queue assumes Poisson arrival and general service distribution. Correlated arrival patterns are analyzed mainly as “batch arrivals” i.e. when an arrival occurs, there is a finite probability for an arrival of more than one customer at the same time. In this note we provide the basic queuing formulas for dependent arrivals with a given correlation factor θ . Exact formulas are developed for the performance measures of the queue. They are based on the mean and variance of the arrival rate. These formulas are simple and easy to use.

2. The GBD and CPD Distributions

The correlated Poisson distribution (CPD)(Drezner and Farnum, 1994) defines the probability of observing r events during time period t for a given arrival rate λ and a correlation factor θ . The CPD is the limit of the generalized binomial distribution (GBD) (Drezner and Farnum, 1993) with n trials, initial probability of success p , and correlation factor θ as $p \rightarrow 0$ and $n \rightarrow \infty$ while np remains constant.

We briefly describe these distributions. The GBD is based on a correlated Bernoulli process. An initial probability of success p is given. The probability of success changes with each additional trial. Suppose that r successes were observed in k trials. The “rate of success” is $\frac{r}{k}$. The probability of success in trial $k + 1$ depends on the rate of success so far and is: $(1 - \theta)p + \theta \frac{r}{k}$ where θ is the correlation factor. When $\theta = 0$, the probability of success remains at p and thus the distribution is the standard binomial distribution. The probability of success changes each trial when $\theta > 0$. For large values of θ the probability distribution may be bi-modal.

The limit of the GBD as $n \rightarrow \infty$ for $p = \frac{\lambda}{n}$ (λ is the arrival rate), is the CPD. For $\theta = 0$ the CPD is the standard Poisson distribution. For positive θ it is more likely that an event occurs right after another event because the recent “rate of success” is increased. After a dormant period, the recent rate of success is lower and the likelihood of an event occurring is reduced compared with that of the standard Poisson distribution. For $\theta > 0$ the distribution has a thicker tail, and a higher probability of zero events than the standard Poisson distribution. The mean of the CPD is λ and its variance is $\frac{\lambda}{1-2\theta}$ for $\theta < 0.5$. For $\theta \geq 0.5$ the variance of the CPD is infinite (Drezner and Farnum, 1994). The probability of r successes in any time period t is (Drezner and Farnum, 1994):

$$P[R(t) = r] = \left\{ \sum_{k=0}^r a_{rk} \right\} e^{-(1-\theta)\lambda t} \quad (1)$$

where $a_{00} = 1$, $a_{rk} = 0$ if either $r = -1$ or $k = -1$ or $k > r$. a_{rk} is calculated by the following recursive relation:

$$a_{rk} = \lambda t \frac{1-\theta}{k+r\theta} a_{r-1,k-1} + \frac{(r-1)\theta}{k+r\theta} a_{r-1,k} \quad (2)$$

Equation (2) leads to $P[R(t) = 0] = e^{-(1-\theta)\lambda t}$; $P[R(t) = 1] = \lambda t \frac{1-\theta}{1+\theta} e^{-(1-\theta)\lambda t}$, $P[R(t) = 2] = \lambda t \frac{\theta(1-\theta)}{(1+\theta)(1+2\theta)} e^{-(1-\theta)\lambda t} + o(t^2)$, and so on. The arrival process is in batches. The batch arrivals are independent events according to a Poisson arrival with an arrival rate of $(1-\theta)\lambda$. While in the standard Poisson distribution, each batch has a size of one, in the CPD each batch may have one, two, or more events. The probabilities, p_r , for $r = 1, 2, \dots$ of getting a “batch” of r arrivals when an arrival of at least one customer occurs define a probability distribution. This probability distribution is termed the *Arrival Probability Distribution* (APD). In the next section we develop the properties of the APD.

3. The Arrival Probability Distribution

For the analysis in this section we need the Gamma function (Abramowitz and Stegun, 1972)

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \quad (3)$$

and the Beta Function (Abramowitz and Stegun, 1972)

$$B(m, n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (4)$$

For integer n $\Gamma(n) = (n-1)!$. Also, the recursive relation $\Gamma(n) = n\Gamma(n-1)$ is useful.

Consider a short period of time t . We find the probability p_r of a batch of r arrival if an arrival occurs at this time period t . The Arrival Probability Distribution (APD) defined by p_r for $r = 1, 2, \dots$ is the probability distribution of the number of events in one batch.

$$\text{THEOREM 1 } p_r = \frac{1}{\theta} B(r, 1 + \frac{1}{\theta})$$

Proof: First we prove that for $r \geq 1$: $a_{r1} = \lambda t \frac{1-\theta}{\theta} B(r, 1 + \frac{1}{\theta})$. By equation (2) $a_{r0} = 0$ for $r \geq 1$. Also, $a_{11} = \lambda t \frac{1-\theta}{1+\theta}$. For $r > 1$, $a_{r1} = \frac{(r-1)\theta}{1+r\theta} a_{r-1,1}$ and therefore

$$\begin{aligned} a_{r1} &= \lambda t \frac{(1-\theta)(r-1)! \theta^{r-1}}{\prod_{j=1}^r (1+j\theta)} = \lambda t \frac{(1-\theta)\Gamma(r)}{\theta \prod_{j=1}^r (\frac{1}{\theta} + j)} \\ &= \lambda t \frac{(1-\theta)\Gamma(r)\Gamma(1 + \frac{1}{\theta})}{\theta \Gamma(r + 1 + \frac{1}{\theta})} = \lambda t \frac{1-\theta}{\theta} B(r, 1 + \frac{1}{\theta}). \end{aligned} \quad (5)$$

By a multiple mathematical induction on k and r it is easy to show that a_{rk} consists of $(\lambda t)^k$ multiplied by a function of r and θ . The property is true for $k = 1$ and every r by (5). Assume that it is true for $k-1$ and every r . By equation (2) it is also true for k and $r = 1$. Now, holding k constant we prove it for every r by mathematical induction on r . This is clearly true by equation (2). Since a_{rk} consists of $(\lambda t)^k$ multiplied by a function of r and θ , $\lim_{t \rightarrow 0} \frac{P[R(t) = r]}{t} = \frac{a_{r1}}{t} = \lambda \frac{1-\theta}{\theta} B(r, 1 + \frac{1}{\theta})$. Define:

$$x_r = \lim_{t \rightarrow 0} \frac{P[R(t) = r]}{t} \quad (6)$$

By equation (1) $P[R(t) = 0] = 1 - (1-\theta)\lambda t + o(t^2)$. Since $P[R(t) = 0] + \sum_{r=1}^{\infty} P[R(t) = r] = 1$ for any t , then for $t \rightarrow 0$, $\sum_{r=1}^{\infty} x_r = (1-\theta)\lambda$. Now, since

$$p_r = \frac{x_r}{\sum_{r=1}^{\infty} x_r} \quad (7)$$

the Theorem follows. \square

$$\text{THEOREM 2 } \sum_{r=1}^{\infty} p_r = 1$$

Proof: By (4):

$$\sum_{r=1}^{\infty} p_r = \frac{1}{\theta} \sum_{r=1}^{\infty} B(r, 1 + \frac{1}{\theta}) = \frac{1}{\theta} \sum_{r=1}^{\infty} \int_0^1 t^{r-1} (1-t)^{\frac{1}{\theta}} dt$$

$$\begin{aligned}
&= \frac{1}{\theta} \int_0^1 \sum_{r=1}^{\infty} t^{r-1} (1-t)^{\frac{1}{\theta}} dt = \frac{1}{\theta} \int_0^1 \frac{1}{1-t} (1-t)^{\frac{1}{\theta}} dt \\
&= \frac{1}{\theta} \int_0^1 (1-t)^{\frac{1}{\theta}-1} dt = \frac{1}{\theta} = 1. \quad \square
\end{aligned} \tag{8}$$

We now find the mean and variance of the Arrival Probability Distribution (APD).

THEOREM 3 *The mean of the APD is $\bar{p}(\theta) = \frac{1}{1-\theta}$.*

Proof:

$$\begin{aligned}
\sum_{r=1}^{\infty} r p_r &= \frac{1}{\theta} \sum_{r=1}^{\infty} r B(r, 1 + \frac{1}{\theta}) = \frac{1}{\theta} \sum_{r=1}^{\infty} r \int_0^1 t^{r-1} (1-t)^{\frac{1}{\theta}} dt \\
&= \frac{1}{\theta} \int_0^1 \sum_{r=1}^{\infty} r t^{r-1} (1-t)^{\frac{1}{\theta}} dt = \frac{1}{\theta} \int_0^1 \frac{1}{(1-t)^2} (1-t)^{\frac{1}{\theta}} dt \\
&= \frac{1}{\theta} \int_0^1 (1-t)^{\frac{1}{\theta}-2} dt = \frac{1}{(\frac{1}{\theta}-1)\theta} = \frac{1}{1-\theta}. \quad \square
\end{aligned} \tag{9}$$

THEOREM 4 *The variance of the APD is $\sigma^2(\theta) = \frac{\theta}{(1-\theta)^2(1-2\theta)}$, for $\theta < 0.5$.*

Proof: Similar to the proof of Theorem 3 we get:

$$\sum_{r=1}^{\infty} r(r+1)p_r = \frac{2}{(\frac{1}{\theta}-2)\theta} = \frac{2}{1-2\theta}$$

The variance is therefore:

$$\frac{2}{1-2\theta} - \frac{1}{1-\theta} - \frac{1}{(1-\theta)^2} = \frac{\theta}{(1-\theta)^2(1-2\theta)}. \quad \square$$

Note that the series converge only for $\theta < 0.5$. The variance of the APD is infinite for $\theta \geq 0.5$.

4. Performance Measures of the M^C/G/1 Queue

The M^C/G/1 queue is defined by a CPD arrival with an average rate of λ and a correlation factor θ . The service time has a general distribution with mean $\frac{1}{\mu}$ and standard deviation σ . The average waiting time in line, W_q , for batch arrivals is (Cox and Smith, 1961):

$$W_q = \frac{1}{\mu} \left\{ \frac{\bar{p}^2(\theta)\lambda(1-\theta)}{2[\mu - \bar{p}(\theta)\lambda(1-\theta)]} \left[1 + \frac{\sigma^2\mu^2}{\bar{p}(\theta)} + \frac{\sigma^2(\theta)}{\bar{p}^2(\theta)} \right] + \frac{\bar{p}(\theta)}{2} \left(1 + \frac{\sigma^2(\theta)}{\bar{p}^2(\theta)} \right) - \frac{1}{2} \right\}$$

Substituting the mean and variance of the APD (Theorem 3 and Theorem 4) leads to:

$$W_q = \frac{1}{\mu} \left\{ \frac{\lambda}{2(1-\theta)(\mu-\lambda)} \left[1 + (1-\theta)\sigma^2\mu^2 + \frac{\theta}{1-2\theta} \right] + \frac{1 + \frac{\theta}{1-2\theta}}{2(1-\theta)} - \frac{1}{2} \right\} \quad (10)$$

Algebraic manipulations simplify (10) to:

$$W_q = \frac{\lambda(\mu^2\sigma^2 + 1) + \frac{2\theta}{1-2\theta}\mu}{2\mu(\mu-\lambda)} \quad (11)$$

Other performance measures of the queue are: $L_q = W_q\lambda$; $L = L_q + \frac{\lambda}{\mu}$; $W = W_q + \frac{1}{\mu}$.

5. Implementation

Assume that arrivals are distributed by a correlated Poisson distribution with a mean of λ and variance of σ_λ^2 . When arrivals are independent, $\sigma_\lambda^2 = \lambda$. However, if arrivals are not independent, then according to the CPD the probability of zero events in a given time interval t is greater than in the corresponding Poisson distribution, and so are the probabilities of large number of arrivals (tail probabilities). Therefore, a correlated arrival process leads to a variance greater than that of the Poisson distribution, namely, $\sigma_\lambda^2 > \lambda$. Note that by assuming a finite variance of arrivals we implicitly assume that $\theta < 0.5$. Otherwise, the variance of the arrivals is infinite. Service is distributed according to a general distribution with a mean of $\frac{1}{\mu}$ and variance of σ^2 . These four values can be estimated from the data.

By the properties of the CPD (Drezner and Farnum, 1994), $\sigma_\lambda^2 = \frac{\lambda}{1-2\theta}$. Solving for θ , substituting in (11), and multiplying by λ to get L_q , yields (note that the factor of λ is actually a product of the arrival rate of batches $\lambda(1-\theta)$ and the average size of batches $\frac{1}{1-\theta}$):

$$L_q = \frac{\lambda^2(\mu^2\sigma^2 + 1) + \mu(\sigma_\lambda^2 - \lambda)}{2\mu(\mu-\lambda)} \quad (12)$$

and the other performance measures L , W_q , and W may be directly calculated.

The queue performance measures are functions of the four values λ , μ , σ^2 , σ_λ^2 . Note how simple is the change from the M/G/1 formula to the M^C/G/1 one: just adding $\mu(\sigma_\lambda^2 - \lambda)$ in the numerator!

In practical situations, it is recommended that the arrival process be analyzed, and its mean and variance be estimated. Unless arrivals are independent, the variance of the arrival process would exceed its mean. The performance measures of the M^C/G/1 queue can be calculated by (12), taking into consideration dependency in the arrival process.

6. Simulating the Queuing Process

In order to simulate the correlated queue process with exponential service time, we use small time intervals Δt and generated the number of arrivals and departures in that time interval. We keep a running record of L , the total number of customers in the system. Suppose that there were A arrivals and D departures during Δt . The numbers are generated using the correlated Poisson distribution and the Poisson distribution, respectively. The number of customers, L , should be updated to $\max\{L + A - D, 0\}$ if the arrivals occurred before the departures, and $\max\{L - D, 0\} + A$ if the departures occurred prior to the arrivals. In most cases these two values are identical. It is possible to get any number between these two values if arrivals and departures are intermingled. If the values are different, we selected one of these two values with a probability of 0.5 for each.

It should be noted that no batch arrival process is assumed. The number of arriving customers A is drawn from the correlated Poisson distribution (1) which is calculated once at the beginning of the simulation. This probability is true for any time period Δt . The time does not have to be small.

The simulations were performed for $\mu = 1, 2, \dots, 5$ and integer $\lambda < \mu$ leading to ten different sets of arrival and service rates. For each set we tried $\theta = 0$, which is M/M/1, and $\theta = 0.1, 0.2, 0.3, 0.4$. We used $\Delta t = 0.01$ and ran the simulation for $10,000\Delta t$ to get to a steady state. The result of the simulation is L_q at the last Δt . Each simulation was repeated 10,000 times for a total of 100 million time periods per case.

This procedure is ideal for a spreadsheet. We first tried the Excel add-on @risk but found that a simple Fortran program is more than two orders of magnitude faster. Therefore, we report the results obtained by a Fortran program even though the same results can be obtained by a standard simulation package such as @risk.

We observed that good results were obtained for $\theta \leq 0.3$ but the results for $\theta = 0.4$ are too low. We first report in Table 1 results for $\theta \leq 0.3$ and then analyze in more detail the case $\theta = 0.4$.

The z -scores reported in Table 1 have a mean of -0.04 and sum of squares 43.94 in perfect correspondence with expected mean of 0 and sum of squares 40.

Table 1. Simulation Results for $\theta \leq 0.3$

λ	μ	θ	$\max\{L_q\}$	$\max\{A\}$	†	‡	z- score
1	2	0.0	14	3	0.5000	0.5025(0.0113)	0.22
1	2	0.1	12	10	0.6250	0.6161(0.0132)	-0.67
1	2	0.2	19	25	0.8333	0.8145(0.0175)	-1.07
1	2	0.3	53	63	1.2500	1.2486(0.0307)	-0.05
1	3	0.0	7	3	0.1667	0.1657(0.0054)	-0.18
1	3	0.1	10	10	0.2292	0.2198(0.0070)	-1.35
1	3	0.2	15	25	0.3333	0.3267(0.0103)	-0.64
1	3	0.3	38	63	0.5417	0.4929(0.0171)	-2.85
1	4	0.0	7	3	0.0833	0.0846(0.0036)	0.35
1	4	0.1	9	10	0.1250	0.1262(0.0050)	0.24
1	4	0.2	24	25	0.1944	0.2053(0.0080)	1.36
1	4	0.3	43	63	0.3333	0.3321(0.0140)	-0.09
1	5	0.0	4	3	0.0500	0.0536(0.0028)	1.27
1	5	0.1	6	10	0.0813	0.0821(0.0037)	0.23
1	5	0.2	16	25	0.1333	0.1328(0.0063)	-0.09
1	5	0.3	63	63	0.2375	0.2324(0.0123)	-0.42
2	3	0.0	21	3	1.3333	1.3221(0.0218)	-0.52
2	3	0.1	23	11	1.5833	1.5629(0.0257)	-0.80
2	3	0.2	28	29	2.0000	1.9743(0.0330)	-0.78
2	3	0.3	66	78	2.8333	2.7434(0.0503)	-1.79
2	4	0.0	11	3	0.5000	0.4909(0.0110)	-0.83
2	4	0.1	15	11	0.6250	0.6315(0.0138)	0.47
2	4	0.2	20	29	0.8333	0.8351(0.0182)	0.10
2	4	0.3	77	78	1.2500	1.2837(0.0311)	1.08
2	5	0.0	9	3	0.2667	0.2696(0.0074)	0.40
2	5	0.1	12	11	0.3500	0.3582(0.0095)	0.87
2	5	0.2	26	29	0.4889	0.4946(0.0131)	0.44
2	5	0.3	46	78	0.7667	0.7790(0.0233)	0.53
3	4	0.0	28	3	2.2500	2.2593(0.0322)	0.29
3	4	0.1	36	12	2.6250	2.5841(0.0376)	-1.09
3	4	0.2	55	32	3.2500	3.2758(0.0474)	0.54
3	4	0.3	91	88	4.5000	4.4382(0.0704)	-0.88
3	5	0.0	19	3	0.9000	0.9318(0.0175)	1.82
3	5	0.1	20	12	1.0875	1.1241(0.0204)	1.79
3	5	0.2	35	32	1.4000	1.4774(0.0268)	2.89
3	5	0.3	65	88	2.0250	2.0390(0.0415)	0.34
4	5	0.0	41	4	3.2000	3.1491(0.0427)	-1.19
4	5	0.1	47	12	3.7000	3.6805(0.0492)	-0.40
4	5	0.2	56	34	4.5333	4.5081(0.0589)	-0.43
4	5	0.3	106	96	6.2000	6.1237(0.0889)	-0.86

† The theoretical L_q

‡ The simulated L_q (with standard error)

Table 2. Simulation Results for $\theta = 0.4$

λ	μ	$\max\{L_q\}$	$\max\{A\}$	\dagger	\ddagger	z- score
1	2	150	161	2.5000	2.2011(0.0698)	-4.28
1	3	101	161	1.1667	0.9751(0.0420)	-4.56
1	4	165	161	0.7500	0.6172(0.0347)	-3.83
1	5	143	161	0.5500	0.4370(0.0262)	-4.32
2	3	216	213	5.3333	4.5500(0.1070)	-7.32
2	4	220	213	2.5000	2.2458(0.0723)	-3.52
2	5	204	213	1.6000	1.4594(0.0612)	-2.30
3	4	272	250	8.2500	7.2682(0.1412)	-6.95
3	5	223	250	3.9000	3.4470(0.0940)	-4.82
4	5	291	281	11.2000	9.7506(0.1684)	-8.61

\dagger The theoretical L_q

\ddagger The simulated L_q (with standard error)

6.1. Simulation for Large θ 's

The results in Table 1 indicate that for $\theta \leq 0.3$ the simulation confirms the theoretical formula (12). However, for $\theta = 0.4$ the same simulation provided L_q values which are much lower than the theoretical ones. Initially, we truncated the correlated Poisson probabilities to 50. Then, we extended the probabilities list up to 100. This improved the results but still resulted in z -scores between -4 and -12, which are unacceptable. Even extending the probability list to 1,000 did not resolve this issue (for results with a list of 1,000 probabilities see Table 2). The maximum observed number of arrivals throughout the simulations was 281, so increasing the list of probabilities beyond this number would not improve the results beyond those reported in Table 2.

It turns out that the correlated Poisson distribution with a large θ has a very significant tail. This tends to underestimate the number of customers in line because part of the theoretical L_q originates with a large number of arrivals which are not encountered in the simulation. When the probability list is truncated (it is effectively truncated at 281), the remaining probability (for arrivals of more than 281) is not negligible, and the variance of the truncated probability distribution is significantly lower than its theoretical value without any truncation. See these values in Table 3

In order to be able to simulate effectively with $\theta = 0.4$ one must increase the number of simulations to such an extent that arrivals in the thousands will be encountered. Also, the random number generator used in our simulation provides for differences of 2^{-31} which are not small enough to generate thousands of arrivals. It is important to note that increasing the number of time periods in each simulation to more than $10,000\Delta t$ did not improve the results. In short, it seems that simulating large values of θ successfully is intractable using this approach.

Finally, we compared the simulation results in Table 2 with equation (12) using σ_λ as the variance calculated for the truncated distribution (truncated at $\max\{A\}$) which is reported in Table 3. This is not a simulation with correlated Poisson ar-

Table 3. The Truncated Correlated Poisson Distribution

λ	θ	Prob $\times 10^8$				Variance Using				
		$A > 50$	$A > 100$	$A > 200$	$A > 300$	$A \leq 50$	$A \leq 100$	$A \leq 200$	$A \leq 300$	$A \leq \infty$
1	0.2	0.2302	0.0083	0.0003	0.0000	0.0167	0.0167	0.0167	0.0167	0.0167
2	0.2	0.4609	0.0166	0.0006	0.0001	0.0333	0.0333	0.0333	0.0333	0.0333
3	0.2	0.6921	0.0249	0.0008	0.0001	0.0500	0.0500	0.0500	0.0500	0.0500
4	0.2	0.9237	0.0332	0.0011	0.0002	0.0666	0.0667	0.0667	0.0667	0.0667
1	0.3	12.236	1.3009	0.1337	0.0350	0.0242	0.0247	0.0249	0.0249	0.0250
2	0.3	24.489	2.6027	0.2674	0.0700	0.0484	0.0493	0.0497	0.0498	0.0500
3	0.3	36.760	3.9054	0.4012	0.1051	0.0726	0.0740	0.0746	0.0748	0.0750
4	0.3	49.048	5.2090	0.5351	0.1401	0.0968	0.0987	0.0995	0.0997	0.1000
1	0.4	103.58	19.100	3.4495	1.2608	0.0364	0.0402	0.0430	0.0443	0.0500
2	0.4	207.28	38.210	6.8999	2.5218	0.0727	0.0804	0.0860	0.0886	0.1000
3	0.4	311.08	57.330	10.351	3.7831	0.1091	0.1206	0.1290	0.1328	0.1500
4	0.4	415.01	76.460	13.803	5.0445	0.1454	0.1608	0.1720	0.1771	0.2000

Table 4. Simulation Results for $\theta = 0.4$ Using Truncated Distributions

λ	μ	$\max\{A\}$	σ_λ^2	\dagger	multicolumn1—c— \ddagger	z-score
1	2	161	0.042224	2.1112	2.2011(0.0698)	1.29
1	3	161	0.042224	0.9723	0.9751(0.0420)	0.07
1	4	161	0.042224	0.6204	0.6172(0.0347)	-0.09
1	5	161	0.042224	0.4528	0.4370(0.0262)	-0.60
2	3	213	0.086445	4.6556	4.5500(0.1070)	-0.99
2	4	213	0.086445	2.1611	2.2458(0.0723)	1.17
2	5	213	0.086445	1.3741	1.4594(0.0612)	1.39
3	4	250	0.131210	7.3105	7.2682(0.1412)	-0.30
3	5	250	0.131210	3.4303	3.4470(0.0940)	0.18
4	5	281	0.176351	10.0176	9.7506(0.1684)	-1.59

$\dagger L_q$ calculated by Equation (12) using σ_λ^2 from column 4.

\ddagger The simulated L_q (with standard error)

ivals. Rather, it is a simulation using the truncated Correlated Poisson arrival. The results are summarized in Table 4. The results confirm that our process simulates the truncated distribution arrival process. The sum of the z-scores is 0.05 and their sum of squares is 8.96 as compared with the expected values of 0 and 10.

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