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Research Article

A Fuzzy Pay-Off Method for Real Option Valuation

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Real option analysis offers interesting insights on the value of assets and on the profitability of investments, which has made real options a growing field of academic research and practical application. Real option valuation is, however, often found to be difficult to understand and to implement due to the quite complex mathematics involved. Recent advances in modeling and analysis methods have made real option valuation easier to understand and to implement. This paper presents a new method (fuzzy pay-off method) for real option valuation using fuzzy numbers that is based on findings from earlier real option valuation methods and from fuzzy real option valuation. The method is intuitive to understand and far less complicated than any previous real option valuation model to date. The paper also presents the use of number of different types of fuzzy numbers with the method and an application of the new method in an industry setting.

1. Introduction

Real option valuation is based on the observation that the possibilities financial options give their holder resemble the possibilities to invest in real investments and possibilities found within real investments, that is, managerial flexibility: "an irreversible investment opportunity is much like a financial call option" [1]. In other words, real option valuation is treating investment opportunities and the different types of managerial flexibility as options and valuing them with option valuation models. Real options are useful both, as a mental model for strategic and operational decision-making, and as a valuation and numerical analysis tool. This paper concentrates on the use of real options in numerical analysis, and particularly on the derivation of the real option value for a given investment opportunity, or identified managerial flexibility.

Real options are commonly valued with the same methods that have been used to value financial options, that is, with Black-Scholes option pricing formula [2], with the binomial option valuation method [3], with Monte-Carlo-based methods [4], and with a number of later methods based on these. Most of the methods are complex and demand a good understanding of the underlying mathematics, issues that make their use difficult in practice. In addition these models are based on the assumption that they can quite accurately mimic the underlying markets as a process, an assumption that may hold for some quite efficiently traded financial securities, but may not hold for real investments that do not have existing markets or have markets that can by no means be said to exhibit even weak market efficiency.

Recently, a novel approach to real option valuation, called the Datar-Mathews method (DMM) was presented in [5–7], where the real option value is calculated from a pay-off distribution, derived from a probability distribution of the net present value (NPV) for a project that is generated with a (Monte-Carlo) simulation. The authors show that the results from the method converge to the results from the analytical Black-Scholes method. The method presented greatly simplifies the calculation of the real option value, making it more transparent and brings real option valuation as a method a big leap closer to practitioners. The most positive issue in the DMM is that it does not suffer from the problems associated with the assumptions connected to the market processes connected to the Black-Scholes and the binomial option valuation methods. The DMM utilizes cash-flow scenarios as an input to a Monte Carlo simulation to derive a distribution for the future investment outcomes. This distribution is then used to create a pay-off distribution for the investment. The DMM is highly compatible with the way cash-flow-based profitability analysis is commonly done in companies, because it can use the same type of inputs as NPV analysis.

All of the afore-mentioned models and methods use probability theory in their treatment of uncertainty, there are, however, other ways than probability to treat uncertainty, or imprecision in future estimates, namely, fuzzy logic and fuzzy sets. In classical set theory an element either (fully) belongs to a set or does not belong to a set at all. This type of bivalue, or true/false, logic is commonly used in financial applications (and is a basic assumption of probability theory). Bivalue logic, however, presents a problem, because financial decisions are generally made under uncertainty. Uncertainty in the financial investment context means that it is in practice impossible, exante to give absolutely correct precise estimates of, for example, future cash-flows. There may be a number of reasons for this, see, for example, [8], however, the bottom line is that our estimations about the future are imprecise.

Fuzzy sets are sets that allow (have) gradation of belonging, such as "a future cash flow at year ten is about x euro". This means that fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain, or imprecise knowledge, for example, future cash-flow estimation, which human reasoning is especially adaptive to. "Fuzzy set-based methodologies blur the traditional line between qualitative and quantitative analysis, since the modeling may reflect more the type of information that is available rather than researchers' preferences" [9], and indeed in economics "the use of fuzzy subsets theory leads to results that could not be obtained by classical methods" [10]. The origins of fuzzy sets date back to an article by Lotfi Zadeh [11] where he developed an algebra for what he called fuzzy sets. This algebra was created to handle imprecise elements in our decision-making processes, and is the formal body of theory that allows the treatment of practically all decisions in an uncertain environment. "Informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not" [12].

In the following subsection we will shortly present fuzzy sets and fuzzy numbers and continue shortly on using fuzzy numbers in option valuation. We will then present a new method for valuation of real options from fuzzy numbers that is based on the previous literature on real option valuation, especially the findings presented in [5] and on fuzzy real option valuation methods, we continue by illustrating the use of the method with a selection of different types of fuzzy numbers and with a case application of the new method in an industry setting, and close with a discussion and conclusions.

1.1. Fuzzy Sets and Fuzzy Numbers

A *fuzzy subset* A of a nonempty X set can be defined as a set of ordered pairs, each with the first element from X, and the second element from the interval $[0,1]$, with exactly one-ordered pair presents for each element of X. This defines a mapping,

$$\mu_A : A \to [0,1], \tag{1.1}$$

between elements of the set X and values in the interval $[0,1]$. The value zero is used to represent complete nonmembership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The set X is referred to as the universe of discourse for the fuzzy subset A. Frequently, the mapping μ_A is described as a function, the membership function of A. The degree to which the statement $x\ is\ in\ A$ is true is determined by finding the ordered pair $(x, \mu_A(x))$. The degree of truth of the statement is the second element of the ordered pair. It is clear that A is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) \mid x \in X\}. \tag{1.2}$$

It should be noted that the terms *membership function* and *fuzzy subset* get used interchangeably and frequently we will write simply $A(x)$ instead of $\mu_A(x)$. A γ-level set (or γ-cut) of a fuzzy set A of X is a nonfuzzy set denoted by $[A]^\gamma$ and defined by

$$[A]^\gamma = \{\mathrm{t} \in \mathrm{X} \mid \mathrm{A(t)} \geq \gamma\}, \tag{1.3}$$

if $\gamma > 0$ and cl(supp A) if $\gamma = 0$, where cl(supp A) denotes the closure of the support of A. A fuzzy set A of X is called *convex* if $[A]^\gamma$ is a convex subset of X for all $\gamma \in [0,1]$. A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex, and continuous membership function of bounded support [13]. Fuzzy numbers can be considered as possibility distributions.

Definition 1.1. Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. Let us introduce the notations

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma \tag{1.4}$$

In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut, $\gamma \in [0,1]$.

Definition 1.2. A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a, \\ 1 - \dfrac{t-a}{\beta} & \text{if } a \leq t \leq \mathrm{a} + \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

and we use the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]^{\gamma} = \left[a - (1-\gamma)\alpha, a + (1-\gamma)\beta\right], \quad \forall \gamma \in [0,1]. \tag{1.6}$$

The support of A is $(a - \alpha, b + \beta)$. A triangular fuzzy number with center a may be seen as a fuzzy quantity "x is approximately equal to a".

Definition 1.3. The *possibilistic (or fuzzy) mean value* of fuzzy number A with $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$ is defined in [13] by

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \, d\gamma \\ &= \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma \, d\gamma. \end{aligned} \tag{1.7}$$

Definition 1.4. A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α, and right width β if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a, \\ 1 & \text{if } a \leq t \leq b, \\ 1 - \dfrac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

and we use the notation

$$A = (a, b, \alpha, \beta). \tag{1.9}$$

It can easily be shown that $[A]^{\gamma} = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$ for all $\gamma \in [0,1]$. The support of A is $(a - \alpha, b + \beta)$.

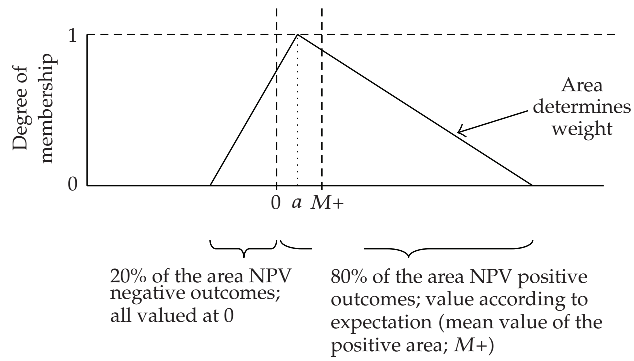


Figure 1: A triangular fuzzy number A, defined by three points $\{a, \alpha, \beta\}$ describing the NPV of a prospective project; (percentages 20% and 80% are for illustration purposes only).

Fuzzy set theory uses fuzzy numbers to quantify subjective fuzzy observations or estimates. Such subjective observations or estimates can be, for example, estimates of future cash flows from an investment. To estimate future cash flows and discount rates "one usually employs educated guesses, based on expected values or other statistical techniques" [14], which is consistent with the use of fuzzy numbers. In practical applications the most used fuzzy numbers are trapezoidal and triangular fuzzy numbers. They are used because they make many operations possible and are intuitively understandable and interpretable.

When we replace nonfuzzy numbers (crisp, single) numbers that are commonly used in financial models with fuzzy numbers, we can construct models that include the inaccuracy of human perception, or ability to forecast, within the (fuzzy) numbers. This makes these models more in line with reality, as they do not simplify uncertain distribution-like observations to a single-point estimate that conveys the sensation of no-uncertainty. Replacing nonfuzzy numbers with fuzzy numbers means that the models that are built must also follow the rules of fuzzy arithmetic.

1.2. Fuzzy Numbers in Option Valuation

Fuzzy numbers (fuzzy logic) have been adopted to option valuation models in (binomial) pricing an option with a fuzzy pay-off, for example, in [15], and in Black-Scholes valuation of financial options in, for example, [16]. There are also some option valuation models that present a combination of probability theory and fuzzy sets, for example, [17]. Fuzzy numbers have also been applied to the valuation of real options in, for example, [18–20]. More recently there are a number of papers that present the application of fuzzy real option models in the industry setting, for example, [21, 22]. There are also specific fuzzy models for the analysis of the value of optionality for very large industrial real investments, for example, [23].

2. New Fuzzy Pay-Off Method for Valuation of Real Options from Fuzzy Numbers

Two recent papers [5, 6] present a practical probability theory-based Datar-Mathews method for the calculation of real option value and show that the method and results from the method are mathematically equivalent to the Black-Sholes formula [2]. The method is based on simulation-generated probability distributions for the NPV of future project outcomes. The project outcome probability distributions are used to generate a pay-off distribution, where the negative outcomes (subject to terminating the project) are truncated into one chunk that will cause a zero pay-off, and where the probability-weighted average value of the resulting pay-off distribution is the real option value. The DMM shows that the real-option value can be understood as the probability-weighted average of the pay-off distribution. We use fuzzy numbers in representing the expected future distribution of possible project costs and revenues, and hence also the profitability (NPV) outcomes. The fuzzy NPV, a fuzzy number, is the pay-off distribution from the project.

The method presented in [5] implies that the weighted average of the positive outcomes of the pay-off distribution is the real option value; in the case with fuzzy numbers the weighted average is the fuzzy mean value of the positive NPV outcomes. Derivation of the fuzzy mean value is presented in [13]. This means that calculating the ROV from a fuzzy NPV (distribution) is straightforward, it is the fuzzy mean of the possibility distribution with values below zero counted as zero, that is, the area-weighted average of the fuzzy mean of the positive values of the distribution and zero (for negative values)

Definition 2.1. We calculate the real option value from the fuzzy NPV as follows:

$$\mathrm{ROV} = \frac{\int_0^\infty A(x)\mathrm{d}x}{\int_{-\infty}^\infty A(x)\mathrm{d}x} \times E(A_+), \tag{2.1}$$

where A stands for the fuzzy NPV, $E(A_+)$ denotes the fuzzy mean value of the positive side of the NPV, and $\int_{-\infty}^\infty A(x)\mathrm{d}x$ computes the area below the whole fuzzy number A while $\int_0^\infty A(x)\mathrm{d}x$ computes the area below the positive part of A.

It is easy to see that when the whole fuzzy number is above zero, then ROV is the fuzzy mean of the fuzzy number, and when the whole fuzzy number is below zero, the ROV is zero.

The components of the new method are simply the observation that real option value is the probability-weighted average of the positive values of a pay-off distribution of a project, which is the fuzzy NPV of the project, and that for fuzzy numbers, the probability-weighted average of the positive values of the pay-off distribution is the weighted fuzzy mean of the positive values of the fuzzy NPV, when we use fuzzy numbers.

2.1. Calculating the ROV with the Fuzzy Pay-Off Method with a Selection of Different Types of Fuzzy Numbers

As the form of a fuzzy number may vary, the most used forms are the triangular and trapezoidal fuzzy numbers. These are very usable forms, as they are easy to understand and can be simply defined by three (triangular) and four (trapezoidal) values.

We should calculate the positive area and the fuzzy mean of the positive area of a triangular fuzzy pay-off $A = (a, \alpha, \beta)$ in the case of $a - \alpha < 0 < a$. Variable z, where $0 \le z \le \alpha$, represents the distance of a general cut point from $a - \alpha$ at which we separate the triangular fuzzy number (distribution) into two parts—for our purposes the variable z gets the value

$\alpha - a$ (we are interested in the positive part of A). Let us introduce the notation

$$(A \mid z)(\mathrm{t}) = \begin{cases} 0 & \text{if} \quad t \le a - \alpha + z, \\ A(t) & \text{otherwise}, \end{cases} \tag{2.2}$$

for the membership function of the right-hand side of a triangular fuzzy number truncated at point $a - \alpha + z$, where $0 \le z \le \alpha$.

Then we can compute the expected value of this truncated triangular fuzzy number:

$$E(A \mid z) = I_1 + I_2 = \int_0^{z_1} \gamma(a - \alpha + z + a + (1-\gamma)\beta)d\gamma + \int_{z_1}^1 \gamma(a - (1-\gamma)\alpha + a + (1-\gamma)\beta)d\gamma, \tag{2.3}$$

where

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}, \tag{2.4}$$

and the integrals are computed by

$$\begin{aligned} I_1 &= \int_0^{z_1} \left[(2a - \alpha + z + \beta)\gamma - \beta\gamma^2\right] d\gamma \\ &= (2a - \alpha + z + \beta)\frac{z^2}{2\alpha^2} - \beta\frac{z^3}{3\alpha^3}, \\ I_2 &= \int_{z_1}^1 \left[(2a + \beta - \alpha)\gamma - \gamma^2(\beta - \alpha)\right] d\gamma \\ &= (2a + \beta - \alpha)\left(\frac{1}{2} - \frac{z^2}{2\alpha^2}\right) - g(\beta - \alpha)\left(\frac{1}{3} - \frac{z^3}{3\alpha^3}\right), \end{aligned} \tag{2.5}$$

that is,

$$\begin{aligned} I_1 + I_2 = (2a - \alpha + z + \beta) \times \frac{z^2}{2\alpha^2} - \beta \times \frac{z^3}{3\alpha^3} + (2a + \beta - \alpha) \times \left(\frac{1}{2} - \frac{z^2}{2\alpha^2}\right) \\ - (\beta - \alpha) \times \left(\frac{1}{3} - \frac{z^3}{3\alpha^3}\right) = \frac{z^3}{2\alpha^2} + \frac{2a - \alpha + \beta}{2} + \frac{\alpha - \beta}{3} - \alpha \times \frac{z^3}{3\alpha^3}, \end{aligned} \tag{2.6}$$

and we get,

$$E(A \mid z) = \frac{z^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}. \tag{2.7}$$

If $z = \alpha - a$, then $A \mid z$ becomes A_+, the positive side of A, and therefore, we get

$$E(A_+) = \frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}. \tag{2.8}$$

To compute the real option value with the afore-mentioned formulas we must calculate the ratio between the positive area of the triangular fuzzy number and the total area of the same number and multiply this by $E(A_+)$, the fuzzy mean value of the positive part of the fuzzy number A, according to (2.1).

For computing the real option value from an NPV (pay-off) distribution of a trapezoidal form we must consider a trapezoidal fuzzy pay-off distribution A defined by

$$A(u) = \begin{cases} \dfrac{u}{\alpha} - \dfrac{a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \le u \le a_1, \\ 1 & \text{if } a_1 \le u \le a_2, \\ \dfrac{u}{-\beta} + \dfrac{a_2 + \beta}{\beta} & \text{if } a_2 \le u \le a_2 + \beta, \\ 0 & \text{otherwise}, \end{cases} \tag{2.9}$$

where the γ-level of A is defined by $[\mathrm{A}]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$ and its expected value is caculated by

$$E(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}. \tag{2.10}$$

Then we have the following five cases.

Case 1. $z < a_1 - \alpha$. In this case we have $E(A \mid z) = E(A)$.

Case 2. $a_1 - \alpha < z < a_1$. Then introducing the notation

$$\gamma_z = \frac{z}{\alpha} - \frac{a_1 - \alpha}{\alpha}, \tag{2.11}$$

we find

$$[A]^\gamma = \begin{cases} (z, -\gamma\beta + a_2 + \beta) & \text{if } \gamma \le \gamma_z, \\ (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta) & \text{if } \gamma_z \le \gamma \le 1, \end{cases} \tag{2.12}$$

$$\begin{aligned} E(A \mid z) &= \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)\,d\gamma + \int_{\gamma_z}^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta)\,d\gamma \\ &= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} + (z - a_1 + \alpha)\frac{\gamma_z^2}{2} - \alpha\frac{\gamma_z^3}{3}. \end{aligned} \tag{2.13}$$

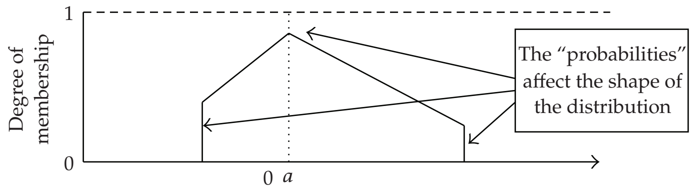


Figure 2: Calculation of the fuzzy mean for the positive part of a fuzzy pay-off distribution of the form of special case.

Case 3. $a_1 < z < a_2$. In this case $\gamma_z = 1$ and

$$[A]^{\gamma} = [z, -\gamma\beta + a_2 + \beta], \tag{2.14}$$

and we get

$$\begin{aligned} E(A \mid z) &= \int_0^1 \gamma(z - \gamma\beta + a_2 + \beta)\,d\gamma \\ &= \frac{z + a_2}{2} + \frac{\beta}{6}. \end{aligned} \tag{2.15}$$

Case 4. $a_2 < z < a_2 + \beta$. In this case we have

$$\gamma_z = \frac{z}{-\beta} + c\frac{a_2 + \beta}{\beta}, \tag{2.16}$$

$$[A]^{\gamma} = [z, -\gamma\beta + a_2 + \beta], \tag{2.17}$$

if $\gamma < \gamma_z$ and we find,

$$\begin{aligned} E(A \mid z) &= \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)\,d\gamma \\ &= (z + a_2 + \beta)\frac{\gamma_z^2}{2} - \beta\frac{\gamma_z^3}{3}. \end{aligned} \tag{2.18}$$

Case 5. $a_2 + \beta < z$. Then it is easy to see that $E(A \mid z) = 0$.

In the following special case, we expect that the managers will have already performed the construction of three cash-flow scenarios and have assigned estimated probabilities to each scenario (adding up to 100%). We want to use all this information and hence will assign the estimated "probabilities" to the scenarios resulting in a fuzzy number that has a graphical presentation of the type presented in Figure 2 (not in scale):

$$A(u)=\begin{cases}(\gamma_3-\gamma_1)\dfrac{u}{\alpha}-(\gamma_3-\gamma_1)\dfrac{a-\alpha}{\alpha}+\gamma_1 & \text{if } a-\alpha\le u\le a,\\ \gamma_3 & \text{if } u=a,\\ (\gamma_2-\gamma_3)\dfrac{u}{\beta}-(\gamma_2-\gamma_3)\dfrac{a}{\beta}+\gamma_3 & \text{if } a\le u\le a+\beta,\\ 0 & \text{otherwise,}\end{cases}$$

$$\begin{aligned}
E(A)&=\int_0^1\gamma(a_1(\gamma)+a_2(\gamma))d\gamma\\
&=\int_0^1\gamma a_1(\gamma)d\gamma+\int_0^1\gamma a_2(\gamma)d\gamma,\\
\int_0^1\gamma a_1(\gamma)d\gamma&=\int_0^{\gamma_1}\gamma(a-\alpha)d\gamma+\int_{\gamma_1}^{\gamma_3}\gamma\left(\frac{\gamma-\gamma_1}{\gamma_3-\gamma_1}\alpha+a-\alpha\right)d\gamma\\
&=(a-\alpha)\frac{\gamma_1^2}{2}+\left(a-\alpha-\frac{\alpha\gamma_1}{\gamma_3-\gamma_1}\right)\left(\frac{\gamma_3^2}{2}-\frac{\gamma_1^2}{2}\right)+\frac{\alpha}{\gamma_3-\gamma_1}\left(\frac{\gamma_3^3}{3}-\frac{\gamma_1^3}{3}\right),\\
\int_0^1\gamma a_2(\gamma)d\gamma&=\int_0^{\gamma_2}\gamma(a+\beta)d\gamma+\int_{\gamma_2}^{\gamma_3}\gamma\left(\frac{\gamma-\gamma_3}{\gamma_2-\gamma_3}\beta+a\right)d\gamma\\
&=(a+\beta)\frac{\gamma_2^2}{2}+\left(a-\frac{\beta\gamma_3}{\gamma_2-\gamma_3}\right)\left(\frac{\gamma_3^2}{2}-\frac{\gamma_2^2}{2}\right)+\frac{\beta}{\gamma_2-\gamma_3}\left(\frac{\gamma_3^3}{3}-\frac{\gamma_2^3}{3}\right),\\
E(A)&=\frac{\gamma_1^2}{2}\frac{\alpha\gamma_1}{\gamma_3-\gamma_1}+\frac{\gamma_2^2}{2}\left(\beta+\frac{\beta\gamma_3}{\gamma_2-\gamma_3}\right)+\frac{\gamma_3^2}{2}\left(2a-\alpha-\frac{\alpha\gamma_1}{\gamma_3-\gamma_1}-\frac{\beta\gamma_3}{\gamma_2-\gamma_3}\right)\\
&\quad-\frac{\gamma_1^3}{3}\frac{\alpha}{\gamma_3-\gamma_1}-\frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2-\gamma_3}+\frac{\gamma_3^3}{3}\left(\frac{\alpha}{\gamma_3-\gamma_1}+\frac{\beta}{\gamma_2-\gamma_3}\right);
\end{aligned}\tag{2.19}$$

(1) $z<a-\alpha: E(A\mid z)=E(A)$,

(2) $a-\alpha<z<a$: $\gamma_z=(\gamma_3-\gamma_1)\dfrac{z}{\alpha}-(\gamma_3-\gamma_1)\dfrac{a-\alpha}{\alpha}+\gamma_1$,

$$\begin{aligned}
E(A\mid z)&=\frac{\gamma_z^2}{2}\left(z-a+\alpha+\frac{\alpha\gamma_1}{\gamma_3-\gamma_1}\right)+\frac{\gamma_2^2}{2}\left(\beta+\frac{\beta\gamma_3}{\gamma_2-\gamma_3}\right)\\
&\quad+\frac{\gamma_3^2}{2}\left(2a-\alpha-\frac{\alpha\gamma_1}{\gamma_3-\gamma_1}-\frac{\beta\gamma_3}{\gamma_2-\gamma_3}\right)-\frac{\gamma_z^3}{3}\frac{\alpha}{\gamma_3-\gamma_1}\\
&\quad-\frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2-\gamma_3}+\frac{\gamma_3^3}{3}\left(\frac{\alpha}{\gamma_3-\gamma_1}+\frac{\beta}{\gamma_2-\gamma_3}\right),
\end{aligned}\tag{2.20}$$

(3) $a < z < a + \beta : \gamma_z = (\gamma_2 - \gamma_3)\dfrac{z}{\beta} - (\gamma_2 - \gamma_3)\dfrac{a}{\beta} + \gamma_3$,

$$E(A \mid z) = \frac{\gamma_z^2}{2}\left(z + a - \frac{\beta}{\gamma_2 - \gamma_3}\right) + \frac{\gamma_2^2}{2}\left(\beta + \frac{\beta\gamma_3}{\gamma_2 - \gamma_3}\right) + \frac{\gamma_z^3}{3}\frac{\beta\gamma_3}{\gamma_2 - \gamma_3} - \frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2 - \gamma_3}, \tag{2.21}$$

(4) $a + \beta < z : E(A \mid z) = 0$.

In the same way as was discussed earlier in connection to the triangular NPV, to compute the real option value with the afore-mentioned formulas we must calculate the ratio between the positive area of the fuzzy number (NPV) and the total area of the same number according to the formula (2.1).

3. A Simple Case: Using the New Method in Analyzing a Business Case

The problem at hand is to evaluate the value of uncertain cash-flows from a business case. The input information available is in the form of three future cash-flow scenarios, good (optimistic), most likely, and bad (pessimistic). The same business case with the same numbers has been earlier presented in [7] and is presented here to allow superficial comparison with the Datar-Mathews method—we are using the same numbers with the fuzzy pay-off method.

The scenario values are given by managers as nonfuzzy numbers, they can, in general, have used any type of analysis tools, or models to reach these scenarios. For more accurate information on the generation of the numbers in this case, see [7] for reference. From the cost and benefit scenarios three scenarios for the NPV are combined (PV benefits - PV investment costs), where the cost cash-flows (CF) are discounted at the risk-free rate and the benefit CF discount rate is selected according to the risk (risk adjusted discount rate). The NPV is calculated for each of the three scenarios separately, see Figures 3 and 4. The resulting fuzzy NPV is the fuzzy pay-off distribution for the investment. To reach a similar probability distribution [7] use Monte Carlo simulation. They point out that a triangular distribution can also be used. The real option value for the investment can be calculated from the resulting fuzzy NPV, which is the pay-off distribution for the project, according to the formula presented in (2.1). We use the formula described in Section 2.1. to calculate the real option value for this business case. We reach the value ROV = 13.56. The work in [7] shows that the value with the same inputs is 8. The difference is caused by the difference in the distributions generated from the inputs.

It is usual that managers are asked to give cash-flow information in the form of scenarios (usually three) and they often have a preselected set of methods for building the scenarios. Usually the scenarios are constructed by trusting past experience and based on looking at, for example, the variables that most contribute to cash-flows and the future market outlook; similar approaches are also reported in [7].

With the fuzzy pay-off method, the scenario approach can be fully omitted and the future cash-flow forecasting can be done fully with fuzzy numbers. The end result will be a fuzzy NPV that is the pay-off distribution for the project. This is the same result that we get if we use scenarios, however, it does not require us to simplify the future to three alternative scenarios.

The detailed calculation used in the case includes present value calculation for the three scenarios of investment cost and revenue cash-flows and then integrates these to form

Investment / cost cash-flows (PV@ "risk free" level) Rf= 0,05

	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6	7	8	9	10
CF Good	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
CF Base	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
CF Bad	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Good	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Base	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Good	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
ΣPV Good	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78
ΣPV Base	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78
ΣPV Bad	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78

Revenue cash-flows (PV@ "risk adjusted" level) Ra = 0,15

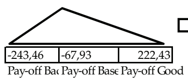
	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6	7	8	9	10
CF Good	0,00	0,00	0,00	0,00	0,00	0,00	80,00	0,00	116,00	0,00	153,00	177,00	223,00	268,00	314,00	0,00
CF Base	0,00	0,00	0,00	0,00	0,00	0,00	52,00	0,00	62,00	0,00	74,00	77,00	89,00	104,00	122,00	0,00
CF Bad	0,00	0,00	0,00	0,00	0,00	0,00	20,00	0,00	23,00	0,00	24,00	18,00	20,00	20,00	22,00	0,00
PV Good	0,00	0,00	0,00	0,00	0,00	0,00	52,60	0,00	66,32	0,00	76,07	76,52	83,83	87,61	89,26	0,00
PV Base	0,00	0,00	0,00	0,00	0,00	0,00	34,19	0,00	35,45	0,00	36,79	33,29	33,46	34,00	34,68	0,00
PV Good	0,00	0,00	0,00	0,00	0,00	0,00	13,15	0,00	13,15	0,00	11,93	7,78	7,52	6,54	6,25	0,00

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
ΣPV Good	0,00	0,00	0,00	0,00	0,00	0,00	52,60	52,60	118,92	118,92	194,99	271,51	355,35	442,96	532,22	532,22
ΣPV Base	0,00	0,00	0,00	0,00	0,00	0,00	34,19	34,19	69,64	69,64	106,43	139,72	173,18	207,18	241,86	241,86
ΣPV Bad	0,00	0,00	0,00	0,00	0,00	0,00	13,15	13,15	26,30	26,30	38,23	46,01	53,53	60,07	66,33	66,33

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
FNPV Good	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-257,18	-257,18	-190,86	-190,86	-114,79	-38,27	45,56	133,17	222,43	222,43
FNPV Base	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-275,59	-275,59	-240,15	-240,15	-203,35	-170,06	-136,61	-102,61	-67,93	-67,93
FNPV Bad	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-296,63	-296,63	-283,48	-283,48	-271,55	-263,77	-256,25	-249,71	-243,46	-243,46

Fuzzy NPV (FNPV) is the pay-distribution from the project. In this case it is calculated by deducting the bad case cost from the good case revenue, the base case cost from the base case revenue, and the good case cost from the bad case revenue - this way the resulting distribution includes the extremes (this is in line with the standard operations with fuzzy numbers).

Pay-off Good 222,43
Pay-off Base -67,93
Pay-off Bad -243,46



13,56
Real Option Value
(by using the fuzzy pay-off method)

Figure 3: Detailed calculations used in the case.

the fuzzy net present value (FNPV). The value of the R&D is directly included in the cost cash-flow table and the resulting ROV is what the work in [7] calls total project value. This is a minor issue, as the [7] project option value is the total project value + the R&D Cost.

4. Discussion and Conclusions

There is a reason to expect that the simplicity of the presented method is an advantage over more complex methods. Using triangular and trapezoidal fuzzy numbers makes very easy implementations possible with the most commonly used spreadsheet software; this opens avenues for real option valuation to find its way to more practitioners. The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis. Fuzzy NPV is a distribution of the possible values that can take place for NPV; this means that it is by definition perceived as impossible at the time of the assessment that values outside of the number can happen. This is in line with the situation that real option value is zero when all the values of the fuzzy NPV are lower than zero. If we compare this to the presented case, we can see that in practice it is often that managers are not interested to use the full distribution of possible outcomes, but rather want to limit their assessment to the most possible alternatives (and leaving out the tails of the distribution). We think that the tails should be included in the real option analysis, because even remote possibilities should be taken into consideration.

The method brings forth an issue that has not gotten very much attention in academia, the dynamic nature of the assessment of investment profitability, that is, the assessment

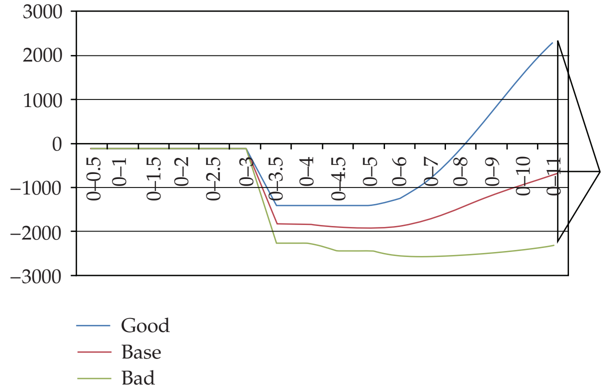


Figure 4: Three NPV scenarios for the duration of the synergies that are used to generate (triangular) fuzzy NPV.

changes when information changes. As cash flows taking place in the future come closer, information changes, and uncertainty is reduced this should be reflected in the fuzzy NPV, the more there is uncertainty the wider the distribution should be, and when uncertainty is reduced, the width of the distribution should decrease. Only under full certainty should the distribution be represented by a single number, as the method uses fuzzy NPV there is a possibility to have the size of the distribution decrease with a lesser degree of uncertainty, this is an advantage vis-à-vis probability-based methods.

The common decision rules for ROV analysis are applicable with the ROV derived with the presented method. We suggest that the single number NPV needed for comparison purposes is derived from the (same) fuzzy NPV by calculating the fuzzy mean value. This means that in cases when all the values of the fuzzy NPV are greater than zero, the single number NPV equals ROV, which indicates immediate investment.

We feel that the presented new method opens possibilities for making simpler generic and modular real option valuation tools that will help construct real options analyses for systems of real options that are present in many types of investments.

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