Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 705814, 4 pages
http://dx.doi.org/10.1155/2013/705814

Research Article

Strong Convergence for Hybrid *S*-Iteration Scheme

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Received 19 November 2012; Accepted 4 February 2013

Academic Editor: D. R. Sahu

We establish a strong convergence for the hybrid S-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces.

1. Introduction and Preliminaries

Let E be a real Banach space and let K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \; \|f^*\| = \|x\| \right\}, \quad \forall x, y \in E, \tag{1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality map by j.

Let $T : K \to K$ be a mapping.

Definition 1. The mapping T is said to be *Lipschitzian* if there exists a constant $L > 1$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in K. \tag{2}$$

Definition 2. The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in K. \tag{3}$$

Definition 3. The mapping T is said to be *pseudocontractive* if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le \|x - y\|^2. \tag{4}$$

Definition 4. The mapping T is said to be *strongly pseudocontractive* if for all $x, y \in K$, there exists $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le k\|x - y\|^2. \tag{5}$$

Let K be a nonempty convex subset C of a normed space E.

(a) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \ge 1, \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, is known as the Ishikawa iteration process [1].
If $\beta_n = 0$ for $n \ge 1$, then the Ishikawa iteration process becomes the Mann iteration process [2].

(b) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

$$x_{n+1} = T y_n, \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \ge 1, \tag{7}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$, is known as the S-iteration process [3, 4].

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive mappings using the *Ishikawa iteration scheme* (see, e.g., [1]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz mappings* have been extended to more general Banach spaces (see, e.g., [5–10] and the references cited therein).

In 1974, Ishikawa [1] proved the following result.

Theorem 5. *Let K be a compact convex subset of a Hilbert space H and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n,$$
$$y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 1, \tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\sum_{n\geq1} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly at a fixed point of T.

In [6], Chidume extended the results of Schu [9] from Hilbert spaces to the much more general class of real Banach spaces and approximated the fixed points of (strongly) pseudocontractive mappings.

In [11], Zhou and Jia gave the more general answer of the question raised by Chidume [5] and proved the following.

If X is a real Banach space with a uniformly convex dual X^*, K is a nonempty bounded closed convex subset of X, and $T : K \to K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration scheme converges strongly at the unique fixed point of T.

In this paper, we establish the strong convergence for the hybrid S-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces. We also improve the result of Zhou and Jia [11].

2. Main Results

We will need the following lemmas.

Lemma 6 (see [12]). *Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$
$$\forall j(x + y) \in J(x + y). \tag{9}$$

Lemma 7 (see [10]). *Let $\{\rho_n\}$ be nonnegative sequence satisfying*

$$\rho_{n+1} \leq (1 - \theta_n) \rho_n + \omega_n, \tag{10}$$

where $\theta_n \in [0, 1]$, $\sum_{n\geq1} \theta_n = \infty$, and $\omega_n = o(\theta_n)$. Then

$$\lim_{n\to\infty} \rho_n = 0. \tag{11}$$

The following is our main result.

Theorem 8. *Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be nonexpansive, and let $T : K \to K$ be Lipschitz strongly pseudocontractive mappings such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \forall x, y \in K,$$
$$\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K. \tag{C}$$

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(iv) $\sum_{n\geq1} \beta_n = \infty$,

(v) $\lim_{n\to\infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$x_{n+1} = S y_n,$$
$$y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 1. \tag{12}$$

Then the sequence $\{x_n\}$ converges strongly at the common fixed point p of S and T.

Proof. For strongly pseudocontractive mappings, the existence of a fixed point follows from Delmling [13]. It is shown in [11] that the set of fixed points for strongly pseudocontractions is a singleton.

By (v), since $\lim_{n\to\infty} \beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\beta_n \leq \min\left\{\frac{1}{4k}, \frac{1-k}{(1+L)(1+3L)}\right\}, \tag{13}$$

where $k < 1/2$. Consider

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \langle x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&= \langle S y_n - p, j(x_{n+1} - p)\rangle \\
&= \langle T x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad + \langle S y_n - T x_{n+1}, j(x_{n+1} - p)\rangle \\
&\leq k\|x_{n+1} - p\|^2 + \|S y_n - T x_{n+1}\| \|x_{n+1} - p\|,
\end{aligned} \tag{14}$$

which implies that

$$\|x_{n+1} - p\| \leq \frac{1}{1-k} \|S y_n - T x_{n+1}\|, \tag{15}$$

where

$$\begin{aligned}
\|S y_n - T x_{n+1}\| &\leq \|S y_n - T y_n\| + \|T y_n - T x_{n+1}\| \\
&\leq \|x_n - S y_n\| + \|x_n - T y_n\| + \|T y_n - T x_{n+1}\| \\
&\leq \|S x_n - S y_n\| + \|T x_n - T y_n\| + \|T y_n - T x_{n+1}\| \\
&\leq \|S x_n - S y_n\| + L(\|x_n - y_n\| + \|y_n - x_{n+1}\|),
\end{aligned} \tag{16}$$

$$\begin{aligned}
\|y_n - x_{n+1}\| &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| \\
&= \|y_n - x_n\| + \|x_n - S y_n\| \\
&\leq \|y_n - x_n\| + \|S x_n - S y_n\|,
\end{aligned} \tag{17}$$

and consequently from (16), we obtain

$$\begin{aligned}\|Sy_n - Tx_{n+1}\| &\le (1+L)\|Sx_n - Sy_n\| + 2L\|x_n - y_n\| \\ &\le (1+3L)\|x_n - y_n\| \\ &= (1+3L)\beta_n\|x_n - Tx_n\| \\ &\le (1+L)(1+3L)\beta_n\|x_n - p\|.\end{aligned} \tag{18}$$

Substituting (18) in (15) and using (13), we get

$$\begin{aligned}\|x_{n+1} - p\| &\le \frac{(1+L)(1+3L)}{1-k}\beta_n\|x_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{19}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}$ is bounded. Since T is Lipschitzian, so $\{Tx_n - p\}$ is also bounded. Let $M_1 = \sup_{n\ge1}\|x_n - p\| + \sup_{n\ge1}\|Tx_n - p\|$. Also by (ii), we have

$$\begin{aligned}\|x_n - y_n\| &= \beta_n\|x_n - Tx_n\| \\ &\le M_1\beta_n \\ &\longrightarrow 0\end{aligned} \tag{20}$$

as $n \to \infty$, implying that $\{x_n - y_n\}$ is bounded, so let $M_2 = \sup_{n\ge1}\|x_n - y_n\| + M_1$. Further,

$$\begin{aligned}\|y_n - p\| &\le \|y_n - x_n\| + \|x_n - p\| \\ &\le M_2,\end{aligned} \tag{21}$$

which implies that $\{y_n - p\}$ is bounded. Therefore, $\{Ty_n - p\}$ is also bounded.

Set

$$M_3 = \sup_{n\ge1}\|y_n - p\| + \sup_{n\ge1}\|Ty_n - p\|. \tag{22}$$

Denote $M = M_1 + M_2 + M_3$. Obviously, $M < \infty$.

Now from (12) for all $n \ge 1$, we obtain

$$\|x_{n+1} - p\|^2 = \|Sy_n - p\|^2 \le \|y_n - p\|^2, \tag{23}$$

and by Lemma 6, we get

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\|^2 \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\|^2 \\ &\le (1-\beta_n)^2\|x_n - p\|^2 + 2\beta_n\langle Tx_n - p, j(y_n - p)\rangle \\ &= (1-\beta_n)^2\|x_n - p\|^2 + 2\beta_n\langle Ty_n - p, j(y_n - p)\rangle \\ &\quad + 2\beta_n\langle Tx_n - Ty_n, j(y_n - p)\rangle \\ &\le (1-\beta_n)^2\|x_n - p\|^2 + 2k\beta_n\|y_n - p\|^2 \\ &\quad + 2\beta_n\|Tx_n - Ty_n\|\|y_n - p\| \\ &\le (1-\beta_n)^2\|x_n - p\|^2 + 2k\beta_n\|y_n - p\|^2 \\ &\quad + 2ML\beta_n\|x_n - y_n\|,\end{aligned} \tag{24}$$

which implies that

$$\begin{aligned}\|y_n - p\|^2 &\le \frac{(1-\beta_n)^2}{1-2k\beta_n}\|x_n - p\|^2 + \frac{2ML\beta_n}{1-2k\beta_n}\|x_n - y_n\| \\ &\le (1-\beta_n)\|x_n - p\|^2 + 4ML\beta_n\|x_n - y_n\|\end{aligned} \tag{25}$$

because by (13), we have $((1-\beta_n)/(1-2k\beta_n)) \le 1$ and $(1/(1-2k\beta_n)) \le 2$. Hence, (23) gives us

$$\|x_{n+1} - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + 4ML\beta_n\|x_n - y_n\|. \tag{26}$$

For all $n \ge 1$, put

$$\begin{aligned}\rho_n &= \|x_n - p\|, \\ \theta_n &= \beta_n, \\ \omega_n &= 4ML\beta_n\|x_n - y_n\|,\end{aligned} \tag{27}$$

then according to Lemma 7, we obtain from (26) that

$$\lim_{n\to\infty}\|x_n - p\| = 0. \tag{28}$$

This completes the proof. □

Corollary 9. *Let K be a nonempty closed convex subset of a real Hilbert space H, let $S : K \to K$ be nonexpansive, and let $T : K \to K$ be Lipschitz strongly pseudocontractive mappings such that $p \in F(S) \cap F(T)$ and the condition (C). Let $\{\beta_n\}$ be a sequence in* $[0, 1]$ *satisfying the conditions* (iv) *and* (v).

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by (12). *Then the sequence $\{x_n\}$ converges strongly at the common fixed point p of S and T.*

Example 10. As a particular case, we may choose, for instance, $\beta_n = 1/n$.

Remark 11. (1) The condition (C) is not new and it is due to Liu et al. [14].

(2) We prove our results for a hybrid iteration scheme, which is simple in comparison to the previously known iteration schemes.

Acknowledgment

This study was supported by research funds from Dong-A University.

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