RELATIVISTIC WAVE EQUATIONS WITH FRACTIONAL DERIVATIVES AND PSEUDODIFFERENTIAL OPERATORS

PETR ZÁVADA

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We study the class of the free relativistic covariant equations generated by the fractional powers of the d'Alembertian operator ($\Box^{1/n}$). The equations corresponding to $n = 1$ and 2 (Klein-Gordon and Dirac equations) are local in their nature, but the multicomponent equations for arbitrary $n > 2$ are nonlocal. We show the representation of the generalized algebra of Pauli and Dirac matrices and how these matrices are related to the algebra of SU(n) group. The corresponding representations of the Poincaré group and further symmetry transformations on the obtained equations are discussed. The construction of the related Green functions is suggested.

1. Introduction

The relativistic covariant wave equations represent an intersection of ideas of the theory of relativity and quantum mechanics. The first and best known relativistic equations, the Klein-Gordon and particularly Dirac equation, belong to the essentials, which our present understanding of the microworld is based on. In this sense, it is quite natural that the searching for and the study of the further types of such equations represent a field of stable interest. For a review see, for example, [5] and the references therein. In fact, the attention has been paid first of all to the study of equations corresponding to the higher spins ($s \geq 1$) and to the attempts to solve the problems, which have been revealed in

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the connection with these equations, for example, the acausality due to external fields introduced by the minimal way.

In this paper, we study the class of equations obtained by the factorization of the d'Alembertian operator, that is, by a generalization of the procedure by which the Dirac equation is obtained. As a result, from each degree of extraction n we get a multicomponent equation, in this way the special case $n = 2$ corresponds to the Dirac equation. However, the equations for $n > 2$ differ substantially from the cases $n = 1,2$ since they contain fractional derivatives (or pseudodifferential operators), so in the effect their nature is nonlocal.

In Section 2, the generalized algebras of the Pauli and Dirac matrices are considered and their properties are discussed, in particular their relation to the algebra of the $SU(n)$ group. The main part (Section 3) deals with the covariant wave equations generated by the roots of the d'Alembertian operator, these roots are defined with the use of the generalized Dirac matrices. In this section, we show the explicit form of the equations, their symmetries, and the corresponding transformation laws. We also define the scalar product and construct the corresponding Green functions. The last section (Section 4) is devoted to the summary and concluding remarks.

Note that the application of the pseudodifferential operators in the relativistic equations is nothing new. The very interesting aspects of the scalar relativistic equations based on the square root of the Klein-Gordon equation are pointed out, for example, in [8, 15, 16]. Recently, an interesting approach for the scalar relativistic equations based on the pseudodifferential operators of the type $f(\Box)$ has been proposed in [1]. We can mention also [7, 17] in which the square and cubic roots of the Dirac equation were studied in the context of supersymmetry. The cubic roots of the Klein-Gordon equation were discussed in the recent papers [10, 13].

It should be observed that our considerations concerning the generalized Pauli and Dirac matrices (Section 2) have much common with the earlier studies related to the generalized Clifford algebras (see, e.g., [2, 3, 12, 14] and the references therein) and with [9], even if our starting motivation is rather different.

2. Generalized algebras of Pauli and Dirac matrices

In the following, by the term *matrix* we mean the square matrix $n \times n$, if not stated otherwise. Considerations of this section are based on the matrix pair introduced as follows.

Definition 2.1. For any $n \geq 2$, we define the matrices

$$
\begin{aligned}
S &= \begin{pmatrix} 0 & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}, \\
T &= \begin{pmatrix} 1 & & & & \\ & \alpha & & & \\ & & \alpha^2 & & \\ & & & \ddots & \\ & & & & \alpha^{n-1} \end{pmatrix},
\end{aligned}
\tag{2.1}
$$

where $\alpha = \exp(2\pi i/n)$, and in the remaining empty positions are zeros.

LEMMA 2.2. *Matrices $X = S,T$ satisfy the following relations:*

$$\alpha ST = TS, \tag{2.2}$$

$$X^n = I, \tag{2.3}$$

$$XX^\dagger = X^\dagger X = I, \tag{2.4}$$

$$\det X = (-1)^{n-1}, \tag{2.5}$$

$$\operatorname{tr} X^k = 0, \quad k = 1,2,\dots,n-1, \tag{2.6}$$

where I denotes the unit matrix.

Proof. All the relations easily follow from Definition 2.1. □

Definition 2.3. Let $\mathcal{A}$ be some algebra on the field of complex numbers, let (p,m) be a pair of natural numbers, $X_1, X_2, \dots, X_m \in \mathcal{A}$ and $a_1, a_2, \dots, a_m \in C$. The pth power of the linear combination can be expanded

$$\left(\sum_{k=1}^{m} a_k X_k\right)^p = \sum_{p_j} a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m} \left\{X_1^{p_1}, X_2^{p_2}, \dots, X_m^{p_m}\right\}; \quad p_1 + \cdots + p_m = p, \tag{2.7}$$

where the symbol $\{X_1^{p_1}, X_2^{p_2}, \dots, X_m^{p_m}\}$ represents the sum of all the possible products created from elements X_k in such a way that each product contains the element X_k just p_k-times. We will call this symbol combinator.

Example 2.4. Some simple combinators read:

$$\begin{aligned}\{X,Y\} &= XY+YX,\\ \{X,Y^2\} &= XY^2+YXY+Y^2X,\\ \{X,Y,Z\} &= XYZ+XZY+YXZ+YZX+ZXY+ZYX.\end{aligned}\tag{2.8}$$

Now, we will prove some useful identities.

Lemma 2.5. *Assume that z is a complex variable, $p,r\ge 0$, and denote*

$$q_p(z)=(1-z)(1-z^2)\cdots(1-z^p),\qquad q_0(z)=1,\tag{2.9}$$

$$F_{rp}(z)=\sum_{k_p=0}^{r}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}z^{k_1}z^{k_2}\cdots z^{k_p},\tag{2.10}$$

$$\begin{aligned}G_p(z)&=\sum_{k=0}^{p}\frac{z^k}{q_{p-k}(z^{-1})q_k(z)},\\ H_p(z)&=\sum_{k=0}^{p}\frac{1}{q_{p-k}(z^{-1})q_k(z)}.\end{aligned}\tag{2.11}$$

Then the following identities hold for $z\neq 0$, $z^j\neq 1$; $j=1,2,\dots,p$:

$$q_p(z)=(-1)^p z^{p(p+1)/2}q_p(z^{-1}),\tag{2.12}$$

$$G_p(z)=0,\tag{2.13}$$

$$H_p(z)=1,\tag{2.14}$$

$$F_{rp}(z)=\sum_{k=0}^{p}\frac{z^{k\cdot r}}{q_{p-k}(z)q_k(z^{-1})}\tag{2.15}$$

and in particular, for $z^{p+r}=1$

$$F_{rp}(z)=0.\tag{2.16}$$

Proof. (1) Relation (2.12) follows immediately from definition (2.9)

$$\begin{aligned}q_r(z)&=(1-z)(1-z^2)\cdots(1-z^r)\\ &=z\cdot z^2\cdots z^r(z^{-1}-1)\cdots(z^{-r}-1)\\ &=(-1)^r z^{r(r+1)/2}q_r(z^{-1}).\end{aligned}\tag{2.17}$$

(2) Relations (2.13) and (2.14): first, if we invert the order of adding in relations (2.11) making substitution, $j = p - k$, then

$$G_p(z) = \sum_{k=0}^{p} \frac{z^k}{q_{p-k}(z^{-1})q_k(z)} = z^p \sum_{j=0}^{p} \frac{z^{-j}}{q_j(z^{-1})q_{p-j}(z)} = z^p G_p(z^{-1}), \quad (2.18)$$

$$H_p(z) = \sum_{k=0}^{p} \frac{1}{q_{p-k}(z^{-1})q_k(z)} = \sum_{j=0}^{p} \frac{1}{q_j(z^{-1})q_{p-j}(z)} = H_p(z^{-1}). \quad (2.19)$$

Now, we calculate

$$\begin{aligned} H_p(z) - H_{p-1}(z) &= \sum_{k=0}^{p} \frac{1}{q_{p-k}(z^{-1})q_k(z)} - \sum_{k=0}^{p-1} \frac{1}{q_{p-1-k}(z^{-1})q_k(z)} \\ &= \frac{1}{q_p(z)} + \sum_{k=0}^{p-1} \frac{1}{q_{p-k}(z^{-1})q_k(z)} - \sum_{k=0}^{p-1} \frac{1}{q_{p-k-1}(z^{-1})q_k(z)} \\ &= \frac{1}{q_p(z)} + \sum_{k=0}^{p-1} \frac{1-(1-z^{k-p})}{q_{p-k}(z^{-1})q_k(z)} \\ &= \sum_{k=0}^{p} \frac{z^{k-p}}{q_{p-k}(z^{-1})q_k(z)} \\ &= G_p(z^{-1}). \end{aligned} \quad (2.20)$$

The last relation combined with (2.19) implies that

$$G_p(z^{-1}) = G_p(z), \quad (2.21)$$

which, compared with (2.18), gives

$$G_p(z^{-1}) = 0; \quad z \neq 0,\ z^j \neq 1,\ j = 1,2,\dots,p. \quad (2.22)$$

So identity (2.13) is proved. Further, relations (2.22) and (2.20) imply that

$$H_p(z) - H_{p-1}(z) = 0, \quad (2.23)$$

therefore,

$$H_p(z) = H_{p-1}(z) = \cdots = H_0(z) = 1, \quad (2.24)$$

and identity (2.14) is proved as well.

(3) Relation (2.15) can be proved by induction, therefore, first assume $p=1$, then its left-hand side reads

$$\sum_{k_1=0}^{k_2} z^{k_1} = \frac{1-z^{k_2+1}}{1-z} \tag{2.25}$$

and the right-hand side gives

$$\frac{1}{q_1(z)} + \frac{z^{k_2}}{q_1\left(z^{-1}\right)} = \frac{1}{1-z} + \frac{z^{k_2}}{1-z^{-1}} = \frac{1-z^{k_2+1}}{1-z}, \tag{2.26}$$

so for $p=1$ the relation is valid. Now, suppose that the relation holds for p and calculate the case $p+1$

$$\begin{aligned}
&\sum_{k_{p+1}=0}^{k_{p+2}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} z^{k_1} z^{k_2} \ldots z^{k_{p+1}} \\
&= \sum_{k_{p+1}=0}^{k_{p+2}} z^{k_{p+1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} z^{k_1} z^{k_2} \ldots z^{k_p} \\
&= \sum_{k_{p+1}=0}^{k_{p+2}} z^{k_{p+1}} \sum_{k=0}^{p} \frac{z^{k \cdot k_{p+1}}}{q_{p-k}(z) q_k\left(z^{-1}\right)} \\
&= \sum_{k=0}^{p} \frac{1}{q_{p-k}(z) q_k\left(z^{-1}\right)} \sum_{k_{p+1}=0}^{k_{p+2}} z^{(k+1)\cdot k_{p+1}} \\
&= \sum_{k=0}^{p} \frac{1}{q_{p-k}(z) q_k\left(z^{-1}\right)} \frac{1-z^{(k+1)\cdot(k_{p+2}+1)}}{1-z^{k+1}} \\
&= \sum_{k=0}^{p} \frac{z^{-k-1}-z^{(k+1)\cdot k_{p+2}}}{q_{p-k}(z) q_k\left(z^{-1}\right)\left(z^{-k-1}-1\right)} \\
&= \sum_{k=0}^{p} \frac{z^{(k+1)\cdot k_{p+2}}-z^{-k-1}}{q_{p-k}(z) q_{k+1}\left(z^{-1}\right)} \\
&= \sum_{k=1}^{p+1} \frac{z^{k\cdot k_{p+2}}-z^{-k}}{q_{p+1-k}(z) q_k\left(z^{-1}\right)} \\
&= \sum_{k=0}^{p+1} \frac{z^{k\cdot k_{p+2}}-z^{-k}}{q_{p+1-k}(z) q_k\left(z^{-1}\right)} \\
&= \sum_{k=0}^{p+1} \frac{z^{k\cdot k_{p+2}}}{q_{p+1-k}(z) q_k\left(z^{-1}\right)} - \sum_{k=0}^{p+1} \frac{z^{-k}}{q_{p+1-k}(z) q_k\left(z^{-1}\right)}.
\end{aligned} \tag{2.27}$$

The last sum equals $G_{p+1}(z^{-1})$, which is zero according to (2.13), so we have proven relation (2.15) for $p+1$. Therefore, the relation is valid for any p.

(4) Relation (2.16) is a special case of (2.15). The denominators in the sum (2.15) can be with the use of the identity (2.12) expressed as

$$q_{p-k}(z)q_k(z^{-1}) = (-1)^p z^s q_{p-k}(z^{-1})q_k(z), \quad s = \left(\frac{p}{2}-k\right)(p+1), \tag{2.28}$$

and since $z^{r\cdot k} = z^{-p\cdot k}$, the sum can be rewritten as

$$\begin{aligned}\sum_{k=0}^{p}\frac{z^{k\cdot r}}{q_{p-k}(z)q_k(z^{-1})} &= (-1)^p\sum_{k=0}^{p}\frac{z^{-s}z^{-p\cdot k}}{q_{p-k}(z^{-1})q_k(z)}\\ &= (-1)^p z^{-p(p+1)/2}\sum_{k=0}^{p}\frac{z^k}{q_{p-k}(z^{-1})q_k(z)}.\end{aligned} \tag{2.29}$$

Obviously, the last sum coincides with $G_p(z)$, which is zero according to the already proven identity (2.13).

Remark that Lemma 2.5 implies also the known formula

$$x^n - y^n = (x-y)(x-\alpha y)(x-\alpha^2 y)\cdots(x-\alpha^{n-1}y), \quad \alpha = \exp\left(\frac{2\pi i}{n}\right). \tag{2.30}$$

The product can be expanded as follows:

$$x^n - y^n = \sum_{j=0}^{n} c_j x^{n-j}(-y)^j, \tag{2.31}$$

and we can easily check that

$$c_0 = 1, \qquad c_n = \alpha\alpha^2\alpha^3\cdots\alpha^{n-1} = (-1)^{n-1}. \tag{2.32}$$

For the remaining j, $0<j<n$, we get

$$c_j = \sum_{k_j=j-1}^{n-1}\cdots\sum_{k_2=1}^{k_3-1}\sum_{k_1=0}^{k_2-1}\alpha^{k_1}\alpha^{k_2}\cdots\alpha^{k_j}, \tag{2.33}$$

and after the shift of the summing limits, we obtain

$$c_j = \alpha\alpha^2\alpha^3\cdots\alpha^{j-1}\sum_{k_j=0}^{n-j}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\alpha^{k_1}\alpha^{k_2}\cdots\alpha^{k_j}. \tag{2.34}$$

This multiple sum is a special case of formula (2.10) and since $\alpha^n = 1$, the identity (2.16) is satisfied. Therefore, for $0 < j < n$ we get $c_j = 0$, and formula (2.30) is proved. □

Definition 2.6. Suppose a matrix product created from some string of matrices X, Y in such a way that matrix X is in total involved p-times, and Y is involved r-times. By the symbol P_j^+ (P_j^-) we denote permutation, which shifts the leftmost (rightmost) matrix to right (left) on the position in which the shifted matrix has j matrices of different kind left (right). (The range of j is restricted by p or r if the shifted matrix is Y or X.)

Example 2.7. Simple case of the permutation defined above reads:

$$P_3^+ \circ XYXYYXY = YXYYXXY. \tag{2.35}$$

Now, we can prove the following theorem.

Theorem 2.8. *Let $p, r > 0$ and $p + r = n$ (i.e., $\alpha^{p+r} = 1$). Then the matrices S, T fulfill*

$$\{S^p, T^r\} = 0. \tag{2.36}$$

Proof. Obviously, all the terms in the combinator $\{S^p, T^r\}$ can be generated, for example, from the string

$$\underbrace{SS\cdots S}_{p}\,\underbrace{TT\cdots T}_{r} = S^pT^r \tag{2.37}$$

by means of the permutations P_j^+

$$\{S^p, T^r\} = \sum_{k_p=0}^{r}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} P_{k_1}^+ \circ P_{k_2}^+ \cdots P_{k_p}^+ \circ S^pT^r. \tag{2.38}$$

Now relation (2.2) implies that

$$P_j^+ \circ S^pT^r = \alpha^j S^pT^r \tag{2.39}$$

and (2.38) can be modified

$$\{S^p, T^r\} = \left(\sum_{k_p=0}^{r}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} \alpha^{k_1}\alpha^{k_2}\cdots\alpha^{k_p}\right) S^pT^r. \tag{2.40}$$

Apparently, the multiple sum in this equation coincides with the right-hand side of (2.10) and satisfies the condition for (2.16), thereby the theorem is proved. □

Remark that an alternative use of permutations P_j^- instead of P_j^+ would lead to the equation

$$\{S^p, T^r\} = \left(\sum_{k_r=0}^{p} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1}\alpha^{k_2}\cdots\alpha^{k_r}\right) S^p T^r. \tag{2.41}$$

The comparison of (2.40) and (2.41) with the relation for F_{pr} defined by (2.10) implies that

$$F_{pr}(\alpha) = F_{rp}(\alpha). \tag{2.42}$$

Obviously, this equation is valid irrespective of the assumption $\alpha^{p+r} = 1$, that is, it holds for any n and $\alpha = \exp(2\pi i/n)$. It follows that (2.42) is satisfied for any α.

Definition 2.9. By the symbols Q_{pr} we denote n^2 matrices,

$$Q_{pr} = S^p T^r, \quad p, r = 1, 2, \ldots, n. \tag{2.43}$$

Lemma 2.10. *The matrices Q_{pr} satisfy the following relations:*

$$Q_{rs}Q_{pq} = \alpha^{s\cdot p}Q_{kl}; \quad k = \operatorname{mod}(r+p-1, n)+1,\ l = \operatorname{mod}(s+q-1, n)+1, \tag{2.44}$$

$$Q_{rs}Q_{pq} = \alpha^{s\cdot p - r\cdot q}Q_{pq}Q_{rs}, \tag{2.45}$$

$$(Q_{rs})^n = (-1)^{(n-1)r\cdot s}I, \tag{2.46}$$

$$Q_{rs}^\dagger Q_{rs} = Q_{rs}Q_{rs}^\dagger = I, \tag{2.47}$$

$$Q_{rs}^\dagger = \alpha^{r\cdot s}Q_{kl}; \quad k = n - r,\ l = n - s, \tag{2.48}$$

$$\det Q_{rs} = (-1)^{(n-1)(r+s)}, \tag{2.49}$$

and for $r \neq n$ or $s \neq n$,

$$\operatorname{tr} Q_{rs} = 0. \tag{2.50}$$

Proof. The relations follow from the definition of Q_{pr} and relations (2.2). □

Theorem 2.11. *The matrices Q_{pr} are linearly independent and any matrix A (of the same dimension) can be expressed as their linear combination*

$$A = \sum_{k,l=1}^{n} a_{kl} Q_{kl}, \quad a_{kl} = \frac{1}{n} \operatorname{tr} \left(Q_{kl}^{\dagger} A \right). \tag{2.51}$$

Proof. Assume that matrices Q_{kl} are linearly dependent, that is, there exists some $a_{rs} \neq 0$, and simultaneously,

$$\sum_{k,l=1}^{n} a_{kl} Q_{kl} = 0, \tag{2.52}$$

which with the use of Lemma 2.10 implies that

$$\operatorname{tr} \sum_{k,l=1}^{n} a_{kl} Q_{rs}^{\dagger} Q_{kl} = a_{rs} n = 0. \tag{2.53}$$

This equation contradicts our assumption, therefore, the matrices are independent and obviously represent a base in the linear space of matrices $n \times n$, which with the use of Lemma 2.10 implies relations (2.51). □

Theorem 2.12. *For any $n \geq 2$, among the n^2 matrices (2.43), there exists the triad Q_λ, Q_μ, Q_ν for which*

$$\{Q_\lambda^p, Q_\mu^r\} = \{Q_\mu^p, Q_\nu^r\} = \{Q_\nu^p, Q_\lambda^r\} = 0; \quad 0 < p, r, \ p + r = n \tag{2.54}$$

and moreover, if $n \geq 3$, then also

$$\{Q_\lambda^p, Q_\mu^r, Q_\nu^s\} = 0; \quad 0 < p, r, s, \ p + r + s = n. \tag{2.55}$$

Proof. We show that the relations hold, for example, for indices $\lambda = 1n$, $\mu = 11$, $\nu = n1$. Denote

$$X = Q_{1n} = S, \qquad Y = Q_{11}, \qquad Z = Q_{n1} = T, \tag{2.56}$$

then relation (2.45) implies that

$$YX = \alpha XY, \qquad ZX = \alpha XZ, \qquad ZY = \alpha YZ. \tag{2.57}$$

Actually, the relation $\{X^p, Z^r\} = 0$ is already proven in Theorem 2.8, obviously the remaining relations (2.54) can be proved exactly in the same way.

The combinator (2.55) can be, as in the proof of Theorem 2.8, expressed as

$$\begin{aligned}\{X^p, Y^r, Z^s\} \\ &= \sum_{j_p=0}^{r+s} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} P_{j_1}^+ \circ P_{j_2}^+ \cdots P_{j_p}^+ \circ X^p \sum_{k_p=0}^{s} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} P_{k_1}^+ \circ P_{k_2}^+ \cdots P_{k_r}^+ \circ Y^r Z^s,\end{aligned} \tag{2.58}$$

which for the matrices obeying relations (2.57) give

$$\begin{aligned}\{X^p, Y^r, Z^s\} \\ &= \left(\sum_{j_p=0}^{r+s} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \alpha^{j_1}\alpha^{j_2}\cdots\alpha^{j_p}\right)\left(\sum_{k_p=0}^{s} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1}\alpha^{k_2}\cdots\alpha^{k_r}\right) X^p Y^r Z^s.\end{aligned} \tag{2.59}$$

Since the first multiple sum (with indices j) coincides with (2.10) and satisfies the condition for (2.16), the right-hand side is zero and the theorem is proved. □

Now we make few remarks to illuminate the content of Theorem 2.12 and meaning of the matrices Q_λ. Obviously, relations (2.54) and (2.55) are equivalent to the statement that any three complex numbers a, b, c satisfy

$$(aQ_\lambda + bQ_\mu + cQ_\nu)^n = (a^n + b^n + c^n)I. \tag{2.60}$$

Further, Theorem 2.12 speaks about the existence of the triad but not about their number. Generally, for $n > 2$ there is more than one triad defined by the theorem, but on the other hand, not any three various matrices from the set Q_{rs} comply with the theorem. Simple example are some X, Y, Z where, for example, $XY = YX$, which happens for $Y \sim X^p$, $2 \le p < n$. Obviously, in this case at least relation (2.54) surely is not satisfied. Computer check of relation (2.58) which has been done with all possible triads from Q_{rs} for $2 \le n \le 20$ suggests that a triad X, Y, Z for which there exist the numbers $p, r, s \ge 1$ and $p + r + s \le n$ so that $X^p Y^r Z^s \sim I$ also does not comply with the theorem. Further, the result on the right-hand side of (2.58) generally depends on the factors β_k in the relations

$$XY = \beta_3 YX, \qquad YZ = \beta_1 ZY, \qquad ZX = \beta_2 XZ, \tag{2.61}$$

and a computer check suggests the sets, in which for some β_k and $p < n$ there is $\beta_k^p = 1$, also contradict the theorem. In this way, the number of different triads obeying relations (2.54) and (2.55) is a rather complicated function of n, as shown in Table 2.1.

TABLE 2.1

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
#3	1	1	1	4	1	9	4	9	4	25	4	36	9	16	16	64	9	81	16

Here the statement that the triad X, Y, Z is different from X', Y', Z' means that after any rearrangement of the symbols X, Y, Z for marking of matrices in the given set, there is always at least one pair $\beta_k \neq \beta'_k$.

Naturally, we can ask if there exists also the set of four or generally N matrices, which satisfy a relation similar to (2.60),

$$\left(\sum_{\lambda=0}^{N-1} a_\lambda Q_\lambda\right)^n = \sum_{\lambda=0}^{N-1} a_\lambda^n. \tag{2.62}$$

For $2 \leq n \leq 10$ and $N = 4$, the computer suggests the negative answer, in the case of matrices generated according to Definition 2.9. However, we can verify that if U_l, $l = 1,2,3$, is the triad complying with Theorem 2.12 (or equivalently with relation (2.60)), then the matrices $n^2 \times n^2$

$$Q_0 = I \otimes T = \begin{pmatrix} I & & & & \\ & \alpha I & & & \\ & & \alpha^2 I & & \\ & & & \ddots & \\ & & & & \alpha^{n-1} I \end{pmatrix}, \tag{2.63}$$

$$Q_l = U_l \otimes S = \begin{pmatrix} 0 & & & & U_l \\ U_l & & & & \\ & U_l & & & \\ & & \ddots & & \\ & & & U_l & 0 \end{pmatrix} \tag{2.64}$$

satisfy relation (2.62) for $N = 4$. Generally, if U_λ are matrices complying with (2.62) for some $N \geq 3$, then the matrices created from them according to the rule (2.63) and (2.64) will satisfy (2.62) for $N + 1$. The last statement follows from the following equalities. Assume that

$$\sum_{k=0}^{N} p_k = n, \tag{2.65}$$

then

$$\begin{aligned}
&\{Q_0^{p_0}, Q_1^{p_1}, \dots, Q_N^{p_N}\} \\
&= \sum_{j_{p_N}=0}^{n-p_N} \cdots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} P_{j_0}^- \circ P_{j_1}^- \cdots P_{j_{p_N}}^- \circ \{Q_0^{p_0}, \dots, Q_{N-1}^{p_{N-1}}\} Q_N^{p_N} \\
&= \sum_{j_{p_N}=0}^{n-p_N} \cdots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} P_{j_0}^- \circ P_{j_1}^- \cdots P_{j_{p_N}}^- \circ \{(U_0 \otimes S)^{p_0}, \dots, (U_{N-1} \otimes S)^{p_{N-1}}\} (I \otimes T)^{p_N} \\
&= \sum_{j_{p_N}=0}^{n-p_N} \cdots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \alpha^{j_0} \alpha^{j_1} \cdots \alpha^{j_{p_N}} \{(U_0 \otimes S)^{p_0}, \dots, (U_{N-1} \otimes S)^{p_{N-1}}\} (I \otimes T)^{p_N} \\
&= \left(\sum_{j_{p_N}=0}^{n-p_N} \cdots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \alpha^{j_0} \alpha^{j_1} \cdots \alpha^{j_{p_N}} \right) \{U_1^{p_1}, \dots, U_{N-1}^{p_{N-1}}\} \otimes S^{n-p_N} T^{p_N},
\end{aligned} \tag{2.66}$$

where the last multiple sum equals zero according to relations (2.10) and (2.16). Obviously, for $n = 2$ matrices (2.56), (2.63), and (2.64) created from them correspond, up to some phase factors, to the Pauli matrices σ_j and Dirac matrices γ_μ.

Obviously, from the set of matrices Q_{rs} (with exception of $Q_{nn} = I$) we can easily make the $n^2 - 1$ generators of the fundamental representation of SU(n) group,

$$G_{rs} = a_{rs} Q_{rs} + a_{rs}^* Q_{rs}^+, \tag{2.67}$$

where a_{rs} are suitable factors. For example, the choice

$$a_{kl} = \frac{1}{\sqrt{2}} \alpha^{[kl + n(k+l-1/4)]/2} \tag{2.68}$$

gives the commutation relations

$$\begin{aligned}
[G_{kl}, G_{rs}] = i \sin\left(\frac{\pi(ks - lr)}{n}\right) \\
\cdot \{ \mathrm{sg}(k+r, l+s, n) \left(G_{k+r,l+s} - (-1)^{n+k+l+r+s} G_{-k-r,-l-s} \right) \\
- \mathrm{sg}(k-r, l-s, n) \left(G_{k-r,l-s} - (-1)^{n+k+l+r+s} G_{r-k,s-l} \right) \},
\end{aligned} \tag{2.69}$$

where

$$\mathrm{sg}(p, q, n) = (-1)^{p \cdot m_q + q \cdot m_p - n}, \qquad m_x = \frac{x - \mathrm{mod}(x-1, n) - 1}{n}, \tag{2.70}$$

and the indices at G (on the right-hand side) in (2.69) are understood in the sense of mod, like in relation (2.44). We can easily check, for example, that for $n = 2$ matrices (2.67) with the factors a_{rs} according to (2.68) are the Pauli matrices, generators of the fundamental representation of the SU(2) group.

3. Wave equations generated by the roots of d'Alembertian operator $\square^{1/n}$

Now, using the generalized Dirac matrices (2.63) and (2.64), we will assemble the corresponding wave equation as follows. These four matrices with the normalization

$$(Q_0)^n = -(Q_l)^n = I, \quad l = 1,2,3, \tag{3.1}$$

allow to write down the set of algebraic equations

$$(\Gamma(p) - \mu I)\Psi(p) = 0, \tag{3.2}$$

where

$$\Gamma(p) = \sum_{\lambda=0}^{3} \pi_\lambda Q_\lambda. \tag{3.3}$$

If the variables μ, π_λ represent the fractional powers of the mass and the momentum components

$$\mu^n = m^2, \qquad \pi_\lambda^n = p_\lambda^2, \tag{3.4}$$

then

$$\Gamma(p)^n = p_0^2 - p_1^2 - p_2^2 - p_3^2 \equiv p^2, \tag{3.5}$$

and after $n - 1$ times-repeated application of the operator Γ on (3.2), we get the set of Klein-Gordon equations in the p-representation,

$$(p^2 - m^2)\Psi(p) = 0. \tag{3.6}$$

Equations (3.2) and (3.6) are the sets of n^2 equations with solution Ψ having n^2 components. Obviously, the case $n^2 = 4$ corresponds to the Dirac equation. For $n > 2$, (3.2) is a new equation, which is more complicated and immediately invoking some questions. In the present paper, we will attempt to answer at least some of them. We can check that the solution of the set (3.2) reads

$$\Psi(p) = \begin{pmatrix} \mathbf{h} \\ \dfrac{U(p)}{\alpha\pi_0 - \mu}\mathbf{h} \\ \dfrac{U^2(p)}{(\alpha\pi_0 - \mu)(\alpha^2\pi_0 - \mu)}\mathbf{h} \\ \vdots \\ \dfrac{U^{n-1}(p)}{(\alpha\pi_0 - \mu)\cdots(\alpha^{n-1}\pi_0 - \mu)}\mathbf{h} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \tag{3.7}$$

where

$$U(p) = \sum_{l=1}^{3} \pi_l U_l, \quad (U_l)^n = -I, \tag{3.8}$$

(U_l is the triad from which the matrices Q_l are constructed in accordance with (2.63) and (2.64)) and $h_1, h_2, \ldots, h_n$ are arbitrary functions of p. At the same time, π_λ satisfy the constraint

$$\pi_0^n - \pi_1^n - \pi_2^n - \pi_3^n = \mu^n = m^2. \tag{3.9}$$

First of all, we can bring to notice that in (3.2) the fractional powers of the momentum components appear, which means that the equation in the x-representation will contain the fractional derivatives

$$\pi_\lambda = (p_\lambda)^{2/n} \longrightarrow (i\partial_\lambda)^{2/n}. \tag{3.10}$$

Our primary considerations will concern p-representation, but afterwards we will show how the transition to the x-representation can be realized by means of the Fourier transformation, in accordance with the approach suggested in [21].

A further question concerning the relativistic covariance of (3.2): how to transform simultaneously the operator

$$\Gamma(p) \longrightarrow \Gamma(p') = \Lambda\Gamma(p)\Lambda^{-1}, \tag{3.11}$$

and the solution

$$\Psi(p) \longrightarrow \Psi'(p') = \Lambda\Psi(p), \tag{3.12}$$

to preserve the equal form of the operator Γ for initial variables p_λ and the boosted ones p'_λ?

3.1. Infinitesimal transformations

First, consider the infinitesimal transformations

$$\Lambda(d\omega) = I + i\,d\omega \cdot L_\omega, \tag{3.13}$$

where $d\omega$ represents the infinitesimal values of the six parameters of the Lorentz group corresponding to the space rotations

$$p_i' = p_i + \epsilon_{ijk} p_j\, d\varphi_k, \quad i = 1,2,3, \tag{3.14}$$

and the Lorentz transformations

$$p_i' = p_i + p_0\, d\psi_i, \quad p_0' = p_0 + p_i\, d\psi_i, \quad i = 1,2,3, \tag{3.15}$$

where $\tanh\psi_i = v_i/c \equiv \beta_i$ is the corresponding velocity. Here, and anywhere in the next we use the convention that in the expressions involving the antisymmetric tensor ϵ_{ijk}, the summation over indices appearing twice is done. From the infinitesimal transformations (3.14) and (3.15), we can obtain the finite ones. For the three space rotations, we get

$$\begin{aligned}
&p_1' = p_1\cos\varphi_3 + p_2\sin\varphi_3, && p_2' = p_2\cos\varphi_3 - p_1\sin\varphi_3, && p_3' = p_3,\\
&p_2' = p_2\cos\varphi_1 + p_3\sin\varphi_1, && p_3' = p_3\cos\varphi_1 - p_2\sin\varphi_1, && p_1' = p_1,\\
&p_3' = p_3\cos\varphi_2 + p_1\sin\varphi_2, && p_1' = p_1\cos\varphi_2 - p_3\sin\varphi_2, && p_2' = p_2
\end{aligned} \tag{3.16}$$

and for the Lorentz transformations, similarly,

$$p_0' = p_0\cosh\psi_i + p_i\sinh\psi_i, \quad i = 1,2,3, \tag{3.17}$$

where

$$\cosh\psi_i = \frac{1}{\sqrt{1-\beta_i^2}}, \qquad \sinh\psi_i = \frac{\beta_i}{\sqrt{1-\beta_i^2}}. \tag{3.18}$$

The definition of the six parameters implies that the corresponding infinitesimal transformations of the reference frame $p \to p'$ changes a function $f(p)$:

$$f(p) \longrightarrow f(p') = f(p+\delta p) = f(p) + \frac{df}{d\omega}d\omega, \tag{3.19}$$

where $d/d\omega$ stands for

$$\frac{d}{d\varphi_i} = -\epsilon_{ijk} p_j \frac{\partial}{\partial p_k}, \quad \frac{d}{d\psi_i} = p_0 \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial p_0}, \quad i = 1,2,3. \tag{3.20}$$

Obviously, the equation

$$p' = p + \frac{dp}{d\omega} d\omega \tag{3.21}$$

combined with (3.20) is identical to (3.14) and (3.15). Further, with the use of formulas (3.13) and (3.20), relations (3.11) and (3.12) can be rewritten in the infinitesimal form

$$\begin{aligned} \Gamma(p') &= \Gamma(p) + \frac{d\Gamma(p)}{d\omega} d\omega = (I + i\, d\omega \cdot L_\omega)\Gamma(p)(I - i\, d\omega \cdot L_\omega), \\ \Psi'(p') &= \Psi'(p) + \frac{d\Psi'(p)}{d\omega} d\omega = (I + i\, d\omega \cdot L_\omega)\Psi(p). \end{aligned} \tag{3.22}$$

If we define

$$\boldsymbol{L_\omega} = L_\omega + i \frac{d}{d\omega}, \tag{3.23}$$

then relations (3.22) imply that

$$[\boldsymbol{L_\omega}, \Gamma] = 0, \tag{3.24}$$

$$\Psi'(p) = (I + i\, d\omega \cdot \boldsymbol{L_\omega})\Psi(p). \tag{3.25}$$

The six operators $\boldsymbol{L_\omega}$ are generators of the corresponding representation of the Lorentz group, so they have to satisfy the commutation relations

$$[\boldsymbol{L_{\varphi_j}}, \boldsymbol{L_{\varphi_k}}] = i\epsilon_{jkl} \boldsymbol{L_{\varphi_l}}, \tag{3.26}$$

$$[\boldsymbol{L_{\psi_j}}, \boldsymbol{L_{\psi_k}}] = -i\epsilon_{jkl} \boldsymbol{L_{\varphi_l}}, \tag{3.27}$$

$$[\boldsymbol{L_{\varphi_j}}, \boldsymbol{L_{\psi_k}}] = i\epsilon_{jkl} \boldsymbol{L_{\psi_l}}, \quad j,k,l = 1,2,3. \tag{3.28}$$

How this representation looks like, in other words, what operators $\boldsymbol{L_\omega}$ satisfy (3.26), (3.27), (3.28), and (3.24)? First, we can easily check that for $n > 2$ there do not exist matrices L_ω with constant elements representing the first term in the right-hand side of equality (3.23) and satisfying (3.24). If we assume that L_ω consist only of constant elements, then the elements of matrix $(d/d\omega)\Gamma(p)$ involving the terms like $p_i^{2/n-1} p_j$ certainly cannot be expressed through the elements of the difference

$L_\omega\Gamma - \Gamma L_\omega$ consisting only of the elements proportional to $p_k^{2/n}$, in contradistinction to the case $n = 2$, that is, the case of the Dirac equation. In this way, (3.24) cannot be satisfied for $n > 2$ and L_ω constant. Nevertheless, we can show that the set of (3.24), (3.26), (3.27), and (3.28) is solvable provided that we accept that the elements of the matrices L_ω are not constants, but the functions of p_i. To prove this, first make a few preparing steps.

Definition 3.1. Let $\Gamma_1(p)$, $\Gamma_2(p)$, and let X be the square matrices of the same dimension and

$$\Gamma_1(p)^n = \Gamma_2(p)^n = p^2. \tag{3.29}$$

Then for any matrix X, we define the form

$$Z(\Gamma_1, X, \Gamma_2) = \frac{1}{np^2}\sum_{j=1}^{n}\Gamma_1^j X \Gamma_2^{n-j}. \tag{3.30}$$

We can easily check that the matrix Z satisfies, for example,

$$\Gamma_1 Z = Z\Gamma_2, \tag{3.31}$$

$$Z(Z(X)) = Z(X), \tag{3.32}$$

and in particular for $\Gamma_1 = \Gamma_2 \equiv \Gamma$,

$$[\Gamma, Z] = 0, \tag{3.33}$$

$$[\Gamma, X] = 0 \Longrightarrow X = Z(X). \tag{3.34}$$

Lemma 3.2. *Equation (3.2) can be expressed in the diagonalized (canonical) form*

$$(\Gamma_0(p) - \mu)\Psi_0(p) = 0; \quad \Gamma_0(p) \equiv (p^2)^{1/n} Q_0, \tag{3.35}$$

where Q_0 is the matrix (2.63), that is, there exists the set of transformations Υ,

$$\Gamma_0(p) = \Upsilon(p)\Gamma(p)\Upsilon^{-1}(p); \quad \Upsilon = Z(\Gamma_0, X, \Gamma), \tag{3.36}$$

and a particular form reads

$$\Upsilon = y \cdot Z(\Gamma_0, I, \Gamma), \qquad \Upsilon^{-1} = y \cdot Z(\Gamma, I, \Gamma_0), \tag{3.37}$$

where

$$y = \sqrt{\frac{n\left[1 - \left(p_0^2/p^2\right)^{1/n}\right]}{1 - p_0^2/p^2}}. \tag{3.38}$$

Proof. Equation (3.31) implies that

$$\Gamma_0 = Z(\Gamma_0, X, \Gamma)\Gamma Z(\Gamma_0, X, \Gamma)^{-1}, \tag{3.39}$$

therefore, if the matrix X is chosen in such a way that $\det Z \neq 0$, then Z^{-1} exists and the transformation (3.39) diagonalizes the matrix Γ. Put $X = I$ and calculate the following product:

$$C = Z(\Gamma_0, I, \Gamma)Z(\Gamma, I, \Gamma_0) = \frac{1}{n^2p^4}\sum_{i,j=1}^{n}\Gamma_0^i\Gamma^{n-i+j}\Gamma_0^{n-j}. \tag{3.40}$$

The last sum can be rearranged, instead of the summation index j we use the new one:

$$k = i - j \quad \text{for } i \geq j, \qquad k = i - j + n \quad \text{for } i < j; \qquad k = 0, \dots, n-1, \tag{3.41}$$

then (3.40) reads

$$C = \frac{1}{n^2p^4}\sum_{k=0}^{n-1}\left(\sum_{i=k+1}^{n}\Gamma_0^i\Gamma^{n-k}\Gamma_0^{n+k-i} + \sum_{i=1}^{k}\Gamma_0^i\Gamma^{2n-k}\Gamma_0^{k-i}\right), \tag{3.42}$$

and if we take into account that $\Gamma_0^n = \Gamma^n = p^2$, then this sum can be simplified as

$$C = \sum_{k=0}^{n-1}C_k = \frac{1}{n^2p^2}\sum_{k=0}^{n-1}\sum_{i=1}^{n}\Gamma_0^i\Gamma^{n-k}\Gamma_0^{k-i}. \tag{3.43}$$

For the term $k = 0$, we get

$$C_0 = \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i\Gamma^n\Gamma_0^{-i} = \frac{1}{n} \tag{3.44}$$

and for $k > 0$, using (3.3), (2.63), (2.64), (3.35), and Definition 2.3 we obtain

$$
\begin{aligned}
C_k &= \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i\Gamma^{n-k}\Gamma_0^{k-i} = \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i\left(\sum_{\lambda=0}^{3}\pi_\lambda Q_\lambda\right)^{n-k}\Gamma_0^{k-i} \\
&= \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i\left(\pi_0\cdot I\otimes T + \left[\sum_{\lambda=1}^{3}\pi_\lambda U_\lambda\right]\otimes S\right)^{n-k}\Gamma_0^{k-i} \\
&= \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i(\pi_0\cdot I\otimes T + U\otimes S)^{n-k}\Gamma_0^{k-i} \qquad (3.45)\\
&= \frac{1}{n^2p^2}\sum_{i=1}^{n}\Gamma_0^i\left(\sum_{p=0}^{n-k}\pi_0^p\cdot U^{n-k-p}\otimes\{T^p,S^{n-k-p}\}\right)\Gamma_0^{k-i} \\
&= \frac{(p^2)^{k/n}}{n^2p^2}\sum_{p=0}^{n-k}\pi_0^p\cdot U^{n-k-p}\otimes\sum_{i=1}^{n}T^i\{T^p,S^{n-k-p}\}T^{k-i}.
\end{aligned}
$$

For $p < n-k \equiv l$, the last sum can be modified with the use of relation (2.2)

$$
\begin{aligned}
\sum_{i=1}^{n}T^i\{T^p,S^{l-p}\}T^{k-i} &= \{T^p,S^{l-p}\}T^k\sum_{i=1}^{n}\alpha^{i\cdot(l-p)} \\
&= \{T^p,S^{l-p}\}T^k\alpha^{(l-p)}\frac{1-\alpha^{n\cdot(l-p)}}{1-\alpha^{(l-p)}} = 0,
\end{aligned}
\tag{3.46}
$$

therefore, only the term $p = n-k$ contributes:

$$
C_k = \frac{(p^2)^{k/n}}{n^2p^2}(p_0^2)^{(n-k)/n}n = \frac{1}{n}\left(\frac{p_0^2}{p^2}\right)^{(n-k)/n}. \tag{3.47}
$$

So the sum (3.43) gives in total

$$
\begin{aligned}
C &= \frac{1}{n}\left[1+\left(\frac{p_0^2}{p^2}\right)^{1/n}+\left(\frac{p_0^2}{p^2}\right)^{2/n}+\cdots+\left(\frac{p_0^2}{p^2}\right)^{(n-1)/n}\right] \\
&= \frac{1-p_0^2/p^2}{n\left[1-(p_0^2/p^2)^{1/n}\right]},
\end{aligned}
\tag{3.48}
$$

therefore, (3.36) is satisfied with Υ, Υ^{-1} given by (3.37) and the proof is completed. □

The solution of (3.35) reads

$$\Psi_0(p) = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{g} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}; \quad \mathbf{g} \equiv \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \ \mathbf{0} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.49}$$

that is, the sequence of nonzero components can be only in one block, whose location depends on the choice of the phase of the power $(p^2)^{1/n}$. The g_j are arbitrary functions of p and simultaneously, the constraint $p^2 = m^2$ is required. Now, we will try to find the generators satisfying the covariance condition for (3.35),

$$\left[\boldsymbol{L}_{\boldsymbol{\omega}}, \Gamma_0(p)\right] = 0, \tag{3.50}$$

together with the commutation relations (3.26), (3.27), and (3.28). Some hint can be obtained from the Dirac equation transformed to the diagonal form in accordance with relations (3.36) and (3.37). We will use the current representation of the Pauli and Dirac matrices:

$$\begin{gathered} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma_0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ -\sigma_j & \mathbf{0} \end{pmatrix}; \quad j = 1,2,3, \end{gathered} \tag{3.51}$$

where the bold **0, 1** stand for zero and unit matrices 2×2. The Dirac equation

$$\big(\Gamma(p) - m\big)\Psi(p) = 0, \quad \Gamma(p) \equiv \sum_{\lambda=0}^{3} p_\lambda \gamma_\lambda \tag{3.52}$$

is covariant under the transformations generated by

$$\begin{gathered} \boldsymbol{L}_{\boldsymbol{\varphi}_j} = \frac{i}{4}\epsilon_{jkl}\gamma_k\gamma_l + i\frac{d}{d\varphi_j} = L_{\varphi_j} + i\frac{d}{d\varphi_j}; \quad L_{\varphi_j} = \frac{1}{2}\begin{pmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix}, \\ \boldsymbol{L}_{\boldsymbol{\psi}_j} = \frac{i}{2}\gamma_0\gamma_j + i\frac{d}{d\psi_j} = L_{\psi_j} + i\frac{d}{d\psi_j}; \quad L_{\psi_j} = \frac{i}{2}\begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}, \end{gathered} \tag{3.53}$$

where $j,k,l = 1,2,3$. Obviously, to preserve covariance, we have also, with the transformation $\Gamma \to \Gamma_0 = \Upsilon\Gamma\Upsilon^{-1}$ perform

$$\boldsymbol{L}_{\boldsymbol{\omega}} \longrightarrow \boldsymbol{M}_{\boldsymbol{\omega}} = \Upsilon(p)\boldsymbol{L}_{\boldsymbol{\omega}}\Upsilon^{-1}(p). \tag{3.54}$$

For the space rotations $\boldsymbol{L}_{\boldsymbol{\varphi}_j}$ commuting with both Γ_0, Γ, and with the Υ from relation (3.37), the result is quite straightforward,

$$\boldsymbol{M}_{\boldsymbol{\varphi}_j} = \boldsymbol{L}_{\boldsymbol{\varphi}_j} = L_{\varphi_j} + i\frac{d}{d\varphi_j}, \tag{3.55}$$

that is, the generators of the space rotations are not changed by the transformation (3.54). The similar procedure with the Lorentz transformations is slightly more complicated, nevertheless, after the calculation of the commutator $[\boldsymbol{L}_{\boldsymbol{\varphi}_j}, \Gamma_0/\sqrt{1+p_0/\sqrt{p^2}}]$ and with a few further steps, we obtain

$$\boldsymbol{M}_{\boldsymbol{\psi}_j} = M_{\psi_j}(p) + i\frac{d}{d\psi_j}; \quad M_{\psi_j}(p) = \epsilon_{jkl}\frac{p_k L_{\varphi_l}}{p_0 + \sqrt{p^2}}. \tag{3.56}$$

So generators (3.55) and (3.56) guarantee the covariance of the diagonalized Dirac equation obtained from (3.52) according to Lemma 3.2. At the same time it is obvious that having the set of generators L_{φ_j} (with constant elements) of space rotations, we can, according to (3.56), construct the generators of Lorentz transformations $M_{\psi_j}(p)$ (or $\boldsymbol{M}_{\boldsymbol{\psi}_j}$), which satisfy commutation relations (3.26), (3.27), and (3.28). Obviously, this recipe is valid for any representation of infinitesimal space rotations. Remark that the algebra given by (3.55) and (3.56) appears in a slightly modified form in [4]. Now, we will show that if we require a linear relation between the generators M_{ψ_j} and L_{φ_l}, like in (3.56), then this relation can have a more general shape than that in (3.56).

Lemma 3.3. *Let L_{φ_j} be matrices with constant elements satisfying commutation relations (3.26). Then the operators*

$$\boldsymbol{M}_{\boldsymbol{\psi}_j} = M_{\psi_j}(p) + i\frac{d}{d\psi_j}; \quad M_{\psi_j}(p) = \frac{\kappa L_{\varphi_j} + \epsilon_{jkl}p_k L_{\varphi_l}}{p_0 + \sqrt{p^2 - \kappa^2}}, \tag{3.57}$$

where κ is any complex constant, satisfy the commutation relations (3.27) and (3.28).

Proof. After the insertion of generators (3.57) into relations (3.27) and (3.28), we can check that the commutation relations are satisfied. In fact, it is sufficient to verify, for example, the commutators $[\boldsymbol{L}_{\varphi_1}, \boldsymbol{L}_{\varphi_2}]$, $[\boldsymbol{L}_{\varphi_1}, \boldsymbol{L}_{\psi_1}]$, and $[\boldsymbol{L}_{\psi_1}, \boldsymbol{L}_{\psi_3}]$, the remaining follow from the cyclic symmetry.

Note that formula (3.57) covers also the limit case $|\kappa| \to \infty$, then

$$M_{\psi_j} = iL_{\varphi_j}. \tag{3.58}$$

On the other hand, relation (3.56) corresponds to $\kappa = 0$. The representations of the Lorentz group defined by generators (3.55) and (3.57) and differing only in the parameter κ should be equivalent in the sense that

$$\boldsymbol{M}_\omega(\kappa') = X^{-1}(p)\boldsymbol{M}_\omega(\kappa)X(p). \tag{3.59}$$

We will not make a general proof of this relation, but rather we will show that the representations, defined in Lemma 3.3 and differing only in κ, can be classified by the same mass $m^2 = p^2$ and spin $\boldsymbol{s}^2 = s(s+1)$. First, note that the six generators considered in the lemma together with the four generators p_α of the space-time translations form the set of generators of the Poincaré group. We can easily check that the corresponding additional commutation relations are satisfied,

$$\begin{gathered}
[p_\alpha, p_\beta] = 0, \qquad [\boldsymbol{M}_{\varphi_j}, p_0] = 0, \qquad [p_\alpha, \Gamma_0] = 0, \\
[\boldsymbol{M}_{\varphi_j}, p_k] = i\epsilon_{jkl}p_l, \qquad [\boldsymbol{M}_{\psi_j}, p_k] = i\delta_{jk}p_0, \qquad [\boldsymbol{M}_{\psi_j}, p_0] = ip_j.
\end{gathered} \tag{3.60}$$

Further, the generators $\boldsymbol{M}_\omega$ can be rewritten in the covariant notation

$$\boldsymbol{M}_{jk} = \epsilon_{jkl}\boldsymbol{M}_{\varphi_l}, \qquad \boldsymbol{M}_{j0} = \boldsymbol{M}_{\psi_j}, \qquad \boldsymbol{M}_{\alpha\beta} = -\boldsymbol{M}_{\beta\alpha}. \tag{3.61}$$

Now the Pauli-Lubanski vector can be constructed:

$$V_\alpha = \frac{\epsilon_{\alpha\beta\gamma\delta}\boldsymbol{M}^{\beta\gamma}p^\delta}{2}, \tag{3.62}$$

which satisfies

$$V_\alpha V^\alpha = -m^2 s(s+1), \tag{3.63}$$

where s is the corresponding spin number. We can check that after inserting the generators (3.61) into relations (3.62) and (3.63), the result does not depend on κ,

$$V_\alpha V^\alpha = -p^2\left(M_{\varphi_1}^2 + M_{\varphi_2}^2 + M_{\varphi_3}^2\right) = -m^2 s(s+1). \tag{3.64}$$

So the generators of the Lorentz group, which satisfy (3.50), can have the form

$$\boldsymbol{R}_{\boldsymbol{\omega}} = R_\omega + i\frac{d}{d\omega}; \quad R_\omega = \begin{pmatrix} M_\omega & & & & \\ & M_\omega & & & \\ & & M_\omega & & \\ & & & \ddots & \\ & & & & M_\omega \end{pmatrix}, \tag{3.65}$$

where M_ω are the $n \times n$ matrices defined in accordance with Lemma 3.3. There are n such matrices on the diagonal and apparently these matrices may not be identical.

Finally, it is obvious that (3.35) is covariant also under any infinitesimal transform,

$$\Lambda(d\xi) = I + i\, d\xi \cdot K_\xi, \tag{3.66}$$

where the generators K_ξ have the similar form as the generators (3.65)

$$K_\xi = \begin{pmatrix} k_\xi & & & & \\ & k_\xi & & & \\ & & k_\xi & & \\ & & & \ddots & \\ & & & & k_\xi \end{pmatrix} \tag{3.67}$$

and generally, their elements may depend on p. Obviously, we can put the question: if the generators L_φ, k_ξ from (3.55) and (3.67) with constant elements represent the algebra of some group (containing the rotation group as a subgroup), then what linear combination $M_{\varphi_j}(p)$ of these generators satisfy the commutation relations (3.27) and (3.28) for the generators of Lorentz transformations? In other words, what are the coefficients in the summation

$$M_{\varphi_j}(p) = \sum_{k=1}^{3} c_{jk}(p) L_{\varphi_k} + \sum_{\xi} c_{j\xi} k_\xi, \tag{3.68}$$

satisfying the commutation relations for the generators of the Lorentz transformations? In this paper we will not discuss this more general task, for our present purpose it is sufficient that we proved the existence of the generators of infinitesimal Lorentz transformations under which (3.35) is covariant. □

3.2. Finite transformations

Now, having the infinitesimal transformations, we can proceed to the finite ones corresponding to the parameters ω and ξ,

$$\Psi_0'(p') = \Lambda(\omega)\Psi_0(p), \qquad \Psi_0'(p) = \Lambda(\xi)\Psi_0(p), \tag{3.69}$$

where $p \longrightarrow p'$ is some of transformations (3.16) and (3.17). The matrices Λ satisfy

$$\Lambda(\omega + d\omega) = \Lambda(\omega)\Lambda(d\omega), \qquad \Lambda(\xi + d\xi) = \Lambda(\xi)\Lambda(d\xi), \tag{3.70}$$

which for the parameters φ (space rotations only) and ξ imply that

$$\frac{d\Lambda(\varphi_j)}{d\varphi_j} = i\Lambda(\varphi_j)R_{\varphi_j}, \qquad \frac{d\Lambda(\xi)}{d\xi} = i\Lambda(\xi)K_\xi. \tag{3.71}$$

Assuming the constant elements of the matrices R_{φ_j} and K_ξ, the solutions of the last equations can be written in the usual exponential form,

$$\Lambda(\varphi_j) = \exp\left(i\varphi_j R_{\varphi_j}\right), \qquad \Lambda(\xi) = \exp\left(i\xi K_\xi\right). \tag{3.72}$$

The space rotation by an angle φ about the axis having the direction $\vec{u}$, $|\vec{u}| = 1$, is represented by

$$\Lambda(\varphi, \vec{u}) = \exp\left[i\varphi(\vec{u}\cdot\vec{R}_\varphi)\right]; \quad \vec{R}_\varphi = \left(R_{\varphi_1}, R_{\varphi_2}, R_{\varphi_3}\right). \tag{3.73}$$

For the Lorentz transformations we get, instead of (3.71),

$$\frac{d\Lambda(\psi_j)}{d\psi_j} = if_j(\psi_j)\Lambda(\psi_j)N_j, \tag{3.74}$$

where, in accordance with (3.17) and (3.57), we have

$$\begin{aligned} f_j(\psi_j) &= \frac{1}{p_0\cosh\psi_j + p_j\sinh\psi_j + \sqrt{p^2 - \kappa^2}}, \\ N_j &= \kappa R_{\varphi_j} + \epsilon_{jkl}p_k R_{\varphi_l}. \end{aligned} \tag{3.75}$$

The solution of (3.74) reads

$$\Lambda(\psi_j) = \exp\left(iF(\psi_j)N_j\right); \quad F(\psi_j) = \int_0^{\psi_j} f_j(\eta)d\eta. \tag{3.76}$$

The Lorentz boost in a general direction $\vec{u}$ with the velocity β is represented by

$$\Lambda(\varphi, \vec{u}) = \exp\left(iF(\varphi)N\right), \qquad \tanh\varphi = \beta, \tag{3.77}$$

where

$$\begin{aligned} F(\varphi) &= \int_0^{\varphi} \frac{d\eta}{p_0 \cosh\eta + \vec{p}\vec{u}\sinh\eta + \sqrt{p^2 - \kappa^2}}, \\ N &= \kappa\vec{u}\vec{R}_{\varphi} + (\vec{u} \times \vec{p}) \cdot \vec{R}_{\varphi}. \end{aligned} \tag{3.78}$$

The corresponding integrals can be found, for example, in the handbook [11].

Note, from the technical point of view, that the solution of the equation

$$\frac{d\Lambda(t)}{dt} = \Omega(t)\Lambda(t), \tag{3.79}$$

where Λ, Ω are some square matrices, can be written in the exponential form

$$\Lambda(t) = \exp\left(\int_0^t \Omega(\eta)d\eta\right) \tag{3.80}$$

only if the matrix Ω satisfies

$$\left[\Omega(t), \int_0^t \Omega(\eta)d\eta\right] = 0. \tag{3.81}$$

This condition is necessary for the differentiation

$$\begin{aligned} \frac{d\Lambda(t)}{dt} &= \frac{d}{dt}\sum_{j=0}^{\infty} \frac{\left(\int_0^t \Omega(\eta)d\eta\right)^j}{j!} \\ &= \Omega(t)\sum_{j=0}^{\infty} \frac{\left(\int_0^t \Omega(\eta)d\eta\right)^j}{j!} \\ &= \Omega(t)\Lambda(t) = \Lambda(t)\Omega(t). \end{aligned} \tag{3.82}$$

Obviously, condition (3.82) is satisfied for the generators of all the considered transformations, including the Lorentz ones in (3.77), since the matrix N does not depend on φ. (N depends only on the momenta components perpendicular to the direction of the Lorentz boost.)

3.3. Equivalent transformations

Now, from the symmetry of (3.35) we can obtain the corresponding transformations for (3.2). The generators (3.65) satisfy relations (3.50), (3.26), (3.27), and (3.28), it follows that the generators

$$\begin{aligned} \boldsymbol{R}_\omega(\Gamma) &= \Upsilon^{-1}(p)\boldsymbol{R}_\omega(\Gamma_0)\Upsilon(p) = R_\omega(\Gamma) + i\frac{d}{d\omega}, \\ R_\omega(\Gamma) &= \Upsilon^{-1}(p)R_\omega(\Gamma_0)\Upsilon(p) + i\Upsilon^{-1}(p)\frac{d\Upsilon(p)}{d\omega}, \end{aligned} \tag{3.83}$$

where $\boldsymbol{R}_\omega(\Gamma_0)$, $R_\omega(\Gamma_0)$ are generators (3.65) and $\Upsilon(p)$ is the transformation (3.37), will satisfy the same conditions, but with relation (3.24) instead of relation (3.50). Similarly, the generators $K_\xi(\Gamma_0)$ in relation (3.67) will be for (3.2) replaced by

$$K_\xi(\Gamma) = \Upsilon^{-1}(p)K_\xi(\Gamma_0)\Upsilon(p). \tag{3.84}$$

The finite transformations of (3.2) and its solutions can be obtained as follows. First, consider the transformations $\Lambda(\Gamma_0, \omega, \vec{u})$ given by (3.73) and (3.77). In accordance with (3.36), we have

$$\Gamma(p) = \Upsilon^{-1}(p)\Gamma_0(p)\Upsilon(p), \qquad \Gamma(p') = \Upsilon^{-1}(p')\Gamma_0(p')\Upsilon(p'), \tag{3.85}$$

and correspondingly for the solutions of (3.2) and (3.35),

$$\Psi(p) = \Upsilon^{-1}(p)\Psi_0(p), \qquad \Psi'(p') = \Upsilon^{-1}(p')\Psi'_0(p'). \tag{3.86}$$

Since

$$\Psi'_0(p') = \Lambda(\Gamma_0, \omega, \vec{u})\Psi_0(p), \tag{3.87}$$

then (3.86) imply that

$$\Psi'(p') = \Lambda(\Gamma, \omega, \vec{u})\Psi(p); \quad \Lambda(\Gamma, \omega, \vec{u}) = \Upsilon^{-1}(p')\Lambda(\Gamma_0, \omega, \vec{u})\Upsilon(p). \tag{3.88}$$

Similarly, the transformations $\Lambda(\Gamma_0, \xi)$ given by (3.72) are for (3.2) replaced by

$$\Lambda(\Gamma, \xi) = \Upsilon^{-1}(p)\Lambda(\Gamma_0, \xi)\Upsilon(p). \tag{3.89}$$

Note that all the symmetries of (3.2) like the transformation (3.89), which are not connected with a change of the reference frame (p), can be, in accordance with relation (3.33), expressed as

$$\Lambda(\Gamma, X) = Z(\Gamma, X, \Gamma), \tag{3.90}$$

where Z is defined by (3.30) and $X(p)$ is any matrix for which there exists $Z(\Gamma,X,\Gamma)^{-1}$. Further, it is obvious that if we have some set of generators $\boldsymbol{R}_\omega(\Gamma)$ satisfying relations (3.24), (3.26), (3.27), and (3.28) then also any set

$$\hat{\boldsymbol{R}}_\omega(\Gamma) = Z(\Gamma,X,\Gamma)^{-1}\boldsymbol{R}_\omega(\Gamma)Z(\Gamma,X,\Gamma) \tag{3.91}$$

satisfies these conditions. For the finite transformations, we get, correspondingly,

$$\hat{\Lambda}(\Gamma,\omega,\vec{u}) = Z(\Gamma(p'),X(p'),\Gamma(p'))^{-1}\Lambda(\Gamma,\omega,\vec{u})Z(\Gamma(p),X(p),\Gamma(p)). \tag{3.92}$$

In the same way, the sets of equivalent generators and transformations can be obtained for the diagonalized equation (3.35).

We remark that according to Lemma 3.2, there exists the set of transformations $\Gamma(p) \leftrightarrow \Gamma_0(p)$ given by relation (3.36). We used its particular form (3.37), but how will the generators

$$\boldsymbol{R}_\omega(\Gamma,X_k) = Z(\Gamma_0,X_k,\Gamma)^{-1}\boldsymbol{R}_\omega(\Gamma_0)Z(\Gamma_0,X_k,\Gamma); \quad k = 1,2 \tag{3.93}$$

differ for the two different matrices X_1 and X_2? The last relation implies that

$$\begin{aligned}\boldsymbol{R}_\omega&(\Gamma,X_1)\\ &= Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma)\boldsymbol{R}_\omega(\Gamma,X_2)Z(\Gamma_0,X_2,\Gamma)^{-1}Z(\Gamma_0,X_1,\Gamma)\end{aligned} \tag{3.94}$$

and according to relation (3.31),

$$\begin{aligned}\Gamma Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma) &= Z(\Gamma_0,X_1,\Gamma)^{-1}\Gamma_0 Z(\Gamma_0,X_2,\Gamma)\\ &= Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma)\Gamma,\end{aligned} \tag{3.95}$$

which means that

$$[Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma),\Gamma] = 0. \tag{3.96}$$

It follows that there must exist a matrix X_3 (e.g., according to implication (3.34) we can put $X_3 = Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma)$) so that

$$Z(\Gamma,X_3,\Gamma) = Z(\Gamma_0,X_1,\Gamma)^{-1}Z(\Gamma_0,X_2,\Gamma), \tag{3.97}$$

then relation (3.94) can be rewritten as

$$\boldsymbol{R}_\omega(\Gamma,X_1) = Z(\Gamma,X_3,\Gamma)\boldsymbol{R}_\omega(\Gamma,X_2)Z(\Gamma,X_3,\Gamma)^{-1}, \tag{3.98}$$

that is, the generators $\boldsymbol{R}_\omega(\Gamma, X_1)$, $\boldsymbol{R}_\omega(\Gamma, X_2)$ are equivalent in the sense of relation (3.91).

3.4. Scalar product and unitary representations

Definition 3.4. The scalar product of the two functions satisfying (3.2) or (3.35) is defined as

$$\left(\Phi(p), \Psi(q)\right) = \begin{cases} 0 & \text{for } p \neq q, \\ \Phi^\dagger(p)W(p)\Psi(q) & \text{for } p = q, \end{cases} \tag{3.99}$$

where the metric W is the matrix, which satisfies

$$W^\dagger(p) = W(p), \tag{3.100}$$

$$R_\omega^\dagger(p)W(p) - W(p)R_\omega(p) + i\frac{dW}{d\omega} = 0, \tag{3.101}$$

$$K_\xi^\dagger(p)W(p) - W(p)K_\xi(p) = 0. \tag{3.102}$$

Conditions (3.101) and (3.102) in the above definition imply that the scalar product is invariant under corresponding infinitesimal transformations. For example, for the Lorentz group the transformed scalar product reads

$$\begin{aligned}\Phi'^\dagger(p')W(p')\Psi'(p') \\ = \Phi^\dagger(p)\left(I - id\omega R_\omega^\dagger(p)\right)\left(W(p) + d\omega\frac{dW}{d\omega}\right)\left(I + id\omega R_\omega(p)\right)\Psi(p)\end{aligned} \tag{3.103}$$

and with the use of condition (3.101), we get

$$\Phi'^\dagger(p')W(p')\Psi'(p') = \Phi^\dagger(p)W(p)\Psi(p). \tag{3.104}$$

According to a general definition, the transformations conserving the scalar product are unitary. In this way, (3.101) and (3.102) represent the condition of unitarity for the representation of the corresponding group generated by $\boldsymbol{R}_\omega$ and K_ξ.

How to choose these generators and the matrix $W(p)$ to solve (3.101) and (3.102)? Similarly, as in the case of the solution of (3.24), (3.26), (3.27), and (3.28), it is convenient to begin with the representation related to the canonical equation (3.35). Apparently, the generators of the space rotations can be chosen Hermitian

$$R_{\varphi_j}^\dagger(\Gamma_0) = R_{\varphi_j}(\Gamma_0). \tag{3.105}$$

Then also for the Lorentz transformations we get

$$R^{\dagger}_{\psi_j}(\Gamma_0) = R_{\psi_j}(\Gamma_0), \tag{3.106}$$

provided that the constant κ in (3.57) is real and $|\kappa| \leq m$. Also the generators K_ξ can be chosen in the same way,

$$K^{\dagger}_{\xi}(\Gamma_0) = K_\xi(\Gamma_0). \tag{3.107}$$

It follows that instead of conditions (3.101) and (3.102), we can write

$$[\boldsymbol{R}_\omega(\Gamma_0), W(\Gamma_0)] = 0, \qquad [K_\xi(\Gamma_0), W(\Gamma_0)] = 0. \tag{3.108}$$

The structure of the generators $\boldsymbol{R}_\omega(\Gamma_0)$, $K_\xi(\Gamma_0)$ given by (3.65) and (3.67) suggests that the metric W satisfying condition (3.108) can have a similar structure, but in which the corresponding blocks on the diagonal are occupied by unit matrices multiplied by some constants. Nevertheless, note that condition (3.108) in general can be satisfied also for some other structures of $W(\Gamma_0)$.

From $W(\Gamma_0)$ we can obtain matrix $W(\Gamma)$, the metric for the scalar product of the two solutions of (3.2). We can check that after the transformations

$$\begin{aligned} \boldsymbol{R}_\omega(\Gamma_0) &\longrightarrow \boldsymbol{R}_\omega(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^{-1}\boldsymbol{R}_\omega(\Gamma_0)Z(\Gamma_0, X, \Gamma), \\ K_\xi(\Gamma_0) &\longrightarrow K_\xi(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^{-1}K_\xi(\Gamma_0)Z(\Gamma_0, X, \Gamma), \end{aligned} \tag{3.109}$$

and simultaneously,

$$W(\Gamma_0) \longrightarrow W(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^{\dagger}W(\Gamma_0)Z(\Gamma_0, X, \Gamma), \tag{3.110}$$

the unitarity in the sense of conditions (3.101) and (3.102) is conserved, in spite of the fact that equalities (3.105), (3.106), and (3.107) may not hold for $R_\omega(\Gamma, X)$, $K_\xi(\Gamma, X)$.

3.5. Space-time representation and Green functions

If we take the solutions of the wave equation (3.2) or (3.35) in the form of the functions $\Psi(p)$, for which there exists the Fourier picture

$$\tilde{\Psi}(x) = \frac{1}{(2\pi)^4}\int \Psi(p)\delta(p^2 - m^2)\exp(-ipx)d^4p, \tag{3.111}$$

then the space of the functions $\tilde{\Psi}(x)$ constitutes the x-representation of wave functions. Correspondingly, for all the operators $D(p)$ given in the

p-representation and discussed in the previous sections, we can formally define their x-representation:

$$\tilde{D}(z) = \frac{1}{(2\pi)^4}\int D(p)\exp(-ipz)d^4p, \tag{3.112}$$

which means that

$$\begin{aligned}\tilde{D}\tilde{\Psi}(x) &= \frac{1}{(2\pi)^4}\int D(p)\Psi(p)\delta(p^2-m^2)\exp(-ipx)d^4p\\ &= \frac{1}{(2\pi)^4}\int D(p)\exp(-ipx)\tilde{\Psi}(y)\exp(ipy)d^4y d^4p\\ &= \frac{1}{(2\pi)^4}\int \tilde{D}(x-y)\tilde{\Psi}(y)d^4y.\end{aligned} \tag{3.113}$$

In this way, we get for our operators

$$\begin{aligned}\Gamma_0(p) &\longrightarrow \tilde{\Gamma}_0(z) = Q_0\frac{1}{(2\pi)^4}\int \left(p^2\right)^{1/n}\exp(-ipz)d^4p; \quad pz \equiv p_0z_0 - \vec{p}\vec{z},\\ \Gamma(p) &\longrightarrow \tilde{\Gamma}(z) = \sum_{\lambda=0}^{3} Q_\lambda \frac{1}{(2\pi)^4}\int p_\lambda^{2/n}\exp(-ipz)d^4p,\\ \boldsymbol{R}_{\varphi_j}(\Gamma_0) &\longrightarrow \tilde{\boldsymbol{R}}_{\varphi_j}(\Gamma_0) = R_{\varphi_j}(\Gamma_0) + i\frac{d}{d\tilde{\varphi}_j}; \quad \frac{d}{d\tilde{\varphi}_j} = -\epsilon_{jkl}x_k\frac{\partial}{\partial x_l},\\ \boldsymbol{R}_{\psi_j}(\Gamma_0) &\longrightarrow \tilde{\boldsymbol{R}}_{\psi_j}(z) = \frac{1}{(2\pi)^4}\int \frac{\kappa R_{\varphi_j}(\Gamma_0) + \epsilon_{jkl}p_k R_{\varphi_l}(\Gamma_0)}{p_0 + \sqrt{p^2-\kappa^2}}\exp(-ipz)d^4p\\ &\qquad + i\frac{d}{d\tilde{\psi}_j}; \quad \frac{d}{d\tilde{\psi}_j} = -x_0\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_0},\end{aligned} \tag{3.114}$$

and in the same way,

$$\begin{aligned}\boldsymbol{R}_\omega(\Gamma) &\longrightarrow \tilde{\boldsymbol{R}}_\omega(z)\\ &= \frac{1}{(2\pi)^4}\int Z(\Gamma_0,\mathrm{X},\Gamma)^{-1}\boldsymbol{R}_\omega(\Gamma_0)Z(\Gamma_0,\mathrm{X},\Gamma)\exp(-ipz)d^4p.\end{aligned} \tag{3.115}$$

Apparently, the similar relations are valid also for the remaining operators K_ξ, W, Z, Z^{-1} and the finite transformations Λ in the x-representation. Concerning the translations, the usual correspondence is valid, $p_\alpha \rightarrow i\partial_\alpha$.

Further, the solutions of the inhomogeneous version of (3.35) and (3.2),

$$(\Gamma_0(p)-\mu)G_0(p)=I, \qquad (\Gamma(p)-\mu)G(p)=I \tag{3.116}$$

can be obtained with the use of formula (2.30),

$$\begin{aligned} G_0(p) &= \frac{(\Gamma_0-\alpha\mu)(\Gamma_0-\alpha^2\mu)\cdots(\Gamma_0-\alpha^{n-1}\mu)}{p^2-m^2}, \\ G(p) &= \frac{(\Gamma-\alpha\mu)(\Gamma-\alpha^2\mu)\cdots(\Gamma-\alpha^{n-1}\mu)}{p^2-m^2}, \end{aligned} \tag{3.117}$$

and (3.36) implies that

$$G(p)=Z(\Gamma_0,X,\Gamma)^{-1}G_0(p)Z(\Gamma_0,X,\Gamma). \tag{3.118}$$

Apparently, the functions

$$\begin{aligned} \tilde{G}_0(x) &= \frac{1}{(2\pi)^4}\int G_0(p)\exp(-ipx)d^4p, \\ \tilde{G}(x) &= \frac{1}{(2\pi)^4}\int G(p)\exp(-ipx)d^4p \end{aligned} \tag{3.119}$$

formally satisfy (3.116) in the x-representation

$$\begin{gathered} \int \tilde{\Gamma}_0(x-y)\tilde{G}_0(y)d^4y-\mu\tilde{G}_0(x)=\delta^4(x), \\ \int \tilde{\Gamma}(x-y)\tilde{G}(y)d^4y-\mu\tilde{G}(x)=\left[\sum_{\lambda=0}^{3}Q_\lambda\left(i\partial_\lambda\right)^{2/n}-\mu\right]\tilde{G}(x)=\delta^4(x). \end{gathered} \tag{3.120}$$

The last equation contains the fractional derivatives defined in [21]. Obviously, the functions $\tilde{G}_0$, $\tilde{G}$ can be identified with the Green functions related to the x-representation of (3.35) and (3.2).

With the exception of the operators $\tilde{\boldsymbol{R}}_{\varphi_j}(\Gamma_0)$, $\tilde{W}(\Gamma_0)$, and $i\partial_\alpha$, all the remaining operators considered above are pseudodifferential ones, which are in general nonlocal. The ways to deal with such operators are suggested in [1, 8, 21]. A more general treatise of the pseudodifferential operators can be found, for example, in [6, 18, 19, 20]. In our case it is significant that the corresponding integrals will depend on the choice of passing about the singularities and the choice of the cuts of the power functions $p^{2j/n}$. This choice should reflect contained physics, however, corresponding discussion would exceed the scope of this paper.

4. Summary and concluding remarks

In this paper, we have first studied the algebra of the matrices $Q_{pr} = S^pT^r$ generated by the pair of matrices S, T with the structure given by Definition 2.1. We have proved that for a given $n \geq 2$ we can in the corresponding set $\{Q_{pr}\}$ always find a triad, for which (2.60) is satisfied, where the Pauli matrices represent particular case $n = 2$. On this base, we have got the rule, to construct the generalized Dirac matrices [(2.63) and (2.64)]. Further, we have shown that there is a simple relation [(2.67) and (2.68)] between the set $\{Q_{pr}\}$ and the algebra of generators of the fundamental representation of the SU(n) group.

In the further part, using the generalized Dirac matrices we have demonstrated how we can, from the roots of the d'Alembertian operator, generate a class of relativistic equations containing the Dirac equation as a particular case. In this context, we have shown how the corresponding representations of the Lorentz group, which guarantee the covariance of these equations, can be found. At the same time, we have found additional symmetry transformations on these equations. Further, we have suggested how we can define the scalar product in the space of the corresponding wave functions and make the unitary representation of the whole group of symmetry. Finally, we have suggested how to construct the corresponding Green functions. In the x-representation, the equations themselves and all the mentioned transformations are in general nonlocal, being represented by the fractional derivatives and pseudodifferential operators in the four space-time dimensions.

In line with the choice of the representation of the rotation group used for the construction of the unitary representation of the Lorentz group according to which the equations transform, we can ascribe to the related wave functions the corresponding spin—and further quantum numbers connected with the additional symmetries. Nevertheless, it is obvious that before more serious physical speculations, we should answer some more questions requiring a further study. Perhaps the first could be the problem how to introduce the interaction. The usual direct replacement

$$\partial_\lambda \longrightarrow \partial_\lambda + igA_\lambda(x) \tag{4.1}$$

would lead to the difficulties, first of all with the rigorous definition of terms like

$$\left(\partial_\lambda + igA_\lambda(x)\right)^{2/n}. \tag{4.2}$$

At the end, still the more general question: is it possible on the base of the discussed wave equations to build up a meaningful quantum field theory?

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Petr Závada: Institute of Physics, Academy of Sciences of Czech Republic, Na Slovance 2, CZ-182 21 Prague 8, Czech Republic

E-mail address: zavada@fzu.cz