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Research Article

Strong Convergence Theorem for Bregman Strongly Nonexpansive Mappings and Equilibrium Problems in Reflexive Banach Spaces

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By using a new hybrid method, a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of Bregman strongly nonexpansive mappings in a reflexive Banach space is proved.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all nonnegative real numbers, respectively. We also assume that E is a real reflexive Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. Let Θ be a bifunction from $C \times C \to \mathbb{R}$. The equilibrium problem is to find

$$x^* \in C \text{ such that } \Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of such solutions x^* is denoted by $\mathrm{EP}(\Theta)$.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

We denote by $F(T)$ the set of fixed points of T.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert spaces; see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3]. Recently, Tada and Takahashi [4, 5] and S. Takahashi and W. Takahashi [6] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [4] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced by Nakajo and Takahashi [7]. The authors also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In this paper, motivated by Takahashi et al. [8], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method. Using this theorem, we obtain two new strong convergence results for finding a solution of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mappings in a real reflexive Banach space.

2. Preliminaries and Lemmas

In the sequel, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. We denote by $\operatorname{dom} f$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$.

Let $x \in \operatorname{int}\operatorname{dom} f$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\}, \quad (3)$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \quad (4)$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*. \quad (5)$$

A function f on E is coercive [9] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty. \quad (6)$$

A function f on E is said to be strongly coercive [10] if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty. \quad (7)$$

For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^{\circ}(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (8)$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t \to 0^+}((f(x + ty) - f(x))/t)$ exists for any y. In this case, $f^{\circ}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp., Fréchet differentiable) on $\operatorname{int}\operatorname{dom} f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp., continuous) on $\operatorname{int}\operatorname{dom} f$ (see also [11, 12]). We will need the following result.

Lemma 1 (see [13]). *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2 (see [14]). The function f is said to be

(i) essentially smooth, if ∂f is both locally bounded and single valued on its domain,

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$,

(iii) Legendre if it is both essentially smooth and essentially strictly convex.

Remark 3. Let E be a reflexive Banach space. Then we have the following.

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [14, Theorem 5.4]).

(ii) $(\partial f)^{-1} = \partial f^*$ (see [12]).

(iii) f is Legendre if and only if f^* is Legendre (see [14, Corollary 5.5]).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$, and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$ (see [14, Theorem 5.10]).

Examples of Legendre functions were given in [14, 15]. One important and interesting Legendre function is $(1/p)\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f is coincident with the generalized duality mapping of E; that is, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined as

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \quad (9)$$

is called the Bregman distance with respect to f [16].

Recall that the Bregman projection [17] of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty closed and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \quad (10)$$

Concerning the Bregman projection, the following are well known.

Lemma 4 (see [18]). *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) *$z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$, for all $y \in C$.*

(b)

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall x \in E,\ y \in C. \quad (11)$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\}. \quad (12)$$

The function f is called totally convex at x if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$ and is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(B,t) := \inf\left\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\right\}. \tag{13}$$

The next lemma will be useful in the proof of our main results.

Lemma 5 (see [19]). *If $x \in \operatorname{dom} f$, then the following statements are equivalent.*

(i) *The function f is totally convex at x.*

(ii) *For any sequence $\{y_n\} \subset \operatorname{dom} f$,*

$$\lim_{n\to+\infty} D_f(y_n,x) = 0 \Longrightarrow \lim_{n\to+\infty}\|y_n - x\| = 0. \tag{14}$$

Recall that the function f is called sequentially consistent [18] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to+\infty} D_f(y_n,x_n) = 0 \Longrightarrow \lim_{n\to+\infty}\|y_n - x_n\| = 0. \tag{15}$$

Lemma 6 (see [20]). *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 7 (see [21]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 8 (see [21]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$, and let C be a nonempty, closed, and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n,x_0) \le D_f(P_C^f x_0, x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_C^f x_0$.*

Let C be a convex subset of $\operatorname{int}\operatorname{dom} f$ and let T be a self-mapping of C. A point $p \in C$ is called an asymptotic fixed point of T (see [22, 23]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T.

Definition 9. A mapping T with a nonempty asymptotic fixed point set $\widehat{F}(T)$ is said to be

(i) Bregman strongly nonexpansive (see [24, 25]) with respect to $\widehat{F}(T)$ if

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C,\ p \in \widehat{F}(T), \tag{16}$$

and if, whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty}\left(D_f(p,x_n) - D_f(p,Tx_n)\right) = 0, \tag{17}$$

it follows that

$$\lim_{n\to\infty} D_f(x_n,Tx_n) = 0. \tag{18}$$

(ii) Bregman firmly nonexpansive [26] if, for all $x, y \in C$,

$$\begin{aligned}&\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\quad \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle\end{aligned} \tag{19}$$

or, equivalently,

$$\begin{aligned}&D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \\ &\quad \le D_f(Tx,y) + D_f(Ty,x).\end{aligned} \tag{20}$$

The existence and approximation of Bregman firmly nonexpansive mappings were studied in [26]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [26]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Lemma 10 (see [27]). *Let E be a real reflexive Banach space and $f : E \to (-\infty,+\infty]$ a proper lower semicontinuous function; then $f^* : E^* \to (-\infty,+\infty]$ is a proper weak* lower semicontinuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^N t_i D_f(z,x_i). \tag{21}$$

In order to solve the equilibrium problem, let us assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions [28]:

(A_1) $\Theta(x,x) = 0$, for all $x \in C$.

(A_2) Θ is monotone; that is, $\Theta(x,y) + \Theta(y,x) \le 0$, for all $x, y \in C$.

(A_3) $\limsup_{t\downarrow 0}\Theta(x + t(z-x), y) \le \Theta(x,y)$ for all $x, z, y \in C$.

(A_4) The function $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

The resolvent of a bifunction Θ [29] is the operator $\operatorname{Res}_\Theta^f : E \to 2^C$ defined by

$$\begin{aligned}\operatorname{Res}_\Theta^f(x) = \{&z \in C : \Theta(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \\ &\ge 0,\ \forall y \in C\}.\end{aligned} \tag{22}$$

From Lemma 1 in [24], if $f : E \to (-\infty,+\infty]$ is a strongly coercive and Gâteaux differentiable function and Θ satisfies conditions $(A_1$–$A_4)$, then $\operatorname{dom}(\operatorname{Res}_\Theta^f) = E$. We also know the following lemma which gives us some characterizations of the resolvent $\operatorname{Res}_\Theta^f$.

Lemma 11 (see [24]). *Let E be a real reflexive Banach space and C a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)–(A_4), then the followings hold:*

(i) *Res^f_Θ is single-valued;*

(ii) *Res^f_Θ is a Bregman firmly nonexpansive operator;*

(iii) *$F(Res^f_\Theta) = EP(\Theta)$;*

(iv) *$EP(\Theta)$ is a closed and convex subset of C;*

(v) *for all $x \in E$ and for all $q \in F(Res^f_\Theta)$, we have*

$$D_f\left(q, Res^f_\Theta(x)\right) + D_f\left(Res^f_\Theta(x), x\right) \le D_f(q, x). \quad (23)$$

3. Strong Convergence Theorem

In this section, we proved a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method.

Theorem 12. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let g be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) and let T be a Bregman strongly nonexpansive mapping from C into itself such that $F(T) = \widehat{F}(T)$ and $G = F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{aligned} & y_n = \nabla f^*\left(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\right), \\ & u_n \in C \text{ such that} \\ & g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n\rangle \ge 0, \\ & \qquad \forall y \in C, \\ & C_{n+1} = \left\{z \in C_n : D_f(z, u_n) \le D_f(z, x_n)\right\}, \\ & x_{n+1} = P^f_{C_{n+1}} x \end{aligned} \quad (24)$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)\cap EP(g)}x$, where $P^f_{F(T)\cap EP(g)}$ is the Bregman projection of E onto $F(T) \cap EP(g)$.

Proof. We divide the proof of Theorem 12 into five steps.

(I) We first prove that G and C_n both are closed and convex subset of C for all $n \ge 0$. In fact, it follows from Lemma 11 and by Reich and Sabach [26] that $\mathrm{EP}(g)$ and $F(T)$ both are closed and convex. Therefore, G is a closed and convex subset in C. Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 1$. Since the inequality $D_f(z, u_n) \le D_f(z, x_n)$ is equivalent to

$$\langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(u_n), z - u_n\rangle \le f(u_n) - f(x_n). \quad (25)$$

Therefore, we have

$$\begin{aligned} C_{n+1} = \{z \in C_n : & \langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(u_n), z - u_n\rangle \\ & \le f(u_n) - f(x_n)\}. \end{aligned} \quad (26)$$

This implies that C_{n+1} is closed and convex. The desired *conclusions* are proved. These in turn show that $P^f_{F(T)\cap \mathrm{EP}(g)}x$ and $P^f_{C_n}x$ are well defined.

(II) we prove that $G := F(T) \cap \mathrm{EP}(g) \subset C_n$ for all $n \ge 0$.

Indeed, it is obvious that $G = F(T) \cap \mathrm{EP}(g) \subset C_0 = C$. Suppose that $G \subset C_n$ for some $n \in \mathbb{N}$. Let $u \in G \subset C_n$; since $u_n = \mathrm{Res}^f_g(y_n)$, by Lemma 11 and (21), we have

$$\begin{aligned} D_f(u, u_n) &= D_f\left(u, \mathrm{Res}^f_g y_n\right) \le D_f(u, y_n) \\ &= D_f\left(u, \nabla f^*\left(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\right)\right) \\ &\le \alpha_n D_f(u, x_n) + (1-\alpha_n) D_f(u, Tx_n) \\ &\le \alpha_n D_f(u, x_n) + (1-\alpha_n) D_f(u, x_n) \\ &= D_f(u, x_n). \end{aligned} \quad (27)$$

Hence, we have $u \in C_{n+1}$. This implies that

$$F(T) \cap \mathrm{EP}(g) \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (28)$$

So, $\{x_n\}$ is well defined.

(III) We prove that $\{x_n\}$ is a bounded sequence in C.

By the definition of C_n, we have $x_n = P^f_{C_n}x$ for all $n \ge 0$. It follows from Lemma 4(b) that

$$\begin{aligned} D_f(x_n, x) &= D_f\left(P^f_{C_n}x, x\right) \le D_f(u, x) - D_f\left(u, P^f_{C_n}x\right) \\ &\le D_f(u, x), \quad \forall n \ge 0,\ u \in G. \end{aligned} \quad (29)$$

This implies that $\{D_f(x_n, x)\}$ is bounded. By Lemma 7, $\{x_n\}$ is bounded. Since $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, by Lemma 1 ∇f is uniformly continuous and bounded on bounded subsets of E. This implies that $\{\nabla f(x_n)\}$ is bounded.

(IV) Now we proved that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P^f_{C_n}x$, we have

$$D_f(x_n, x) \le D_f(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (30)$$

Thus, $\{D_f(x_n, x)\}$ is nondecreasing. So, the limit of $\{D_f(x_n, x)\}$ exists. Since $D_f(x_{n+1}, x_n) = D_f(x_{n+1}, P^f_{C_n}x) \le D_f(x_{n+1}, x) - D_f(P^f_{C_n}x, x) = D_f(x_{n+1}, x) - D_f(x_n, x)$ for all $n \ge 0$, we have $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$. From $x_{n+1} = P^f_{C_{n+1}} x \in C_{n+1}$, we have

$$D_f(x_{n+1}, u_n) \le D_f(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{31}$$

Therefore, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{32}$$

From Lemma 5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{33}$$

So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{34}$$

This means that the sequence $\{u_n\}$ is bounded. Since f is uniformly Fréchet differentiable, it follows from Lemma 1 that ∇f is uniformly continuous. Therefore, we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{35}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [30, Theorem 1.8]). It follows that

$$\lim_{n\to\infty} |f(x_n) - f(u_n)| = 0. \tag{36}$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(u, x_n) - D_f(u, u_n) &= [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle] \\ &\quad - [f(u) - f(u_n) - \langle \nabla f(u_n), u - u_n\rangle] \\ &= (f(u_n) - f(x_n)) + \langle \nabla f(u_n) - \nabla f(x_n), u - u_n\rangle \\ &\quad + \langle \nabla f(x_n), x_n - u_n\rangle \end{aligned} \tag{37}$$

for any $u \in G$.

It follows from (34)–(37) that

$$\lim_{n\to\infty} (D_f(u, x_n) - D_f(u, u_n)) = 0. \tag{38}$$

On the other hand, from $u_n = \mathrm{Res}^f_g y_n$ and Lemma 11(v), for any $u \in G$ we have that

$$\begin{aligned} D_f(u_n, y_n) &= D_f(\mathrm{Res}^f_g y_n, y_n) \\ &\le D_f(u, y_n) - D_f(u, \mathrm{Res}^f_g y_n) \\ &\le D_f(u, x_n) - D_f(u, \mathrm{Res}^f_g y_n) \\ &= D_f(u, x_n) - D_f(u, u_n). \end{aligned} \tag{39}$$

So, we have from (38) that

$$\lim_{n\to\infty} D_f(u_n, y_n) = 0. \tag{40}$$

From Lemma 5, we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{41}$$

So, from (34) and (41),we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{42}$$

This means that the sequence $\{y_n\}$ is bounded. Since f is uniformly Fréchet differentiable, it follows from Lemma 1 that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{43}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [30]). It follows that

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0. \tag{44}$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(u, y_n) - D_f(u, x_n) &= [f(u) - f(y_n) - \langle \nabla f(y_n), u - y_n\rangle] \\ &\quad - [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle] \\ &= (f(x_n) - f(y_n)) - \langle \nabla f(y_n) - \nabla f(x_n), u - y_n\rangle \\ &\quad + \langle \nabla f(x_n), y_n - x_n\rangle \end{aligned} \tag{45}$$

for any $u \in G$.

It follows from (42) to (45) that

$$\lim_{n\to\infty} (D_f(u, y_n) - D_f(u, x_n)) = 0. \tag{46}$$

On the other hand, for any $u \in G$ we have

$$\begin{aligned} D_f(u, y_n) - D_f(u, x_n) &= D_f(u, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Tx_n))) \\ &\quad - D_f(u, x_n) \\ &\le \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, Tx_n) - D_f(u, x_n) \\ &= (1 - \alpha_n)(D_f(u, Tx_n) - D_f(u, x_n)). \end{aligned} \tag{47}$$

This together with (46), (16), and $\lim_{n\to\infty} \alpha_n < 1$ shows that

$$\lim_{k\to\infty} (D_f(u, Tx_n) - D_f(u, x_n)) = 0. \tag{48}$$

Since T is Bregman strongly nonexpansive, it follows from (48) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{49}$$

(V) Next, we prove that every weak subsequential limit of $\{x_n\}$ belongs to $G = F(T) \cap EP(g)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Since T is a Bregman strongly nonexpansive mapping with $F(T) = \widehat{F}(T)$, we have $x^* \in F(T)$.

From $x_{n_k} \rightharpoonup x^*$ and (34), we have $u_{n_k} \rightharpoonup x^*$.

By $u_n = \mathrm{Res}_g^f y_n$, we have

$$g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \tag{50}$$

Replacing n by n_k, we have from (A_2) that

$$\left\langle \nabla f\left(u_{n_k}\right) - \nabla f\left(y_{n_k}\right), y - u_{n_k} \right\rangle \geq -g\left(u_{n_k}, y\right) \geq g\left(y, u_{n_k}\right), \quad \forall y \in C. \tag{51}$$

Since $g(x, \cdot)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. So, letting $k \to \infty$, we have from (35), (43), and (A_4) that

$$g(y, x^*) \leq 0, \quad \forall y \in C. \tag{52}$$

For $t \in (0,1]$ and $y \in C$, letting $y_t = ty + (1-t)x^*$, there are $y_t \in C$ and $g(y_t, x^*) \leq 0$. By condition (A_1) and (A_4), we have

$$0 = g(y_t, y_t) \leq tg(y_t, y) + (1-t)\, g(y_t, x^*) \leq tg(y_t, y). \tag{53}$$

Dividing both sides of the above equation by t, we have $g(y_t, y) \geq 0$, for all $y \in C$. Letting $t \downarrow 0$, from condition (A_3), we have

$$g(x^*, y) \geq 0, \quad \forall y \in C. \tag{54}$$

Therefore, $x^* \in \mathrm{EP}(g)$.

(VI) Now, we prove $x_n \to P^f_{F(T)\cap \mathrm{EP}(g)} x$.

Let $w = P^f_{F(T)\cap \mathrm{EP}(g)} x$. From $w \in F(T) \cap \mathrm{EP}(g) \subset C_{n+1}$, we have $D_f(x_{n+1}, x) \leq D_f(w, x)$. Therefore, Lemma 8 implies that $\{x_n\}$ converges strongly to $w = P^f_{F(T)\cap \mathrm{EP}(g)} x$, as claimed. This completes the proof of Theorem 12. □

Corollary 13. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, and*

$$\begin{gathered} u_n \in C \text{ such that} \\ g(u_n, y) + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\}, \\ x_{n+1} = P^f_{C_{n+1}} x \end{gathered} \tag{55}$$

for every $n \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges strongly to $P^f_{EP(g)} x$, where $P^f_{EP(g)}$ is the Bregman projection of E onto $EP(g)$.

Proof. Putting $T = I$ in Theorem 12, we obtain Corollary 13. □

Corollary 14. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. let T be a Bregman strongly nonexpansive mapping from C into itself such that $F(T) = \widehat{F}(T)$ and $G = F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, and*

$$\begin{gathered} u_n = P^f_C \nabla f^* \left(\alpha_n \nabla f(x_n) + (1-\alpha_n) \nabla f(Tx_n)\right), \\ C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\}, \\ x_{n+1} = P^f_{C_{n+1}} x \end{gathered} \tag{56}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)} x$, where $P^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.

Proof. Putting $g(x, y) = 0$ for all $x, y \in C$ in Theorem 12, we obtain Corollary 14. □

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