Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 271978, 5 pages
http://dx.doi.org/10.1155/2013/271978

Research Article

Completing a 2×2 Block Matrix of Real Quaternions with a Partial Specified Inverse

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Received 4 December 2012; Revised 23 February 2013; Accepted 20 March 2013

Academic Editor: K. Sivakumar

This paper considers a completion problem of a nonsingular 2×2 block matrix over the real quaternion algebra $\mathbb{H}$: Let m_1, m_2, n_1, n_2 be nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, and $A_{12} \in \mathbb{H}^{m_1 \times n_2}$, $A_{21} \in \mathbb{H}^{m_2 \times n_1}$, $A_{22} \in \mathbb{H}^{m_2 \times n_2}$, $B_{11} \in \mathbb{H}^{n_1 \times m_1}$ be given. We determine necessary and sufficient conditions so that there exists a variant block entry matrix $A_{11} \in \mathbb{H}^{m_1 \times n_1}$ such that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{H}^{n \times n}$ is nonsingular, and B_{11} is the upper left block of a partitioning of A^{-1}. The general expression for A_{11} is also obtained. Finally, a numerical example is presented to verify the theoretical findings.

1. Introduction

The problem of completing a block-partitioned matrix of a specified type with some of its blocks given has been studied by many authors. Fiedler and Markham [1] considered the following completion problem over the real number field $\mathbb{R}$. Suppose m_1, m_2, n_1, n_2 are nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, and $B_{22} \in \mathbb{R}^{n_2 \times m_2}$. Determine a matrix $A_{22} \in \mathbb{R}^{m_2 \times n_2}$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{1}$$

is nonsingular and B_{22} is the lower right block of a partitioning of A^{-1}. This problem has the form of

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & ? \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & B_{22} \end{pmatrix}, \tag{2}$$

and the solution and the expression for A_{22} were obtained in [1]. Dai [2] considered this form of completion problems with symmetric and symmetric positive definite matrices over $\mathbb{R}$.

Some other particular forms for 2×2 block matrices over $\mathbb{R}$ have also been examined (see, e.g., [3]), such as

$$\begin{gathered} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & ? \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & ? \\ ? & ? \end{pmatrix}, \\ \begin{pmatrix} A_{11} & ? \\ ? & ? \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & B_{22} \end{pmatrix}, \\ \begin{pmatrix} A_{11} & ? \\ ? & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} ? & B_{12} \\ B_{21} & ? \end{pmatrix}. \end{gathered} \tag{3}$$

The real quaternion matrices play a role in computer science, quantum physics, and so on (e.g., [4–6]). Quaternion matrices are receiving much attention as witnessed recently (e.g., [7–9]). Motivated by the work of [1, 10] and keeping such applications of quaternion matrices in view, in this paper we consider the following completion problem over the real quaternion algebra:

$$\begin{aligned} \mathbb{H} = \{ & a_0 + a_1 i + a_2 j + a_3 k \mid \\ & i^2 = j^2 = k^2 = ijk = -1 \text{ and } a_0, a_1, a_2, a_3 \in \mathbb{R} \}. \end{aligned} \tag{4}$$

Problem 1. Suppose m_1, m_2, n_1, n_2 are nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, and $A_{12} \in \mathbb{H}^{m_1 \times n_2}$,

$A_{21} \in \mathbb{H}^{m_2\times n_1}$, $A_{22} \in R^{m_2\times n_2}$, $B_{11} \in \mathbb{H}^{n_1\times m_1}$. Find a matrix $A_{11} \in \mathbb{H}^{m_1\times n_1}$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{H}^{n\times n} \tag{5}$$

is nonsingular, and B_{11} is the upper left block of a partitioning of A^{-1}. That is

$$\begin{pmatrix} ? & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & ? \\ ? & ? \end{pmatrix}, \tag{6}$$

where $\mathbb{H}^{m\times n}$ denotes the set of all $m \times n$ matrices over $\mathbb{H}$ and A^{-1} denotes the inverse matrix of A.

Throughout, over the real quaternion algebra $\mathbb{H}$, we denote the identity matrix with the appropriate size by I, the transpose of A by A^T, the rank of A by $r(A)$, the conjugate transpose of A by $A^* = (\overline{A})^T$, a reflexive inverse of a matrix A over $\mathbb{H}$ by A^+ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. Moreover, $L_A = I - A^+A$, $R_A = I - AA^+$, where A^+ is an arbitrary but fixed reflexive inverse of A. Clearly, L_A and R_A are idempotent, and each is a reflexive inverse of itself. $\mathscr{R}(A)$ denotes the right column space of the matrix A.

The rest of this paper is organized as follows. In Section 2, we establish some necessary and sufficient conditions to solve Problem 1 over $\mathbb{H}$, and the general expression for A_{11} is also obtained. In Section 3, we present a numerical example to illustrate the developed theory.

2. Main Results

In this section, we begin with the following lemmas.

Lemma 1 (singular-value decomposition [9]). *Let $A \in \mathbb{H}^{m\times n}$ be of rank r. Then there exist unitary quaternion matrices $U \in \mathbb{H}^{m\times m}$ and $V \in \mathbb{H}^{n\times n}$ such that*

$$UAV = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where $D_r = \operatorname{diag}(d_1, \ldots, d_r)$ and the d_j's are the positive singular values of A.

Let $\mathbb{H}_c^n$ denote the collection of column vectors with n components of quaternions and A be an $m \times n$ quaternion matrix. Then the solutions of $Ax = 0$ form a subspace of $\mathbb{H}_c^n$ of dimension $n(A)$. We have the following lemma.

Lemma 2. *Let*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{8}$$

be a partitioning of a nonsingular matrix $A \in \mathbb{H}^{n\times n}$, and let

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{9}$$

be the corresponding (i.e., transpose) partitioning of A^{-1}. Then $n(A_{11}) = n(B_{22})$.

Proof. It is readily seen that

$$\begin{pmatrix} B_{22} & B_{21} \\ B_{12} & B_{11} \end{pmatrix}, \qquad \begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix} \tag{10}$$

are inverse to each other, so we may suppose that $n(A_{11}) < n(B_{22})$.

If $n(B_{22}) = 0$, necessarily $n(A_{11}) = 0$ and we are finished. Let $n(B_{22}) = c > 0$, then there exists a matrix F with c right linearly independent columns, such that $B_{22}F = 0$. Then, using

$$A_{11}B_{12} + A_{12}B_{22} = 0, \tag{11}$$

we have

$$A_{11}B_{12}F = 0. \tag{12}$$

From

$$A_{21}B_{12} + A_{22}B_{22} = I, \tag{13}$$

we have

$$A_{21}B_{12}F = F. \tag{14}$$

It follows that the rank $r(B_{12}F) \geq c$. In view of (12), this implies

$$n(A_{11}) \geq r(B_{12}F) \geq c = n(B_{22}). \tag{15}$$

Thus

$$n(A_{11}) = n(B_{22}). \tag{16}$$

□

Lemma 3 (see [10]). *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{p\times q}$, $D \in \mathbb{H}^{m\times q}$ be known and $X \in \mathbb{H}^{n\times p}$ unknown. Then the matrix equation*

$$AXB = D \tag{17}$$

is consistent if and only if

$$AA^+DB^+B = D. \tag{18}$$

In that case, the general solution is

$$X = A^+DB^+ + L_AY_1 + Y_2R_B, \tag{19}$$

where Y_1, Y_2 are any matrices with compatible dimensions over $\mathbb{H}$.

By Lemma 1, let the singular value decomposition of the matrix A_{22} and B_{11} in Problem 1 be

$$A_{22} = Q\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}R^*, \tag{20}$$

$$B_{11} = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^*, \tag{21}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_s)$ is a positive diagonal matrix, $\lambda_i \neq 0$ $(i = 1, \dots, s)$ are the singular values of A_{22}, $s = r(A_{22})$, $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$ is a positive diagonal matrix, $\sigma_i \neq 0$ $(i = 1, \dots, r)$ are the singular values of B_{11} and $r = r(B_{11})$.

$Q = (Q_1 \ Q_2) \in \mathbb{H}^{m_2 \times m_2}$, $R = (R_1 \ R_2) \in \mathbb{H}^{n_2 \times n_2}$, $U = (U_1 \ U_2) \in \mathbb{H}^{n_1 \times n_1}$, $V = (V_1 \ V_2) \in \mathbb{H}^{m_1 \times m_1}$ are unitary quaternion matrices, where $Q_1 \in \mathbb{H}^{m_2 \times s}$, $R_1 \in \mathbb{H}^{n_2 \times s}$, $U_1 \in \mathbb{H}^{n_1 \times r}$, and $V_1 \in \mathbb{H}^{m_1 \times r}$.

Theorem 4. *Problem 1 has a solution if and only if the following conditions are satisfied:*

(a) $r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = n_2$,

(b) $n_2 - r(A_{22}) = m_1 - r(B_{11})$, *that is* $n_2 - s = m_1 - r$,

(c) $\mathcal{R}(A_{21}B_{11}) \subset \mathcal{R}(A_{22})$,

(d) $\mathcal{R}(A_{12}^* B_{11}^*) \subset \mathcal{R}(A_{22}^*)$.

In that case, the general solution has the form of

$$A_{11} = B_{11}^{+} + A_{12}R\begin{pmatrix} \Lambda^{-1}Q_1^* A_{21} U_1 \Sigma & 0 \\ H & -(V_2^* A_{12} R_2)^{-1} \end{pmatrix} \times V^* B_{11}^{+} + Y - Y B_{11} B_{11}^{+}, \quad (22)$$

where H is an arbitrary matrix in $\mathbb{H}^{(n_2 - s)\times r}$ and Y is an arbitrary matrix in $\mathbb{H}^{m_1 \times n_1}$.

Proof. If there exists an $m_1 \times n_1$ matrix A_{11} such that A is nonsingular and B_{11} is the corresponding block of A^{-1}, then (a) is satisfied. From $AB = BA = I$, we have that

$$\begin{aligned} A_{21}B_{11} + A_{22}B_{21} &= 0, \\ B_{11}A_{12} + B_{12}A_{22} &= 0, \end{aligned} \quad (23)$$

so that (c) and (d) are satisfied.

By (11), we have

$$r(A_{22}) + n(A_{22}) = n_2, \qquad r(B_{11}) + n(B_{11}) = m_1. \quad (24)$$

From Lemma 2, Notice that $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is the corresponding partitioning of B^{-1}, we have

$$n(B_{11}) = n(A_{22}), \quad (25)$$

implying that (b) is satisfied.

Conversely, from (c), we know that there exists a matrix $K \in \mathbb{H}^{n_2 \times m_1}$ such that

$$A_{21}B_{11} = A_{22}K. \quad (26)$$

Let

$$B_{21} = -K. \quad (27)$$

From (20), (21), and (26), we have

$$A_{21}U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = Q\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}R^* K. \quad (28)$$

It follows that

$$Q^* A_{21} U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* V = Q^* Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} R^* K V. \quad (29)$$

This implies that

$$\begin{aligned} &\begin{pmatrix} Q_1^* A_{21} U_1 & Q_1^* A_{21} U_2 \\ Q_2^* A_{21} U_1 & Q_2^* A_{21} U_2 \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_1^* K V_1 & R_1^* K V_2 \\ R_2^* K V_1 & R_2^* K V_2 \end{pmatrix}. \end{aligned} \quad (30)$$

Comparing corresponding blocks in (30), we obtain

$$Q_2^* A_{21} U_1 = 0. \quad (31)$$

Let $R^* K V = \widehat{K}$. From (29), (30), we have

$$\widehat{K} = \begin{pmatrix} \Lambda^{-1} Q_1^* A_{21} U_1 \Sigma & 0 \\ H & K_{22} \end{pmatrix}, \quad H \in \mathbb{H}^{(n_2 - s)\times r}, \ K_{22} \in \mathbb{H}^{(n_2 - s)\times(m_1 - r)}. \quad (32)$$

In the same way, from (d), we can obtain

$$V_1^* A_{12} R_2 = 0. \quad (33)$$

Notice that $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ in (a) is a full column rank matrix. By (20), (21), and (33), we have

$$\begin{pmatrix} 0 & Q^* \\ V^* & 0 \end{pmatrix}\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} R = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \\ V_1^* A_{12} R_1 & V_1^* A_{12} R_2 \\ V_2^* A_{12} R_1 & V_2^* A_{12} R_2 \end{pmatrix}, \quad (34)$$

so that

$$\begin{aligned} n_2 &= r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = r\left(\begin{pmatrix} 0 & Q^* \\ V^* & 0 \end{pmatrix}\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} R\right) \\ &= r\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \\ V_1^* A_{12} R_1 & V_1^* A_{12} R_2 \\ V_2^* A_{12} R_1 & V_2^* A_{12} R_2 \end{pmatrix} \\ &= r(\Lambda) + r(V_2^* A_{12} R_2) \\ &= s + r(V_2^* A_{12} R_2). \end{aligned} \quad (35)$$

It follows from (b) and (35) that $V_2^T A_{12} R_2$ is a full column rank matrix, so it is nonsingular.

From $AB = I$, we have the following matrix equation:

$$A_{11}B_{11} + A_{12}B_{21} = I, \quad (36)$$

that is

$$A_{11}B_{11} = I - A_{12}B_{21}, \quad I \in \mathbb{H}^{m_1 \times m_1}, \quad (37)$$

where B_{11}, A_{12} were given, $B_{21} = -K$ (from (27)). By Lemma 3, the matrix equation (37) has a solution if and only if

$$\left(I - A_{12}B_{21}\right) B_{11}^{+}B_{11} = I - A_{12}B_{21}. \tag{38}$$

By (21), (27), (32), and (33), we have that (38) is equivalent to:

$$\left(I + A_{12}K\right) V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{*}U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^{*} = I + A_{12}K. \tag{39}$$

We simplify the equation above. The left hand side reduces to $(I + A_{12}K)V_1V_1^{*}$ and so we have

$$A_{12}KV_1V_1^{*} - A_{12}K = I - V_1V_1^{*}. \tag{40}$$

So,

$$A_{12}R\widehat{K}V^{*}V_1V_1^{*} - A_{12}R\widehat{K}V^{*} = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \begin{pmatrix} V_1^{*} \\ V_2^{*} \end{pmatrix} - V_1V_1^{*}. \tag{41}$$

This implies that

$$A_{12}R\widehat{K} \begin{pmatrix} V_1^{*}V_1 \\ V_2^{*}V_1 \end{pmatrix} V_1^{*} - A_{12}R\widehat{K} \begin{pmatrix} V_1^{*} \\ V_2^{*} \end{pmatrix} = V_2V_2^{*}, \tag{42}$$

so that

$$A_{12}R\widehat{K} \begin{pmatrix} I \\ 0 \end{pmatrix} V_1^{*} - A_{12}R\widehat{K} \begin{pmatrix} V_1^{*} \\ V_2^{*} \end{pmatrix} = V_2V_2^{*}. \tag{43}$$

So,

$$-A_{12}R\widehat{K} \begin{pmatrix} 0 \\ V_2^{*} \end{pmatrix} = V_2V_2^{*}, \tag{44}$$

and hence,

$$-\begin{pmatrix} A_{12}R_1 & A_{12}R_2 \end{pmatrix} \begin{pmatrix} \Lambda^{-1}Q_1^{*}A_{21}U_1\Sigma & 0 \\ H & K_{22} \end{pmatrix} \begin{pmatrix} 0 \\ V_2^{*} \end{pmatrix} = V_2V_2^{*}. \tag{45}$$

Finally, we obtain

$$A_{12}R_2K_{22}V_2^{*} = -V_2V_2^{*}. \tag{46}$$

Multiplying both sides of (46) by V^{*} from the left, considering (33) and the fact that $V_2^{*}A_{12}R_2$ is nonsingular, we have

$$K_{22} = -\left(V_2^{*}A_{12}R_2\right)^{-1}. \tag{47}$$

From Lemma 3, (38), (47), Problem 1 has a solution and the general solution is

$$A_{11} = B_{11}^{+} + A_{12}R \begin{pmatrix} \Lambda^{-1}Q_1^{*}A_{21}U_1\Sigma & 0 \\ H & -\left(V_2^{*}A_{12}R_2\right)^{-1} \end{pmatrix} \times V^{*}B_{11}^{+} + Y - YB_{11}B_{11}^{+}, \tag{48}$$

where H is an arbitrary matrix in $\mathbb{H}^{(n_2-s)\times r}$ and Y is an arbitrary matrix in $\mathbb{H}^{m_1\times n_1}$. □

3. An Example

In this section, we give a numerical example to illustrate the theoretical results.

Example 5. Consider Problem 1 with the parameter matrices as follows:

$$\begin{gathered} A_{12} = \begin{pmatrix} 2 + j & \frac{1}{2}k \\ -k & 1 + \frac{1}{2}j \end{pmatrix}, \\ A_{21} = \begin{pmatrix} \frac{3}{2} + \frac{1}{2}i & -\frac{1}{2}j - \frac{1}{2}k \\ \frac{1}{2}j + \frac{1}{2}k & \frac{3}{2} + \frac{1}{2}i \end{pmatrix}, \\ A_{22} = \begin{pmatrix} 2 & i \\ 2j & k \end{pmatrix}, \qquad B_{11} = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}. \end{gathered} \tag{49}$$

It is easy to show that (c), (d) are satisfied, and that

$$\begin{gathered} n_2 = r \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 2, \\ n_2 - r\left(A_{22}\right) = m_1 - r\left(B_{11}\right) = 0, \end{gathered} \tag{50}$$

so (a), (b) are satisfied too. Therefore, we have

$$\begin{gathered} B_{11}^{+} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}j \\ -\frac{1}{2}i & -\frac{1}{2}k \end{pmatrix}, \\ A_{22} = Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} R^{*}, \qquad B_{11} = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^{*}, \end{gathered} \tag{51}$$

where

$$\begin{gathered} Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \\ R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \\ \Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{52}$$

We also have

$$\begin{gathered} Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{53}$$

By Theorem 4, for an arbitrary matrices $Y \in \mathbb{H}^{2\times 2}$, we have

$$A_{11} = B_{11}^{+} + A_{12}R\left(\Lambda^{-1}Q_1^{*}A_{21}U_1\Sigma\right)V^{*}B_{11}^{+} + Y - YB_{11}B_{11}^{+} = \begin{pmatrix} \frac{3}{2}+\frac{1}{4}j+\frac{1}{4}k & \frac{3}{4}+\frac{1}{4}i-\frac{3}{2}j \\ \frac{1}{2}-i+\frac{1}{4}j-\frac{1}{4}k & \frac{1}{4}-\frac{3}{4}i-\frac{1}{2}j-k \end{pmatrix}, \tag{54}$$

it follows that

$$A = \begin{pmatrix} \frac{3}{2}+\frac{1}{4}j+\frac{1}{4}k & \frac{3}{4}+\frac{1}{4}i-\frac{3}{2}j & 2+j & \frac{1}{2}k \\ \frac{1}{2}-i+\frac{1}{4}j-\frac{1}{4}k & \frac{1}{4}-\frac{3}{4}i-\frac{1}{2}j-k & -k & 1+\frac{1}{2}j \\ \frac{3}{2}+\frac{1}{2}i & -\frac{1}{2}j-\frac{1}{2}k & 2 & i \\ \frac{1}{2}j+\frac{1}{2}k & \frac{3}{2}+\frac{1}{2}i & 2j & k \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} 1 & i & -1 & -1 \\ j & k & 0 & -1 \\ -1 & 0 & \frac{3}{4} & \frac{1}{2}-\frac{3}{4}j \\ -1 & -1 & \frac{1}{2}-i & \frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-k \end{pmatrix}. \tag{55}$$

The results verify the theoretical findings of Theorem 4.

Acknowledgments

The authors would like to give many thanks to the referees and Professor K. C. Sivakumar for their valuable suggestions and comments, which resulted in a great improvement of the paper. This research was supported by Grants from the Key Project of Scientific Research Innovation Foundation of Shanghai Municipal Education Commission (13ZZ080), the National Natural Science Foundation of China (11171205), the Natural Science Foundation of Shanghai (11ZR1412500), the Discipline Project at the corresponding level of Shanghai (A. 13-0101-12-005), and Shanghai Leading Academic Discipline Project (J50101).

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